3Y03 - PROBABILITY AND STATISTICS FOR ENGINEERING

WS19 Lecture 34

Last Time Single Factor Experiments

for now assume n in each (so $N = na$)

→ N observations Y_{ij} grouped into a "treatments"

→ Model: $Y_{ij} = \overbrace{\mu + \tau_i}^{\mu_i \,\leftarrow\, \text{mean of } i\text{th treatment}} + \varepsilon_{ij}$ where $\varepsilon_{ij} \sim N(0, \sigma^2)$

μ: underlying mean; τ_i: ith "treatment effect"; ε_{ij}: random error

→ • Notation: $\bar{Y}_{i\cdot}$ = ith treatment sample mean $\left(\frac{1}{n}\sum_{j=1}^{n} Y_{ij}\right)$

(sum over subscript replaced by •)

$\bar{Y}_{\cdot\cdot}$ = overall sample mean $= \frac{1}{N}\left(\sum_{i=1}^{a}\sum_{j=1}^{n} Y_{ij}\right)$

→ Estimators: $\hat{\mu} = \bar{Y}_{\cdot\cdot}$; $\hat{\tau}_i = \bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}$; $\hat{\varepsilon}_{ij} = Y_{ij} - \bar{Y}_{i\cdot}$

We want to test $H_0: \mu_1 = \mu_2 = \ldots = \mu_a$

H_1: at least one pair $i \neq k$ has $\mu_i \neq \mu_k$

It can be shown that:

$$\underbrace{\sum_{i=1}^{a}\sum_{j=1}^{n}\left(Y_{ij} - \bar{Y}_{\cdot\cdot}\right)^2}_{SS_T} = \underbrace{\sum_{i=1}^{a}\sum_{j=1}^{n}\left(Y_{ij} - \bar{Y}_{i\cdot}\right)^2}_{SS_E} + \underbrace{\sum_{i=1}^{a}\sum_{j=1}^{n}\left(\bar{Y}_{i\cdot} - \bar{Y}_{\cdot\cdot}\right)^2}_{SS_{Treatments}}$$

SS_T: Total variation in the data

SS_E: Random variation "within treatments sum of squares" - comes from $e_{ij} = \hat{\varepsilon}_{ij}$

$SS_{Treatments}$: Variation due to differences between treatment means

Shortcut formulas

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n} Y_{ij}^2 - N\bar{Y}_{\cdot\cdot}^2$$

"between treatments sum of squares" — comes from τ_i

$$SS_{Treatments} = \sum_{i=1}^{a}\boxed{\sum_{j=1}^{n}\bar{Y}_{i\cdot}^2} - N\bar{Y}_{\cdot\cdot}^2$$

$$\boxed{\sum_{j=1}^{n}\bar{Y}_{i\cdot}^2} \;\text{"}\; n\left(\frac{Y_{i\cdot}^2}{n^2}\right) \qquad \bar{Y}_{i\cdot} = \frac{Y_{i\cdot}}{n}$$

$$= \sum_{i=1}^{a}\frac{Y_{i\cdot}^2}{n} - N\bar{Y}_{\cdot\cdot}^2$$

$$SS_E = SS_T - SS_{Treatments}$$

$$E(SS_T) = \underbrace{E(SS_E)}_{=\,(N-a)\sigma^2} + \underbrace{E(SS_{Treatments})}_{=\,(a-1)\sigma^2 + n\sum_{i=1}^{a}\tau_i^2}$$

$$= \underbrace{(N-a)}_{\substack{\text{d.o.f. of } SS_E\\ = N-a}}\sigma^2 + \underbrace{(a-1)}_{\substack{\text{d.o.f. of } SS_{Treatments}\\ = a-1}}\sigma^2 + n\sum_{i=1}^{a}\tau_i^2$$

We define in "usual" way the "mean squares" by dividing SS by d.o.f. :

$$MS_E = \frac{SS_E}{N-a} \quad \& \quad MS_{Treatments} = \frac{SS_{Treatments}}{a-1}.$$

$$E(MS_E) = \sigma^2 \quad \& \quad E(MS_{Treatments}) = \sigma^2 + \underbrace{\frac{n}{a-1}\sum_{i=1}^{a}\tau_i^2}$$

This says $MS_{Tr.}$ is a biased estimator for σ^2 which overshoots σ^2 on average, as this term added on at the end is always positive

Relate this to $H_0 : \mu_1 = \mu_2 = \ldots = \mu_a$

First note that this says $\mu + \tau_1 = \mu + \tau_2 = \ldots = \mu + \tau_a$

i.e. $\tau_1 = \tau_2 = \ldots = \tau_a$

Then note that

We also know: overall mean $\mu = \frac{1}{a}\sum_{i=1}^{a}\mu_i$ ← the mean of the treatment means

$$= \frac{1}{a}\sum_{i=1}^{a}(\mu + \tau_i)$$

$$= \mu + \underbrace{\frac{1}{a}\sum_{i=1}^{a}\tau_i}_{\Rightarrow\ = 0}$$

i.e. the treatments effects are modelled so that they average out

So under $H_0 : \tau_1 = \tau_2 = \ldots = \tau_a = \underline{\underline{0}}$ ← If τ_i all equal & sum to 0, they must all be 0.

