3Y03 - PROBABILITY AND STATISTICS FOR ENGINEERING

WS19 Lecture 15

GAME PLAN

This week:

4.8 Exponential Distribution

(some of) 5.1 ; 5.2 Two Random Variables

$\boxed{5.4}$ Linear Functions of Random Variables

When we return: 6 etc. STATISTICS.

Recall : Poisson Distribution - discrete

- models # incidents/flaws in given time/region.

Exponential Distribution - continuous

- models distance between incidents

X = amount of time (from start) until an incident

$P(X > x) = P$(in time window of duration x no incidents)

Let N be a discrete r.v. with Poisson distribution parameter λ/unit time ; in interval of length x. N has mean λx

So $P(X > x) = P(N = \boxed{0}) = e^{-\lambda x} \dfrac{(\lambda x)^{\boxed{0}}}{\boxed{0}!}$

$$= e^{-\lambda x}, \quad x \geq 0.$$

We want $F(x) = P(X \leq x) = 1 - e^{-\lambda x}, \quad x \geq 0.$

↑ Prob. there's an incident in the first x seconds/minutes/etc.

So $\boxed{f(x) =} \dfrac{d}{dx} F(x) = \boxed{\lambda e^{-\lambda x}} \boxed{, \quad x \geq 0}$ ←

The pdf of the Exponential Distribution.

<u>Notice</u> X depends only on waiting time (x) not <u>from when</u> (start time) we started observing.

If X has exp. distr. with parameter λ,

$$\mu = E(X) = \int_0^\infty x \lambda e^{-\lambda x} \, dx \overset{\text{I.P.}}{=} \ldots = \frac{1}{\lambda}.$$

Similarly, $\sigma^2 = E(X^2) - (E(X))^2 = \dfrac{1}{\lambda^2}$.

<u>Example</u> Time between emails arriving in your inbox is exp. distributed with mean 3 minutes.

(a) What is the prob. you receive an email in the next 2 minutes?

(b) What is the prob. you receive an email in the next 2 minutes given that you've already waited 5 mins for an email?

$X =$ waiting time What's λ?

Time between arrivals has mean $3 = \frac{1}{\lambda} \Rightarrow \lambda = \frac{1}{3}$.

So pdf $f(x) = \frac{1}{3} e^{-x/3}$, $x \geq 0$.

(a) $P(X \leq 2) = F(2) = 1 - e^{-2/3} = 0.49$.

$$\int_0^2 \frac{1}{3} e^{-x/3}\, dx$$

(b) $$P(\underbrace{X \leq 7}_{A} \mid \underbrace{X > 5}_{B}) = \frac{P(\overbrace{5 \leq X \leq 7}^{A \cap B})}{P(\underbrace{X \geq 5}_{B})}$$

$$= \frac{\int_5^7 \frac{1}{3} e^{-x/3}\, dx}{\int_5^\infty \frac{1}{3} e^{-x/3}\, dx} = \frac{\left[-e^{-x/3}\right]_5^7}{\left[-e^{-x/3}\right]_5^\infty}$$

$$= \frac{-e^{-7/3} + e^{-5/3}}{e^{-5/3}} = 1 - e^{-2/3}$$

$$= 0.49.$$

In general, if X has exp. distr.

$$P(X < t_1 + t_2 \mid X > t_1) = P(X < t_2)$$

LACK OF MEMORY PROPERTY.

Chapter 5 More than 1 Random Variable

Start with 2 random measurements

- might (not) be related
- relationship might not be obvious

We work with <u>continuous</u> r.v.s from now on.

<u>Definition</u> The <u>joint</u> p.d.f. for 2 continuous r.v.s X and Y is denoted $f_{XY}(x,y)$ & satisfies

(1) $f_{XY}(x,y) \geq 0$

(2) $$\iint f_{XY}(x,y)\,dx\,dy = 1$$

(Volume under graph of f_{XY} $= 1$)

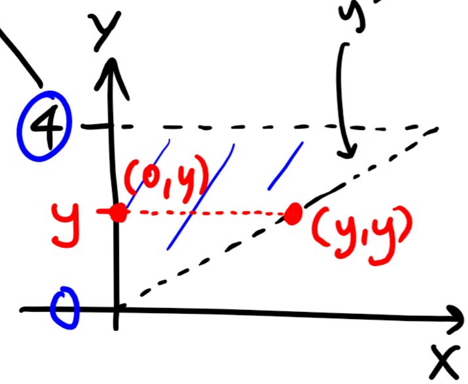
(3) $P((X,Y) \in R)$

$$= \iint_R f_{XY}(x,y)\,dx\,dy$$

e.g. $$P(X \in A, Y \in B) = \int_B \int_A f_{XY}(x,y)\,dx\,dy$$

<u>Example</u> Suppose X & Y are random variables with $0 < X < Y < 4$ and

$$f_{XY}(x,y) = c(x+y).$$



(a) Find c.

(b) Find $P(X<3,\ Y<2)$.

(c) Find $P(X<2,\ Y<3)$.

<u>Solution</u> (a) We know $\displaystyle\iint_{\text{domain}} c(x+y)\,dx\,dy = 1$

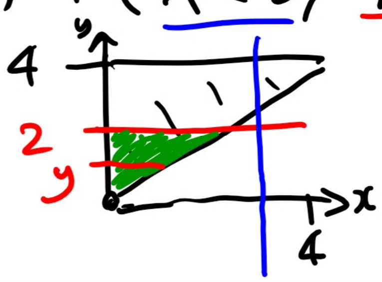
i.e. $$1 = \int_0^4\int_0^y c(x+y)\,dx\,dy = \int_0^4 \left[\frac{cx^2}{2} + cyx\right]_0^y dy$$

$$= \int_0^4 \frac{cy^2}{2} + cy^2\ dy$$

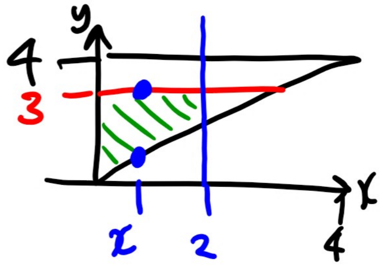
$$= \left[\frac{cy^3}{2}\right]_0^4 = 32c$$

$$\Rightarrow c = \frac{1}{32}.$$

(b) $$P(\underline{X<3},\ \underline{Y<2}) = \int_0^2\int_0^y \frac{1}{32}(x+y)\,dx\,dy = \ldots = \frac{1}{8}.$$



(c) $P(X<2,\ Y<3)$



$$\int_0^2 \int_x^3 \frac{1}{32}(x+y)\,dy\,dx = \frac{11}{32}.$$