3Y03 - 3J04 PROBABILITY & STATISTICS FOR (CIVIL) ENGINEERING

C01 - Lecture 28

Last time: HYPOTHESIS TEST on MEAN μ of Normal Distribution, Variance UNKNOWN

$H_0: \mu = \mu_0$

H_1: (I) $\mu \neq \mu_0$

(II) $\mu > \mu_0$

(III) $\mu < \mu_0$

$$T_0 = \frac{\overline{X} - \mu_0}{S/\sqrt{n}}$$ has t-distribution with $n-1$ degrees of freedom

Reject H_0 if

(I) $|t_0| > t_{\alpha/2, n-1}$ OR $-|t_0| < -t_{\alpha/2, n-1}$

(II) $t_0 > t_{\alpha, n-1}$

(III) $t_0 < -t_{\alpha, n-1}$

Compute $t_0 = \dfrac{\overline{x} - \mu_0}{s/\sqrt{n}}$.

P-values $H_0: \mu = \mu_0$

H_1	p-value
I) $\mu \neq \mu_0$	$P(T_0 > \lvert t_0 \rvert) + P(T_0 < -\lvert t_0 \rvert) = 2P(T_0 > \lvert t_0 \rvert)$
II) $\mu > \mu_0$	$P(T_0 > t_0)$
III) $\mu < \mu_0$	$P(T_0 < t_0)$

Example For a hypothesis test on mean μ of Normal population unknown variance, approximate the p-value when (a) $H_0: \mu = \mu_0$, $H_1: \mu \neq \mu_0$ with $t_0 = 2.537$, $n = 10$

(b) $H_0 : \mu = \mu_0$ with $t_0 = 1.863$
$H_1 : \mu > \mu_0$ and $n = 16$.

Solution (a) $n = 10$ so look up $n-1 = 9$ row in t-table
$t_0 = 2.537$ lies in $\frac{\alpha}{2}$ range $(0.01, 0.025)$
↳ tail prob. above $t_0 = 2.537$

So p-value lies in $(0.02, 0.05)$
(2-sided case).

(b) $n = 16$ so go to $n-1 = 15$ row in t-table
Find $t_0 = 1.863$ lies in α range $(0.025, 0.05)$

So p-value lies in $(0.025, 0.05)$.

9.5 Tests on Population Proportion

Recall $\hat{p} = \dfrac{X}{n} = \dfrac{\text{\# observations in class of interest}}{\text{\# observations}}$

is an estimator of proportion of total pop. in class

where $X \sim Bin(n, p)$

By Normal Approx. to Binomial $X \sim N(np, np(1-p))$
if $np > 5$
& $n(1-p) > 5$.

Test $\quad H_0: p = p_0$

$$H_1: \begin{cases} p \neq p_0 \\ p > p_0 \\ p < p_0 \end{cases}$$

If H_0 true (and $np_0 > 5$, $n(1-p_0) > 5$) then

$$X \sim N(np_0, np_0(1-p_0))$$

$$\Rightarrow \underbrace{\hat{p}}_{= \frac{X}{n}} \sim N\left(p_0, \frac{p_0(1-p_0)}{n}\right) \quad \& \text{ so}$$

$$Z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} \sim N(0,1)$$

Test Statistic.

$$\left[= \frac{X - np_0}{\sqrt{np_0(1-p_0)}} \quad \text{if you prefer} \right]$$

Calculate $z_0 = \dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$.

H_1	p-value
I) $p \neq p_0$	$P(Z_0 > \lvert z_0 \rvert) + P(Z_0 < -\lvert z_0 \rvert) = 2P(Z_0 > \lvert z_0 \rvert)$
II) $p > p_0$	$P(Z_0 > z_0)$
III) $p < p_0$	$P(Z_0 < z_0)$

Example A company claims that 90% of people who use their widget are satisfied with it. A random sample of 80 people are asked & 66 say that they are satisfied. Using signifance level $\alpha = 0.01$ is there evidence to support the company's claim?

Solution

$$H_0 : p = 90\% = 0.9 = p_0$$
$$H_1 : p < 90\% = 0.9.$$

Find $\hat{p} = \frac{66}{80} = 0.825.$

$$z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} = \frac{0.825 - 0.9}{\sqrt{\frac{(0.9)(0.1)}{80}}} = -2.24.$$

1-sided test

Compare $z_0 = -2.24$ to $-z_\alpha = -z_{0.01} = -2.33.$

$z_0 > -z_\alpha$ so do <u>not</u> reject H_0.

Type II Error & Sample Size

Recall $\beta = P(\text{do not reject } H_0 \text{ when } H_1 \text{ true})$

We again find β as a function of true value of pop. proportion p

e.g. if true value of $p = p' > p_0$:

$$\beta = P(Z_0 < z_\alpha)$$

$$= P\left(\frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}} < z_\alpha\right)$$

$$= P\left(\hat{p} < p_0 + z_\alpha\sqrt{\frac{p_0(1-p_0)}{n}}\right)$$

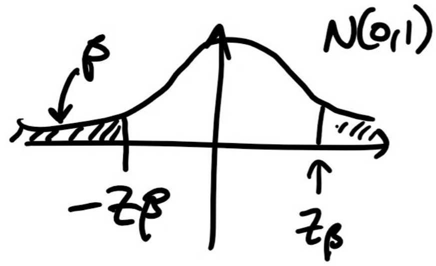
$$= P\Bigg(\underbrace{\frac{\hat{p} - p'}{\sqrt{\frac{p'(1-p')}{n}}}}_{\sim N(0,1) \text{ if } H_1 : p > p_0 \text{ is true}} < \frac{p_0 + z_\alpha\sqrt{\frac{p_0(1-p_0)}{n}} - p'}{\sqrt{\frac{p'(1-p')}{n}}}\Bigg)$$

So $\beta(p') = \Phi\left(\dfrac{p_0 - p' + z_\alpha\sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{\frac{p'(1-p')}{n}}}\right)$ where

p_0 is hypothesised value in H_0

& p' is true value of p.

i.e. $\dfrac{p_0 - p' + z_\alpha\sqrt{\frac{p_0(1-p_0)}{n}}}{\sqrt{\frac{p'(1-p')}{n}}} = -z_\beta$



Rearranging we get $n = \left(\dfrac{z_\alpha\sqrt{\frac{p_0(1-p_0)}{\cancel{n}}} + z_\beta\sqrt{\frac{p'(1-p')}{\cancel{n}}}}{p_0 - p'}\right)^2$

This is Case II, $H_1: p > p_0$ & Case III is same by symmetry; for the formula for n in 2-sided Case I see textbook.

Example In our earlier widget satisfaction example how many people should we sample to test

$H_0: p = 0.9$ at $\alpha = 0.05$

$H_1: p < 0.9$ when true proportion $p' = 0.85$ to get $\beta = 0.1$?

Solution $z_\alpha = z_{0.05} = 1.64$

$z_\beta = z_{0.1} = 1.18$

$p_0 = 0.9$

$p' = 0.85$

$$\text{So } n = \left(\frac{1.64\sqrt{(0.9)(0.1)} + 1.18\sqrt{(0.85)(0.15)}}{0.9 - 0.85} \right)^2$$

$$= 333.67 \rightarrow \underline{\underline{n = 334}}.$$