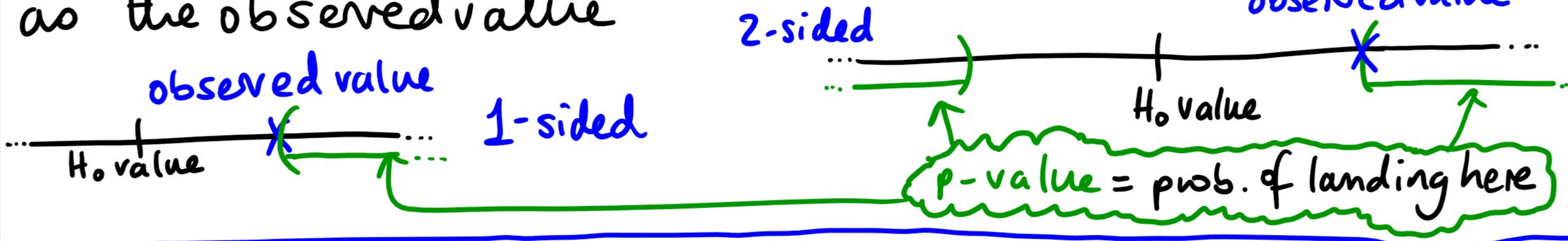
# 3Y03 - PROBABILITY AND STATISTICS FOR ENGINEERING

WS19 **Lecture 26**

**Last Time** p-value of test statistic

Probability that the test stat. is "at least as extreme" as the observed value




## 2 approaches to Hypothesis Testing at level $\alpha$:

(1) Find critical values (prob. of test stat. being in critical region = $\alpha$); is observed test stat. value in critical region?

(2) Calculate p-value based on observed value; is p-value $\leq \alpha$ ? ← If YES, reject $H_0$ (for $H_1$).

## Normal pop., Tests on Mean, Variance Known

$\sigma^2$

$$H_0 : \mu = \mu_0$$

$$H_1 : \text{(I) } \mu \neq \mu_0 \text{ or (II) } \mu > \mu_0 \text{ or (III) } \mu < \mu_0$$

Test statistic: $$Z_0 = \frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0,1)$$ if $H_0$ true

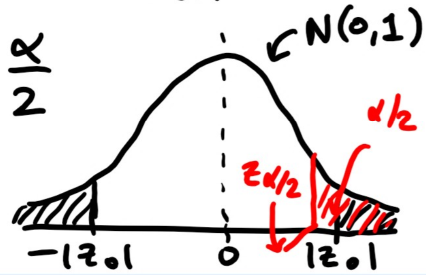
**Case I**

p-value

$$= 2P(Z_0 \geq |z_0|) \qquad z_0 = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

So reject $H_0$ if $P(Z_0 < |z_0|) \leq \frac{\alpha}{2}$

i.e. reject $H_0$ if $|z_0| \geq z_{\alpha/2}$ ← critical value

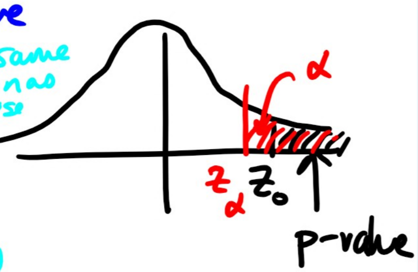


**Case II** $\quad$ $p\text{-value} = P(Z_0 \geq z_0)$ $\leftarrow z_0$ +ve (for same reason as we choose $H_1$)

$H_1: \mu > \mu_0$

Reject $H_0$ if $P(Z_0 \geq z_0) \leq \alpha$

i.e. if $z_0 \geq z_\alpha$ $\leftarrow$ critical value

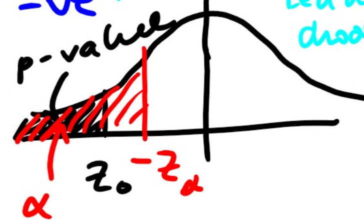



**Case III** $\quad$ $p\text{-value} = P(Z_0 \leq z_0)$ $\leftarrow z_0$ −ve (for whatever reason led us to choose $H_1$)

$H_1: \mu < \mu_0$

Reject $H_0$ if $P(Z_0 \leq z_0) \leq \alpha$

i.e. if $z_0 \leq -z_\alpha$. $\leftarrow$ critical value.




