3Y03 - PROBABILITY AND STATISTICS FOR ENGINEERING

WS19 Lecture 12

Last time POISSON DISTRIBUTION

- X = # of "incidents"/"flaws" in some time period/region.
- p.m.f. (discrete r.v.) is $f(x) = \frac{\lambda^x}{x!} e^{-\lambda}$ where $\lambda = \mu = E(X)$ $(= \sigma^2)$

So "λ" = "λT" = μ !!!

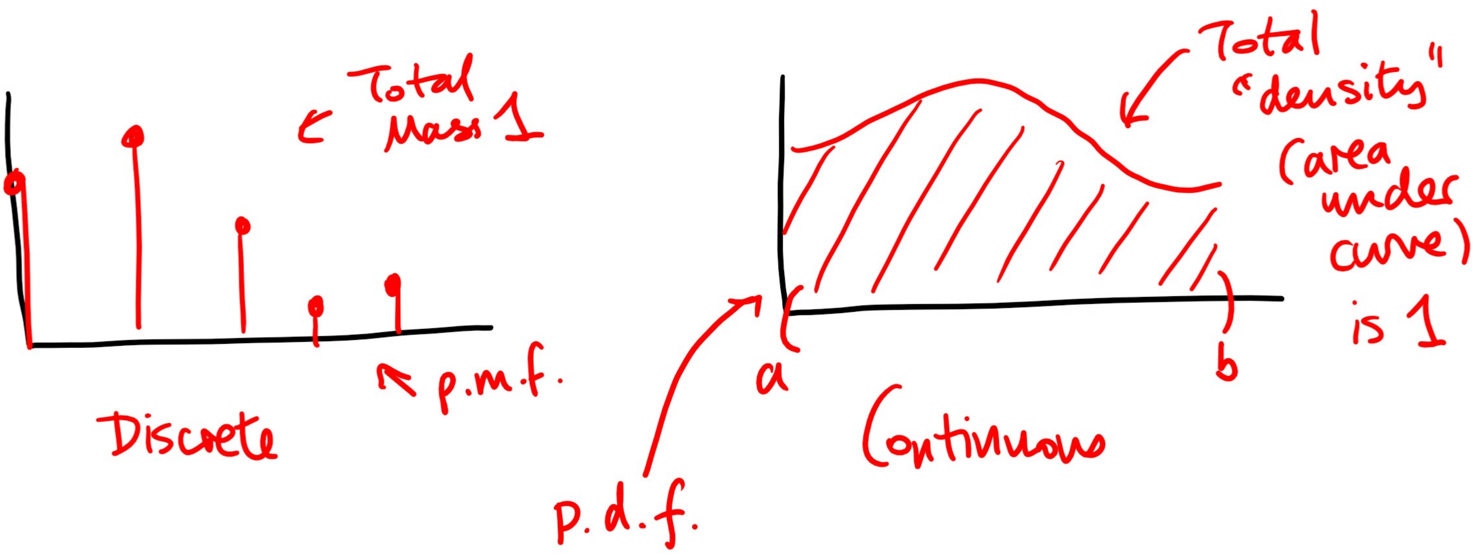
- The parameter "λ" in $f(x)$ is the PRODUCT of:

(also called λ) average rate of incidents/flaws per unit time/area

& T total time window/region area (in which X is counted)

CONTINUOUS RANDOM VARIABLES

X Recall $P(X = x) = 0$



A probability density function (pdf) $f(x)$ of a cont. r.v. X is a function with

(1) $f(x) \geqslant 0$, $f(x) < \infty$

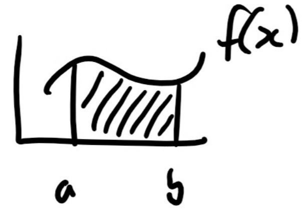
(2) $\displaystyle\int f(x)\,dx = 1$ ("total prob. $=1$")

Region on which X takes values

(3) $P(a \leq X \leq b) = \displaystyle\int_a^b f(x)\,dx$

$\left(= P(a < X < b)\right)$

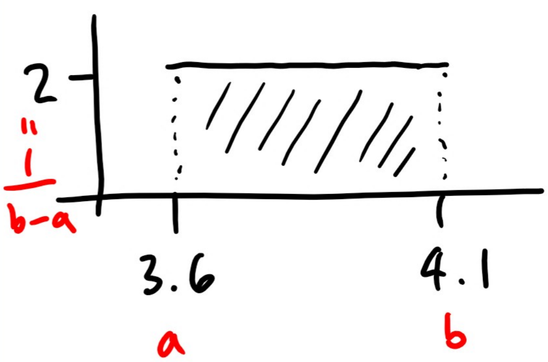
$P(X=a)=0$ $\quad$ $P(X=b)=0$



Example (Uniform Distribution)

Say electric current X in a copper wire with possible values between 3.6 and 4.1. X distributed uniformly with p.d.f.

$$f(x) = \begin{cases} 2 & 3.6 < x < 4.1 \\ 0 & \text{otherwise} \end{cases}$$



$P(3.7 < X < 3.8)$

$$= \int_{3.7}^{3.8} 2\,dx = \left[2x\right]_{3.7}^{3.8} = 0.2.$$

Example Waiting time in hours at a hospital for admission to Emerge. is given by distribution with p.d.f. $f(x) = \begin{cases} 0.5\, e^{-0.5x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

