3Y03 - 3J04 PROBABILITY & STATISTICS FOR (CIVIL) ENGINEERING

C01 - Lecture 9

Last time

BINOMIAL DISTRIBUTION

- n Bernoulli trials < success / failure, Prob. p / $1-p$

X = # successes. X is a binomial random variable

$$P(X=x) = f(x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for } x = 0, 1, \dots, n.$$

For ith trial define new random variable

$$X_i = \begin{cases} 1 & \text{if } i\text{th trial is success} \\ 0 & \text{—— '' —— failure} \end{cases}$$

Then $X = X_1 + \dots + X_n$

Each X_i binomial ($n=1$ & $p=p$) so

$$E(X_i) = \sum_{x=0}^{1} x \binom{1}{x} p^x (1-p)^{1-x}$$

$$= 0 + 1 \binom{1}{1} p^1 (1-p)^{1-1}$$

$$= p$$

Later we'll justify $E(X) = E(X_1) + \dots + E(X_n)$

So $\mu = E(X) = np$.

We can show that $V(X_i) = \sum_{x=0}^{1} (x-p)^2 \binom{1}{x} p^x (1-p)^{1-x}$

$= \dots = p(1-p)$.

Again later we'll justify: $V(X) = V(X_1) + \dots + V(X_n)$

$\sigma^2 = V(X) = np(1-p)$.

Geometric Distribution

Again successive (independent) Bernoulli trials, with prob. p of success.

we always assume this with Bernoulli trials

Now NOT a fixed # of trials, INSTEAD keep going until one success.

e.g. play succession of video games, independently of one another; prob. of losing $= p$

careful; here success = loss

Keep playing until lose

X = # games until loss.

$$P(X = x) = f(x) = \underbrace{(1-p)\cdot(1-p)\cdot\ldots\cdot(1-p)}_{x-1}\, p$$

$$P(X=x) = f(x) = \underbrace{(1-p)^{x-1}\, p}_{\text{notice}}$$

$$p\sum_{x=1}^{\infty}(1-p)^{x-1}$$

$$\mu = E(X) = \sum_{x=1}^{\infty} x\, f(x) = \sum_{x=1}^{\infty} \underbrace{x\,(1-p)^{x-1}}_{-\frac{d}{dp}(1-p)^{x}}\, p$$

$$= \left[-\frac{d}{dp}\left(\sum_{x=1}^{\infty}(1-p)^{x-1}\right)\right]p$$

$$= -\frac{d}{dp}\left(\frac{1}{1-(1-p)}\right)p$$

$$= -\frac{d}{dp}\left(\frac{1}{p}\right)\cdot p = \frac{1}{p^2}\cdot p = \frac{1}{p}.$$

Also can show using 2nd derivative that

$$\sigma^2 = V(X) = \frac{1-p}{p^2}.$$

<u>Example</u> In Video Game example above $\underset{\substack{\|\\ p}}{}\;p(\text{loss}) = 0.3$

$X =$ # games played until loss

$$E(X) = \frac{1}{p} = \frac{1}{0.3} = 3\tfrac{1}{3}.$$

Can also think Expected Value of X = "return rate"

e.g. P(storm hits in a given year) = p (years independent)

$\rightarrow$ Expect a storm on average once every $\frac{1}{p}$ years.

Negative Binomial Distribution

Generalization of Geometric Distribution

- successive Bernoulli trials with prob. of success = p
- keep going until success # r

Binomial: # trials fixed (n)
successes counted

Negative Binomial: # trials counted to get to fixed # of successes (r).

X = # trials until rth success

$$P(X=x) = f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \quad \text{for } x = r, r+1, \ldots$$

$$\underbrace{\overbrace{\_\ \_\ \_\ \_\ \_\ \_}^{(r-1)\text{ successes}}\ \underline{S}}_{x \text{ trials}} \leftarrow r\text{th success}$$

Can think of negative binomial random var. as sum of geometric random variables:

\# trials till 1st success

\# extra " " 2nd "

⋮

\# extra trials till rth success

Later we'll use this to see $\mu = E(X) = \frac{r}{p}$

$$V(X) = \frac{r(1-p)}{p^2}.$$

Example Keep buying lightbulbs until you have exactly enough for 5 light fixtures.

But prob. a lightbulb still works when you get it home is $p = 0.75$. (Each bulb independent.)

What is prob. you need to buy at most 6 bulbs?

Solution $X = $ # bulbs tried

$r = 5, \quad p = 0.75$

$$P(X \le 6) = \sum_{x=5}^{6} \binom{x-1}{5-1} p^5 (1-p)^{x-5}$$

$$= \binom{4}{4}(0.75)^5(0.25)^{\cancel{5-5}} + \binom{5}{4}(0.75)^5(0.25)$$

$$= 0.5339\ldots$$

Hypergeometric Distribution

Fundamentally different : trials NOT independent

Here based on sampling without replacement

N items in 2 classes ($\overset{K}{\text{"successes"}}$, $\overset{N-K}{\text{"failures"}}$)

Sample n times without replacement randomly.

$X = \#$ successes

$$P(X=x) = f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}.$$