3Y03 - PROBABILITY AND STATISTICS FOR ENGINEERING

WS19 **Lecture 20**

Last Time

Random Sample: $\{\underbrace{X_1, \dots, X_n}_{\text{independent \& same distribution}}\}$

Parameter

θ A quantity associated with the underlying distribution (of the X_i)

e.g. mean μ, variance σ^2

Statistic

Any function $h(X_1, \dots, X_n)$ of the random sample

e.g. $\sum_{i=1}^{2} X_i^2$, $\bar{X} = \frac{1}{n}(X_1 + \dots + X_n)$

Estimator

$\hat{\Theta}$ A statistic used to estimate a parameter

e.g. $\bar{X}$ or median for μ

S^2 for σ^2, S for σ

Estimate

Output of an estimator $\hat{\theta}$

e.g. $\bar{x}$ for $\bar{X}$, s^2 for S^2, s for S, m for median

Examples of Parameters (aside from μ & σ^2!)

→ proportion p of population in some class

Good (point) estimator: $\hat{p} = \dfrac{\text{\# items in sample in class}}{n}$

n ← total sample size

→ difference in means of 2 populations $\mu_1 - \mu_2$

Good estimate: $\bar{x}_1 - \bar{x}_2$

→ difference in proportions of 2 populations $p_1 - p_2$

Good estimate: $\hat{p}_1 - \hat{p}_2$

Not only one choice for an estimator e.g.

to estimate μ : - sample mean $\bar{X}$

- median

- trimmed mean: $\bar{X}_{tr(k)}$ is k% timmed mean:

throw away smallest k% & lagest k% of data points, the calculate 'sample mean'

- X_i - ith observation

Will return to 7.2

7.3 Concepts of Point Estimation

↪ how to choose an estimator.

How "good" is $\hat{\Theta}$ as estimator of θ?

Bias We can talk about $E(\hat{\Theta})$. ← f^n of r.v.s

If $E(\hat{\Theta}) = \theta$, then $\hat{\Theta}$ is an unbiased estimator of θ

If $E(\hat{\Theta}) \neq \theta$, then $E(\hat{\Theta}) - \theta$ = bias of $\hat{\Theta}$.

Example $\bar{X}$ unbiased estimator for μ (see 5-4)

✓ S^2 " " " σ^2

$\downarrow$

$$E(S^2) = E\left(\frac{1}{n-1}\left(\sum_{i=1}^{n} X_i^2 - n\overline{X}^2\right)\right)$$ ← linear comb. of r.v.s X_i^2 & $\overline{X}^2$

$$= \frac{1}{n-1}\left(\sum_{i=1}^{n} E(X_i^2) - n\,E(\overline{X}^2)\right)$$ ← To work out $E(X_i^2)$ & $E(\overline{X}^2)$ use that for any r.v. Y

$$= \frac{1}{n-1}\left(n(\sigma^2+\mu^2) - n\left(\frac{\sigma^2}{n}+\mu^2\right)\right)$$

$$= \frac{1}{n-1}\left(n\sigma^2 - \sigma^2 + \cancel{n\mu^2} - \cancel{n\mu^2}\right)$$

$$= \sigma^2$$

$$!\ \begin{cases} V(Y) = E(Y^2) - E(Y)^2 \\ E(Y^2) = V(Y) + E(Y)^2 \end{cases}$$

So $E(X_i^2) = \sigma^2 + \mu^2$

$E(\overline{X}^2) = \frac{\sigma^2}{n} + \mu^2$

$\boxed{\text{So remember } E(S^2) = \sigma^2}$

<u>Non-example</u> S is <u>biased</u> estimator of σ.

<u>Example</u> $E(X_i) = \mu$ so also unbiased est. of μ.

Median also unbiased est. of μ.

Comparison of Estimators

e.g. lots of unbiased ests. for μ – which is "best"?

One method: the one with the smallest variance.

Idea: smaller variance = more likely to

produce estimate close to true parameter value.

For a param. θ, amongst unbiased estimators of θ, the one with smallest variance is the MVUE : Minimum Variance Unbiased Estimator

Example $V(\bar{X}) = \dfrac{\sigma^2}{n}$ ← smaller variance.

than

$V(X_i) = \sigma^2$

In fact, the MVUE for μ is $\bar{X}$.

Error – way to capture how precise est. $\hat{\Theta}$ is.

↳ Again use variation as measure of "goodness":

Standard Error of $\hat{\Theta}$ is $\sqrt{V(\hat{\Theta})} = \sigma_{\hat{\Theta}}$.

Potential problem: to calculate this, you need the value of some (other) parameter

But – can still estimate such a parameter

↳ The estimated standard error is $\hat{\sigma}_{\hat{\Theta}}$ or $se(\hat{\Theta})$

& is obtained by substituting into formula for $\sigma_{\hat{\Theta}}$ any estimates for unknown parameters

↳ i.e. computed using a data set.

<u>Example</u> If $X_i \sim N(\mu, \sigma^2)$, then $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

So standard error of $\bar{X}$ is $\sigma_{\bar{X}} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$

We can estimate σ with S so $se(\bar{X}) = \frac{S}{\sqrt{n}}$.

(Biased) Estimators

Here we have the <u>mean square error</u> (MSE) of $\hat{\Theta}$ which incorporates bias:

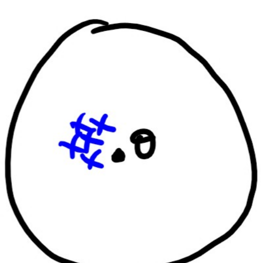
$$MSE(\hat{\Theta}) = E((\hat{\Theta} - \theta)^2) = \ldots = V(\hat{\Theta}) + (\text{bias})^2$$

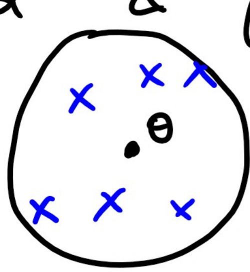
Generalizes $(\text{standard error})^2$:

$$MSE(\hat{\Theta}) = V(\hat{\Theta}) = \sigma^2_{\hat{\Theta}} \quad \text{if } \hat{\Theta} \text{ is unbiased.}$$

With MSE we can compare any 2 estimators & maybe $MSE(\hat{\Theta}_1) < MSE(\hat{\Theta}_2)$ even if

$\hat{\Theta}_1$ is biased & $\hat{\Theta}_2$ is unbiased





More precisely the relative efficiency of $\hat{\Theta}_1$ & $\hat{\Theta}_2$ is $\dfrac{MSE(\hat{\Theta}_1)}{MSE(\hat{\Theta}_2)}$ if $\begin{cases} < 1 & \hat{\Theta}_1 \text{ more efficient} \\ > 1 & \hat{\Theta}_2 \text{ more efficient} \end{cases}$

An optimal estimator for θ is $\hat{\Theta}$ ~~that~~ with $\dfrac{MSE(\hat{\Theta})}{MSE(\hat{\Theta}_1)} < 1$ for any other est. $\hat{\Theta}_1$ for θ.