3Y03 – 3J04 PROBABILITY & STATISTICS FOR (CIVIL) ENGINEERING

C01 - Lecture 38

~~Today~~ REVIEW

- One last thought on bias.
- Confidence intervals / tests → when to use which one?
- Normal Approximation to Binomial → don't forget the continuity correction!

Example Suppose $\left(\frac{X}{n}\right)^2 = \hat{p}^2$ is used to estimate p^2, where p = proportion of underlying population in a certain category.

Find the bias in $\hat{p}^2$.

Solution We need to find $E(\hat{p}^2) - p^2 = \text{bias}$.

$$E\left(\left(\frac{X}{n}\right)^2\right) = E\left(\frac{X^2}{n^2}\right) = \frac{1}{n^2}E(X^2)$$

linear function of r.v. X^2

Cannot use our "linear functions of r.v.s" formulas to find $E(X^2)$.

$$V(X) = E\left((X-\underset{\substack{\uparrow \\ E(X)}}{\mu})^2\right) = E(X^2) - (E(X))^2$$

So for any X: $\boxed{E(X^2) = V(X) + (E(X))^2}$

Here $X \sim \text{Bin}(n,p)$

So $E(X) = np$

$V(X) = np(1-p)$

So $E(X^2) = np(1-p) + n^2p^2$.

Hence $E\left(\frac{X^2}{n^2}\right) = \frac{1}{n^2}E(X^2) = \frac{1}{n^2}\left(np(1-p) + n^2p^2\right)$

$= \frac{p(1-p)}{n} + p^2$

$$\text{Bias} = E\left(\left(\frac{X}{n}\right)^2\right) - p^2 = \frac{p(1-p)}{n} + \cancel{p^2} - \cancel{p^2}.$$

Confidence Intervals & Test Statistics

(How to navigate lines 20-29 of Formula Sheet.)

In pairs

	<u>C.I.</u>	<u>Test</u>
	20) z-C.I. for mean μ	23) z-test for μ
	21) t-C.I. for μ	24) t-test for μ
	22) (z-) C.I. for proportion p	25) z-test for p

26), 27) (t-) C.I. for difference of 2 means $\mu_1 - \mu_2$ with variance unknown

28), 29) t-test for $\mu_1 - \mu_2$ with var. unknown.

First check what your C.I. / test is for:

mean μ, proportion p, difference in 2 means $\mu_1 - \mu_2$

(1) C.I.s/Tests for mean μ

z-tests/C.I.s

(A) Underlying pop: Normal

Variance σ^2: Known

Sample size n: irrelevant

C.I.: $\bar{x} \pm z_{\frac{\alpha}{2}} \frac{\sigma}{\sqrt{n}}$ (20)

Test: $z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$ (23) ← assumed in H_0

t-tests/C.I.s

(B) Underlying pop: Normal

Variance σ^2: UNknown

Sample size n: small ($n < 40$)

(21) C.I.: $\bar{x} \pm t_{\frac{\alpha}{2}, n-1} \frac{S}{\sqrt{n}}$

(24) Test: $t_0 = \frac{\bar{x} - \mu_0}{S/\sqrt{n}}$

(S^2 used to estimate σ^2)

(C) Underlying pop: ~~Normal~~ irrelevant (because use C.L.T.)

Variance σ^2: UNknown (est. with s^2)

Sample size n: large ($n > 40$)

Because of C.L.T., $n > 30$; because we est. σ^2 with s^2, goes up to 40.

C.I. $\bar{x} \pm z_{\frac{\alpha}{2}} \frac{s}{\sqrt{n}}$ ← Not on formula sheet — recover from l.20 with s in place of σ

(Test: $z_0 = (\bar{x} - \mu_0)/s/\sqrt{n}$) ← H_0-value

② C.I.s / Tests for proportion

→ No choice !

z - C.I. (22) $\hat{p} \pm z_{\frac{\alpha}{2}} \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$

z-test (25) $z_0 = \frac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$ under H_0

③ C.I.s / Tests on difference in means $\mu_1 - \mu_2$ v. 0 when variance σ^2 is unknown

→ Always a t - test

<u>Question</u> : Are we allowed to assume $\sigma_1^2 = \sigma_2^2$ ("variances equal")
or not allowed to assume $\sigma_1^2 = \sigma_2^2$ so have to assume $\sigma_1^2 \neq \sigma_2^2$ ("variances unequal") ?

