# 3Y03 - 3J04 PROBABILITY & STATISTICS FOR (CIVIL) ENGINEERING

## C01 - Lecture 36

## Last time: ANOVA F-TEST FOR SINGLE-FACTOR EXPERIMENT

$H_0 : \mu_1 = \mu_2 = \dots = \mu_a$

$H_1 : \mu_i \neq \mu_k$ for at least one pair $i \neq k$

$\mu_i$ = mean of ith treatment

Test statistic:

$$F_0 = \frac{SS_{Treatments}/(a-1)}{SS_E/a(n-1)} = \frac{MS_{Treatments}}{MS_E}$$

$MS_{Treatments}$: unbiased estimator for $\sigma^2$ $\underline{\underline{\text{IFF } H_0 \text{ TRUE}}}$

$MS_E$: unbiased estimator for $\sigma^2$

Reject $H_0$ at level $\alpha$ if $F_0 > f_{\alpha, a-1, a(n-1)}$ ($F_\alpha$-table)

Example from last time: 3 methods of brewing espresso.

| Method | 1 | 2 | 3 |
|---|---|---|---|
| | 36.64 | 70.84 | 56.19 |
| | 39.65 | 46.68 | 36.67 |
| | 37.74 | 73.19 | 35.35 |
| | 35.96 | 57.78 | 40.11 |
| | 38.52 | 48.61 | 33.52 |
| | 21.02 | 72.77 | 37.12 |
| | 24.81 | 65.04 | 37.33 |
| | 34.18 | 62.53 | 32.68 |
| | 23.08 | 54.26 | 48.33 |

**Anova: Single Factor**

SUMMARY

| *Groups* | *Count* | *Sum* | *Average* | *Variance* |
|---|---|---|---|---|
| Method 1 | 9 | 291.6 | 32.4 | 53.29087 |
| Method 2 | 9 | 551.7 | 61.3 | 102.0222 |
| Method 3 | 9 | 357.3 | 39.7 | 59.30182 |

ANOVA

| | *Source of Variation* | *SS* | *df* | *MS* | *F* | *P-value* | *F crit* |
|---|---|---|---|---|---|---|---|
| (Treatments) | Between Groups | 4065.18 | 2 | 2032.59 | 28.41261 | 4.69864E-07 | 3.402826 |
| (Error) | Within Groups | 1716.919 | 24 | 71.5383 | | | |
| | | | | | | | |
| | Total | 5782.099 | 26 | | | | |

at $\alpha = 0.05$

$H_0 : \mu_1 = \mu_2 = \mu_3$

$H_1$ : at least one pair $(i,k)$ has $\mu_i \neq \mu_k$

$$F_0 = \frac{MS_{Treatments}}{MS_E} = \frac{4065.18/2}{1716.919/24} = \frac{2032.59}{71.5383} = \underline{\underline{28.4}}$$

If we test at level $\alpha = 0.05$, need to compare with $f_{0.05,2,24} = \underline{\underline{3.40}}$.

Since $28.4 > 3.40$ reject $H_0$ for $H_1$.

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<u>Notice</u> (v. important!) This test does NOT tell us which of the claims in $H_0: \mu_1 = \mu_2, \mu_2 = \mu_3, \mu_1 = \mu_3$ is false — only that at least one of them is.

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<u>~~Recall~~</u> $\hat{\mu}_i$ is an estimator of $\mu_i$ and $MS_E$ is an unbiased estimator for $\sigma^2$

$\bar{Y}_{i\cdot}$

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So

$$T = \frac{\bar{Y}_{i\cdot} - \mu_i}{\sqrt{MS_E / n}} \sim t\text{-distribution with } a(n-1) \text{ degrees of freedom}$$

So we can get a $100(1-\alpha)\%$ C.I. for individual treatment means $\mu_i$:

$$\bar{y}_{i\cdot} \pm t_{\frac{\alpha}{2}, a(n-1)} \sqrt{\frac{MS_E}{n}}.$$

(& could also do t-tests about <u>individual</u> $\mu_i$s)

Example For our espresso example find a 95% C.I. for $\mu_2$, mean of method 2.