**Example** Sample of 25 adults give mean body temp. reading of 36.8°C. Assume body temp. is normally distributed with $\sigma = 0.34$°C. Test $H_0: \mu = 37$°C
$H_1: \mu \neq 37$°C
with $\alpha = 0.01$.

**Solution**

$$z_0 = \frac{\bar{x} - \mu_0}{(\sigma/\sqrt{n})} = \frac{36.8 - 37}{0.34/5} = -2.94.$$

p-value (2-sided) $= 2P(Z \leq -2.94)$ where $Z \sim N(0,1)$

$= 0.003282 \leq 0.01 = \alpha$

So reject $H_0$ in favour of $H_1$.

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## Controlling $\beta$ (P(Type II Error)) using sample size)

Recall: $\beta$ depends on true value of $\mu$

$\uparrow$ In the context of "Normal pop., Tests on the mean, variance known."

Say that true value $\mu = \mu_0 + \delta$

$\uparrow$ $H_0$ value

We'll find $\beta$ in terms of $\delta$ (We'll assume $\delta > 0$ but $\delta < 0$ case symmetric.)

Remember, we're ~~assuming~~ using $H_0 : \mu = \mu_0$ ~~(i.e. assume true mean $= \mu_0$)~~ & 2-sided $H_1$.

This is what $H_0$ says but more helpful to think of "true mean $\mu$ is $\mu_0 + \delta$"

$$\beta(\delta) = P\left(-z_{\frac{\alpha}{2}} < Z_0 < z_{\frac{\alpha}{2}}\right)$$

If $H_0$ false, $Z_0$ no longer $N(0,1)$ so what is its distribution?

$$Z_0 = \frac{\bar{X} - \mu_0}{(\sigma/\sqrt{n})} = \frac{\bar{X} - (\mu - \delta)}{\sigma/\sqrt{n}}$$

$$= \underbrace{\underbrace{\frac{\bar{X} - \mu}{\sigma/\sqrt{n}}}_{\sim N(0,1)\text{ since true mean } = \mu} + \underbrace{\frac{\delta}{\sigma/\sqrt{n}}}_{\#\text{ (not varying randomly)}}}_{Z_0 \sim N\left(\frac{\delta}{\sigma/\sqrt{n}},\, 1\right)}$$

(see picture below)

