3Y03 - 3J04 PROBABILITY & STATISTICS FOR (CIVIL) ENGINEERING

C01 - Lecture 8

Last time PROBABILITY MASS FUNCTION (p.m.f.):

$$f(x) = P(X = x)$$

CUMULATIVE DISTRIBUTION FUNCTION (c.d.f.):

$$F(x) = P(X \leq x)$$

MEAN: $\mu = E(X) = \sum_x x f(x)$

VARIANCE: $\sigma^2 = V(X) = E\left((X-\mu)^2\right) = \sum_x (x-\mu)^2 f(x)$

$$= \sum_x x^2 f(x) - 2\mu \underbrace{\sum_x x f(x)}_{\mu} + \mu^2 \underbrace{\sum_x f(x)}_{1}$$

$$= \sum_x x^2 f(x) \overset{-2\mu^2}{} - \mu^2$$

Example X = price difference in \$ with p.m.f. $f(x)$ given by:

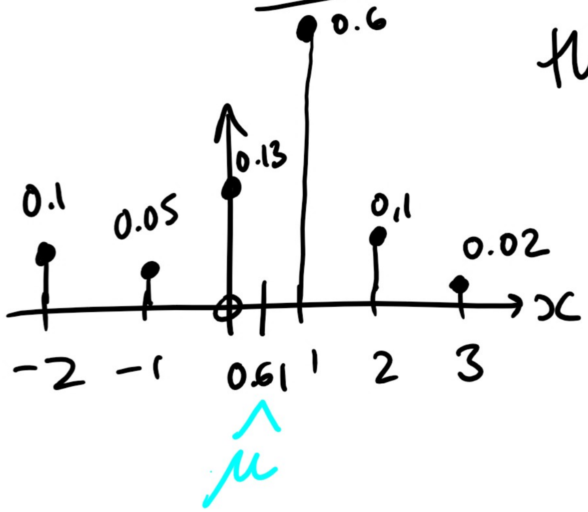
x	-2	-1	0	1	2	3
$f(x)$	0.1	0.05	0.13	0.6	0.1	0.02

Determine μ, σ^2, σ ← standard deviation.

Solution $\mu = \sum_x x f(x) = (-2)(0.1) + (-1)(0.05) + 0 + (1)(0.6) + (2)(0.1) + 3(0.02)$

$= \underline{\underline{\$0.61}}.$

<u>Notice</u> X never takes value 0.61 but this is still the mean.



$$\sigma^2 = V(X) = \sum_x x^2 f(x) - \mu^2$$

$$= 4(0.1) + 1(0.05) + 0 + 1(0.6) + 4(0.1) + 9(0.02) - (0.61)^2$$

$$= \underline{\underline{1.2579}} \ \$^2$$

$$\sigma = \sqrt{V(X)} = \sqrt{1.2579} = \underline{\underline{\$1.12}}$$

In general, the <u>expected value</u> of any function $h(X)$ of our discrete r.v. X with p.m.f. $f(x)$ is given by

$$\boxed{E(h(X)) = \sum_x h(x) f(x).}$$

<u>Notice</u> $h(X)$ is itself a (discrete) r.v. Y with its own

p.m.f. $f_h(y)$. So $E(Y) = \sum_y y f_h(y)$.

It can be shown that these give same answer.

Also notice: Our alt. formula for variance is

$$\sigma^2 = V(X) = \sum_x x^2 f(x) - \mu^2$$

$$\Rightarrow \boxed{\sigma^2 = V(X) = E(X^2) - [E(X)]^2}$$

Observation If a, b are constants, then

(1) $E(aX+b) = aE(X) + b$ (Scaling X by a scales the mean, and shifting by b shifts it by b.)

(2) $V(aX+b) = a^2 V(X)$.

(Scaling X by a scales variances by a^2 but shifting by b does not change how spread out the distribution is.)

Special Probability Distributions

[3.6] Binomial Distribution

- Some number n of independent "trials"

(single experiment that can be repeated)

→ each trial has 2 mutually exclusive possible outcomes (e.g. 0 or 1, arrival or not, Heads or Tails, win or loss, yes or no, faulty or not, success or failure)

↳ This is called a Bernoulli trial

→ probability of success is p (same every time)

X = # successes is called a binomial random variable with parameters (n, p)

[and has a binomial (probability) distribution]

and has p.m.f. $f(x) = \underbrace{\binom{n}{x}}\, p^x (1-p)^{n-x}$ for each $x = 0, \ldots, n$.

C^n_x : # sequences with x successes (& $n-x$ failures) from n trials

Example Multiple Choice Test with $n = 6$ questions

5 options for each question; randomly guess answers

so $p = 0.2 = 1/5$. X = # correct answers;

$$P(X=0) = f(0) = \binom{6}{0}(0.2)^0(0.8)^6 = 0.26$$

$$P(X=1) = f(1) = \binom{6}{1}(0.2)^1(0.8)^5 = 0.39$$

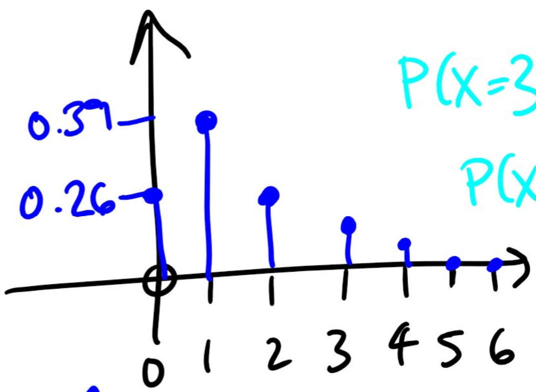
$$P(X=2) = f(2) = \binom{6}{2}(0.2)^2(0.8)^4 = 0.25$$

$$P(X=3) = f(3) = \binom{6}{3}(0.2)^3(0.8)^3 = 0.08$$

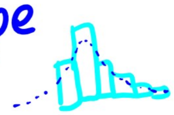
$$P(X=4) = f(4) = \binom{6}{4}(0.2)^4(0.8)^2 = 0.02$$

$$P(X=5) = f(5) = \binom{6}{5}(0.2)^5(0.8)^1 = 0.00$$

$$P(X=6) = f(6) = \binom{6}{6}(0.2)^6(0.8)^0 = 0.00$$



Notice the shape



Notice that for binomial distribution ← prob. distr. of a bin. r.v.

the c.d.f. $(P(X \le x_0) =) F(x_0) = \sum_{x \le x_0} f(x)$

$$= \sum_{x \le x_0} \binom{n}{x} p^x (1-p)^{n-x}.$$

Example In the above example,

find $P(X=4)$, $P(X \le 3)$, $P(2 \le X \le 5)$ and $P(X \ge 3)$

Solution $P(X=4) = f(4) = \underline{\underline{0.02}}.$

$$P(X \le 3) = \sum_{x \le 3} f(x) = f(0) + \dots + f(3)$$

$$= \underline{\underline{0.98}}$$

$$P(2 \leq X \leq 5) = \sum_{x=2}^{5} f(x) = \underline{\underline{0.35}}$$

$$P(X \geqslant 3) = 1 - P(X \leq 2) = 1 - (f(0) + f(1) + f(2))$$

We could also take $\sum_{x=3}^{6} f(x) = f(3) + f(4) + f(5) + f(6)$.

(Know that both options are available so that you can do whichever is easiest!!)