$a = 3$
$n = 9$

Solution Answer is $\bar{y}_{2\cdot} \pm t_{0.025,\, \underbrace{3(8)}_{=24}} \sqrt{\dfrac{MS_E}{9}}$

$$= 61.3 \pm 2.064 \sqrt{\frac{71.5383}{9}}$$

$$= 61.3 \pm 5.82 \quad \text{i.e. } (55.48, 67.12).$$

We can also get $100(1-\alpha)\%$ C.I.s for the difference between 2 treatment means by the ANOVA approach, by noting

$\mu_i - \mu_k$ say

$$\bar{Y}_{i\cdot} - \bar{Y}_{k\cdot} \quad \text{is an estimator for } \mu_i - \mu_k.$$

and
$$V(\bar{Y}_{i\cdot} - \bar{Y}_{k\cdot}) = V(\bar{Y}_{i\cdot}) + V(\bar{Y}_{k\cdot})$$
$$= \frac{\sigma^2}{n} + \frac{\sigma^2}{n} = \frac{2\sigma^2}{n}.$$

We use $MS_E$ as an estimator for $\sigma^2$ again.

A $100(1-\alpha)\%$ C.I. on difference $\mu_i - \mu_k$:

$$(\bar{y}_{i\cdot} - \bar{y}_{k\cdot}) \pm t_{\frac{\alpha}{2},\, a(n-1)} \sqrt{\frac{2\,MS_E}{n}}.$$

**Example** Find a 95% C.I. for difference in $\mu_1, \mu_2$ for our espresso example.

**Solution** Answer: $\left(\bar{y}_{1\cdot} - \bar{y}_{2\cdot}\right) \pm t_{0.025,24} \sqrt{\frac{2 MS_E}{9}}$

$$= (32.4 - 61.3) \pm 2.064 \sqrt{\frac{2 \times 71.5385}{9}}$$

$$= -28.9 \pm 8.23$$

i.e. $(-37.13, -20.67)$.

**Multiple Comparisons** Rejecting $H_0$ with ANOVA F-test means at least one pair $i \neq k$ has $\mu_i \neq \mu_k$.

Now we go back & test all pairs but using ANOVA setup. i.e. C.I. setup for difference between 2 means

$$\left.\begin{array}{l} H_0 : \mu_i = \mu_k \\ H_1 : \mu_i \neq \mu_k \end{array}\right\}$$ at level $\alpha$ : see if 0 lies in $100(1-\alpha)\%$ C.I. for $\mu_i - \mu_k$

If 0 does NOT, reject $H_0$.

Here we have a uniform way of forming these CIs:

$$(\bar{y}_{i\cdot} - \bar{y}_{k\cdot}) \pm \underbrace{t_{\frac{\alpha}{2}, a(n-1)} \sqrt{\frac{2 MS_E}{n}}}$$ ← $MS_E$ est. for $\sigma^2$ based on all

ME (Margin of Error) does not depend on $i, k$.

a treatments not just treatments $i$ and $k$

(it pools info from all sample variances, $i = 1, \ldots, a$)

So computations becomes easier (& uniform across all pairs) so more accurate

and Reject $H_0$ if $|\bar{Y}_{i\cdot} - \bar{Y}_{k\cdot}| > ME$

$$= \underbrace{t_{\frac{\alpha}{2}, a(n-1)} \sqrt{\frac{2MS_E}{n}}}$$

This common ME is called

<u>least significant difference</u> (LSD)

## Fisher's LSD Test/Method

For each pair $(i, k)$ test $H_0 : \mu_i = \mu_k$

$H_1 : \mu_i \neq \mu_k$

using the rule : if $|\bar{Y}_{i\cdot} - \bar{Y}_{k\cdot}| > LSD$, then reject $H_0$.

<u>Example</u> For our espresso example, we had

$$\bar{y}_{1\cdot} = 32.4, \quad \bar{y}_{2\cdot} = 61.3, \quad \bar{y}_{3\cdot} = 39.7$$

with $MS_E = 71.5385$, $n = 9$. Run an LSD test on the underlying means at $\alpha = 0.05$.

Solution First we find $LSD = t_{0.025,24}\sqrt{\frac{2MS_E}{9}}$

$= 8.23$ (from above)

$\mu_1$ v. $\mu_2$ : $|32.4 - 61.3| = 28.9 > 8.23$ !!!

$\mu_1$ v. $\mu_3$ : $|32.4 - 39.7| = 7.3 < 8.23$

$\mu_2$ v. $\mu_3$ : $|61.3 - 39.7| = 21.6 > 8.23$ !!!

So no significant difference between $\mu_1$ and $\mu_3$, but a significant difference between $\mu_1$ and $\mu_2$, and $\mu_2$ and $\mu_3$.

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Since we only carry out an LSD-test/method when we already know $H_0: \mu_1 = \mu_2 = \dots = \mu_a$ is to be rejected, we get a lower EER (Experiment-Wise Error Rate) = P(at least one Type I Error).

(But we don't get into the complexities of working out what that is in this course.)