H_1 : at least two of $\tau_i \neq 0$.

(if one of the $\tau_i \neq 0$, and they sum to 0, then at least two are $\neq 0$)

Recall $\underbrace{E(MS_E) = \sigma^2}_{\text{unbiased}}$, $\underbrace{E(MS_{Treatments}) = \sigma^2 + \frac{n}{a-1}\sum_{i=1}^{a}\tau_i^2}_{\text{unbiased exactly when all } \tau_i = 0 \text{ i.e. } H_0 \text{ true}}$

$MS_{Treatments}$ overshoots σ^2 by an amount dependent on the treatment effects τ_i if H_0 false

- we check to see if it does so by a significant amount

Test stat. $F_0 = \dfrac{MS_{Treatments}}{MS_E} \sim f_{a-1, N-a}$

We reject H_0 at level α if $f_0 > f_{\alpha, a-1, N-a}$

Example – see separate file.

For our example:

$H_0 : \mu_1 = \mu_2 = \mu_3$ (or $\tau_1 = \tau_2 = \tau_3 = 0$)

H_1 : at least one pair $i \neq k$ satisfies $\mu_i \neq \mu_k$

(or at least on $\tau_i \neq 0$ (and hence at least 2, since the τ_i sum to 0))

Tested at $\alpha = 0.01$.

Source: A. Parenti, L. Guerrini, P. Masella, S. Spinelli, L. Calamai,
P. Spugnoli (2014). "Comparison of Espresso Coffee Brewing Techniques,"
Journal of Food Engineering, Vol. 121, pp. 112-117.

Description: Comparison of foam index (Y, in %) for 3 methods of brewing espresso
Method 1 = Bar Machine (BM), **a=3**
Method 2 = Hyper-Espresso Method (HIP),
Method 3 = I-Espresso System (IT).

9 replicates/treatment. **n=9**

Variables: foamIndx (response)
method (factor)

method		1			2			3			
		y_{1j}			y_{2j}			y_{3j}			
foamIndx		36.64	1342.4896		70.84	5018.31		56.19	3157.32		
		39.65	1572.1225		46.68	2179.02		36.67	1344.69		
		37.74	1424.3076		73.19	5356.78		35.35	1249.62		
		35.96	1293.1216		57.78	3338.53		40.11	1608.81		
		38.52	1483.7904		48.61	2362.93		33.52	1123.59		
		21.02	441.8404		72.77	5295.47		37.12	1377.89		
		24.81	615.5361		65.04	4230.2		37.33	1393.53		
		34.18	1168.2724		62.53	3910		32.68	1067.98		
		23.08	532.6864		54.26	2944.15		48.33	2335.79		
	y_1.	**291.6**		**y_2.**	**551.7**		**y_3.**	**357.3**		**y..**	**1200.6**
	ȳ_1.	**=291.6/9**		**ȳ_2.**	**=551.7/9**		**ȳ_3.**	**=357.3/9**		**ȳ..**	**=1200.6/27**
		32.4			**61.3**			**39.7**			**44.4666667**
	$(y\_1.)^2$	**85030.56**		**$(y\_2.)^2$**	**304373**		**$(y\_3.)^2$**	**127663**		**$(\bar{y}..)^2$**	**1977.28444**

$SS_{Treatments}$ 4065.18 $= \sum\sum \frac{y_{i.}^2}{9} - 27\bar{y}_{..}^2$

SS_E 1716.9192

SS_T 5782.0992 $= \sum\sum y_{ij}^2 - 27\bar{y}_{..}^2$

$\Sigma(y_{ij})^2$ = 59168.7792

ANOVA Table

Source of Variation	SS	d.o.f.	MS
Treatments	4065. ...	2	2032.59
Error	1716. ..	24	71.5383
Total	5782...	26	

$= F_0 = 28.41$ Compare to $f_{0.01,2,24} = 5.61$

Test at $\alpha = 0.01$ level

$28.41 > 5.61$

So reject H_0