Find $P(1 < X < 2) = \int_1^2 0.5\, e^{-0.5x}\, dx = \left[-e^{-0.5x} \right]_1^2$

$= e^{-0.5} - e^{-1}$

$= 0.2386.$

Find $P(X > 3) = 1 - P(0 < X < 3)$

$= 1 - \int_0^3 0.5\, e^{-0.5x}\, dx$

$= 1 - \left[-e^{-0.5x} \right]_0^3 = 0.2231$

$1 = \int_0^\infty 0.5 e^{-0.5x}\, dx$

So $P(X > 3) = \int_0^\infty 0.5 e^{-0.5x}\, dx - \int_0^3 0.5 e^{-0.5x}\, dx$

$= \int_3^\infty 0.5 e^{-0.5x}\, dx$

What about $P(X \leq x)$ (for a cont. r.v. with p.d.f. $f(x)$).

$= \int_{\text{bottom of range of } X}^{x} f(t)\, dt$ = anti-derivative of $f(x)$

So <u>cumulative distribution function</u> is

$$F(x) \left(= P(X \le x) \right) \quad \text{with} \quad \frac{d}{dx} F(x) = f(x)$$

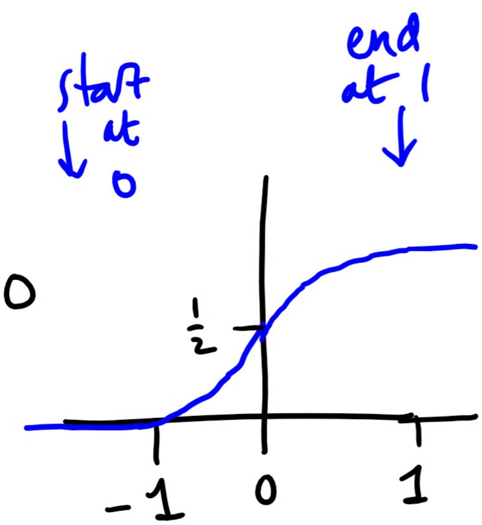
and $P(a < X < b) = F(b) - F(a)$.

(<u>Notice</u> $F(x)$ continuous here — in discrete r.v. world c.d.f. $F(x)$ is step function)

<u>Example</u>

Suppose c.d.f. of X is given by

$$F(x) = \begin{cases} 0 & \text{for } x \le -1 \\ \frac{1}{2}(x+1)^2 & \text{for } -1 < x \le 0 \\ 1 - \frac{1}{2}(x-1)^2 & \text{for } 0 < x < 1 \\ 1 & \text{for } x \ge 1 \end{cases}$$



Find $P(X \le \frac{1}{2})$, $P(X > 0)$ & $f(x)$.

$$P(X \le \tfrac{1}{2}) = F(\tfrac{1}{2}) = 1 - \tfrac{1}{2}\left(\tfrac{1}{2} - 1\right)^2 = 1 - \tfrac{1}{8} = \tfrac{7}{8}.$$

$$P(X > 0) = 1 - P(X \le 0) = 1 - F(0) = 1 - \tfrac{1}{2} = \tfrac{1}{2}.$$

$$f(x) = \frac{d}{dx} F(x) = \begin{cases} 0 & \text{for } x \le -1 \\ x+1 & \text{for } -1 < x \le 0 \\ 1-x & \text{for } 0 < x < 1 \\ 0 & \text{for } x \ge 1. \end{cases}$$

Example

Aside for now: The **median** is the value m s.t. $P(X \le m) = P(X \ge m) = \frac{1}{2}$.

(If this defines a unique value ; more general definition will come later.)

Distribution for amount of salt sold is given by a cont. r.v. with p.d.f. $f(x) = \begin{cases} 2(1-x) & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

Find the median.

Solve $\underbrace{P(X \le m)}_{= F(m)} = 0.5$

$$0.5 = F(m) = \int_0^m f(x)\,dx = \int_0^m 2(1-x)\,dx = \left[2x - x^2\right]_0^m$$

$$= 2m - m^2$$

$\Rightarrow$ find m with $2m - m^2 - 0.5 = 0$

$\Rightarrow m^2 - 2m + 0.5 = 0$

$\Rightarrow m = 1 \pm \frac{1}{\sqrt{2}}$ $\rightarrow$ the only valid solution in $[0,1]$ is $m = 1 - \frac{1}{\sqrt{2}}$

$\approx 0.2929.$

4.4 Mean & Variance of Continuous r.v.s

$\int$ replaces $\sum$

If X is a cont. r.v. with p.d.f. $f(x)$ then the mean

$$\mu = E(X) = \int_{\text{Region}} x f(x)\, dx$$

In general, for any function $h(X)$ the expected value of $h(X)$ is

$$E(h(X)) = \int_{\text{Region}} h(x) f(x)\, dx.$$

Another special case:

Variance of X $\quad V(X) = \sigma^2 = E((X-\mu)^2)$

$$= \int_{\text{Region}} (x-\mu)^2 f(x)\, dx$$

$$= \ldots = E(X^2) - (E(X))^2$$

(as in discrete case).