We want

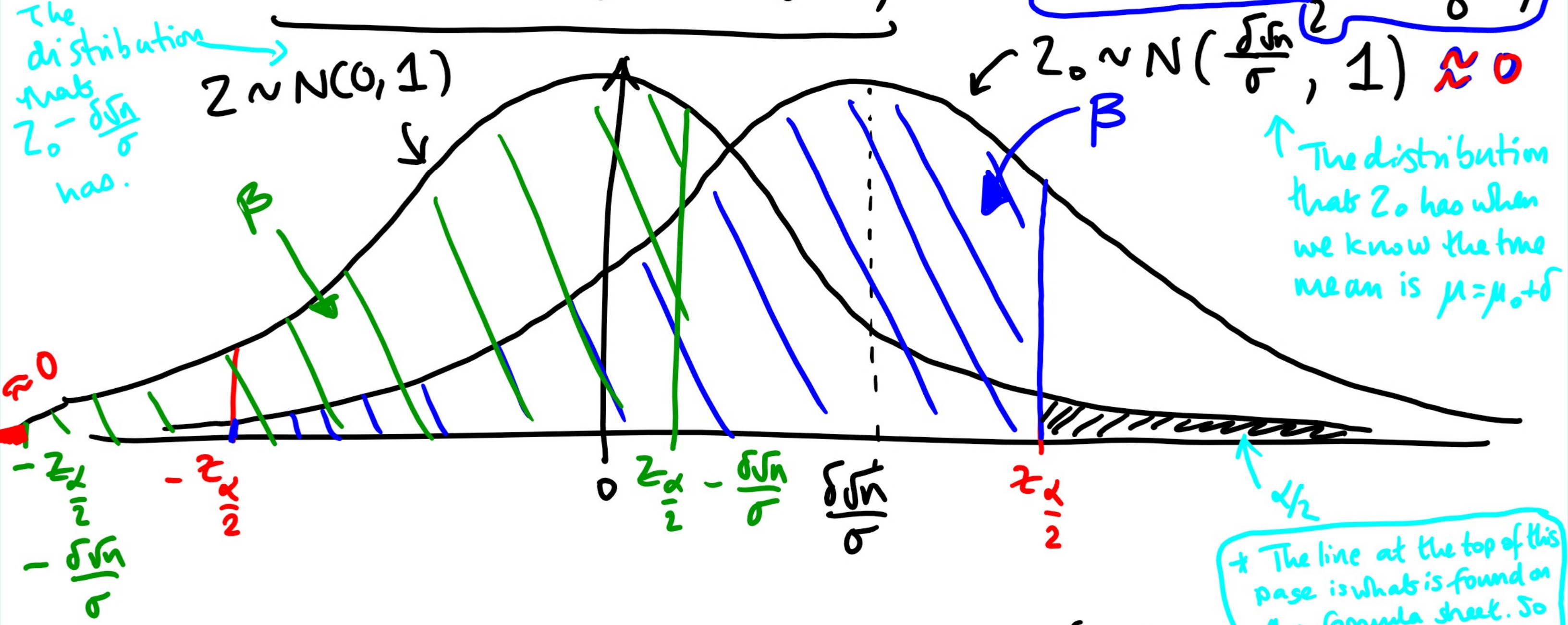
$$\beta(\delta) = P\left(-z_{\alpha/2} < Z_0 < z_{\alpha/2}\right)$$

$$= P\left(\frac{-z_{\alpha/2} - \delta/(\sigma/\sqrt{n})}{1} < Z < \frac{z_{\alpha/2} - \delta/(\sigma/\sqrt{n})}{1}\right)$$

(standardize using $Z_0 \sim N\left(\frac{\delta}{\sigma/\sqrt{n}}, 1\right)$).

$$= \underbrace{P\left(Z < z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)} - \boxed{P\left(Z < -z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}\right)}$$

tiny area in left-hand tail




* The line at the top of this page is what is found on the formula sheet. So you need to remember how to get the formula for n below from that.

So $\beta(\delta) \approx P\left(Z < z_{\alpha/2} - \frac{\delta\sqrt{n}}{\sigma}\right)$

i.e. $-z_\beta \approx z_{\frac{\alpha}{2}} - \frac{\delta\sqrt{n}}{\sigma}$ (The # that puts $\beta$ in left-hand tail.) ($\beta$ = area = prob.)

rearrange to find n $\quad n \approx \frac{(z_{\frac{\alpha}{2}} + z_\beta)^2}{\delta^2}\sigma^2$ (variance) [2-sided case]

(get same formula in the case $\delta < 0$)

[1-sided case] $\quad n \approx \frac{(z_\alpha + z_\beta)^2}{\delta^2}\sigma^2$ (variance) (derived similarly)

<u>Example</u> Batteries have lifetime in hours $\sim N(\mu, 1.5)$

Assumption is that the mean lifetime is 40 hours

If the true mean is 43 hours, how big a sample do we need to ensure $\beta < 0.05$ in a 2-sided test with $\alpha = 0.025$?

Solution

$$n \approx \frac{\left(z_{\frac{\alpha}{2}} + z_{\beta}\right)^2}{\delta^2} \sigma^2$$

delta Sigma

$$= \frac{\left(z_{0.0125} + z_{0.05}\right)^2}{(43-40)^2}(1.5)$$

$$= \frac{(2.24+1.64)^2}{9}(1.5) = 2.51.$$

Take $n \geq 3$.