Equal : (26.) C.I : $\bar{x}_1 - \bar{x}_2 \pm t_{\frac{\alpha}{2}, n_1+n_2-2} \, s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$

s_p^2 = pooled sample variance (26.)

(28.) Test : $t_{n_1+n_2-2} = \frac{\bar{x}_1 - \bar{x}_2}{s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}}$

Unequal : (27.) C.I. $\bar{x}_1 - \bar{x}_2 \pm t_{\frac{\alpha}{2}, \nu} \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}$

ν : degrees of freedom (27.)

(28.) Test : $t_\nu = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}}}$

Critical Values & P-Values

$z_{\frac{\alpha}{2}}$ or $t_{\frac{\alpha}{2},\dots}$ or z_α or $t_{\alpha,\dots}$ ← compare test stat. value to this

(or $-z_{\alpha/2}$ or $-t_{\alpha/2},\dots$) or $-z_\alpha$ or $-t_\alpha$

2-sided (The one that is relevant

1-sided

P-value = P(test stat. takes a value at least as extreme as its observed value)

↳ compare to α.

All the steps are listed on formula sheet → even if test stat. is not.

Review ANOVA steps !!!

Normal Approximation to Binomial

$X \sim Bin(n,p)$ (mean : np variance : $np(1-p)$)

and $np > 5$, $n(1-p) > 5$ we have :

$$\frac{X - np}{\sqrt{np(1-p)}} \sim N(0,1).$$

To use this in practice : continuity correction

$$P(X \le x) = P(X \le x + 0.5)$$

$$= P\left(Z \leq \frac{(x+0.5)-np}{\sqrt{np(1-p)}}\right)$$

$$P(X = x) = P(X \leq x) - P(X \leq x-1)$$

$$= P\left(Z \leq \frac{(x+0.5)-np}{\sqrt{np(1-p)}}\right) - P\left(Z \leq \frac{\overset{(x-0.5)}{\cancel{(x-1+0.5)}}-np}{\sqrt{np(1-p)}}\right)$$

$$P(x_1 \leq X \leq x_2) = P(X \leq x_2) - P(X \leq x_1 - 1)$$

$$= P\left(Z \leq \frac{x_2+0.5-np}{\sqrt{np(1-p)}}\right) - P\left(Z \leq \frac{x_1-0.5-np}{\sqrt{np(1-p)}}\right)$$

$$= P\left(\frac{(x_1-0.5)-np}{\sqrt{np(1-p)}} \leq Z \leq \frac{x_2+0.5-np}{\sqrt{np(1-p)}}\right).$$

$\longrightarrow$ Expand the region when making the continuity correction e.g. if the region would be $[x_1, x_2]$ as in the last example, expand to $[x_1 - 0.5, x_2 + 0.5]$ and then standardize.

e.g. if the region is $\{x\}$, expand to $[x-0.5, x+0.5]$ & standardize.

& e.g. if the region is $[x, \infty)$ or $(-\infty, x]$

expand to $[x-0.5, \infty)$ or $(-\infty, x+0.5]$.
& standardize.

Final Thoughts:

1. COURSE EVALUATIONS: I'm sure you're very busy, but please take a 10-minute break before the end of Thursday Dec. 6th to review this course — help us to make it better (as well as tell us what we're doing right)!

2. You've got this — work hard, but keep your energy levels up for the long haul. Practice, practice, practice — but in the end, be confident you'll do your best, and remember that's anyway all anyone can ask of you. GOOD LUCK!!!