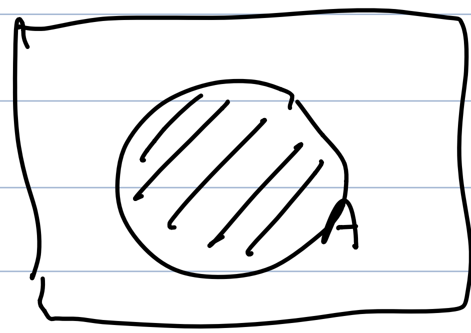
Conditional Probability

A, B, assume $P(A)>0$

Define $P(B|A) = \begin{cases} \dfrac{P(A \cap B)}{P(A)} & \text{if } P(A)>0 \\ \text{not relevant} & \text{if } P(A)=0 \end{cases}$

$P(\cdot|A)$ where $\cdot$ = any event
is a new probability on S



Example: Puzzle of Two Aces

4 playing cards A♠, A♣, 2♡, 2♢
Shuffle & select two at random

$B = \{$both Aces selected$\}$ $\quad P(B)=\frac{1}{6}$

$A_1 = \{$Ace of spades selected$\}$

$A_2 = \{\text{at least one Ace selected}\}$

$$P(B|A_1) = \frac{P(A_1 \cap B)}{P(A_1)} = \frac{P(B)}{P(A_1)} = \frac{1/6}{1/2} = \boxed{\frac{1}{3}}$$

$$P(B|A_2) = \frac{P(A_2 \cap B)}{P(A_2)} = \frac{P(B)}{P(A_2)} = \frac{1/6}{5/6} = \boxed{\frac{1}{5}}$$

Puzzle because why should name of suit affect conditional probability

Resolution: Condition instead

$A = \{$"I tell you I have A♠"$\}$

Rewrite definition as

$$\boxed{P(A \cap B) = P(B|A)\, P(A)}$$

Generalized

$$P(A \cap \underbrace{B \cap C}) = P(B \cap C)\, P(A|B \cap C)$$

$$= P(C)\, P(B|C)\, P(A|B \cap C)$$

multiplication rule for probabilities

Independence

A, B independent if

$$P(B|A) = P(B)$$

Since $P(B|A) = \frac{P(A \cap B)}{P(A)}$

$$\Longrightarrow \boxed{P(A \cap B) = P(A) \cdot P(B)} \quad (*)$$

Equivalent to definition of conditional probability but more useful in practice

Independent family $\{B_1, B_2, \ldots, B_n\}$

if

$$P\left(\bigcap_{j=1}^{k} B_{i_j}\right) = \prod_{j=1}^{k} P(B_{i_j})$$

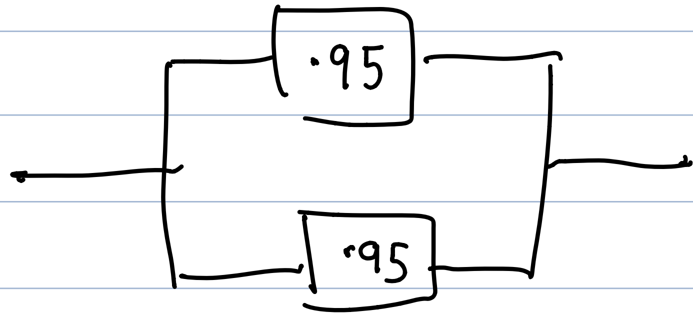
for any subset $\{i_1, i_2, \ldots, i_k\}$, $k \geq 2$

of $\{1, 2, \ldots, n\}$

Reliability of a device or system is the probability that device or system will operate for specified duration.

Parallel circuit

– assume each device operates independently



$A_1 = \{$ upper circuit works $\}$

$A_2 = \{$ lower circuit works $\}$

reliability $= P[A_1 \cup A_2]$

$= P(A_1) + P(A_2) - P(A_1 \cap A_2)$

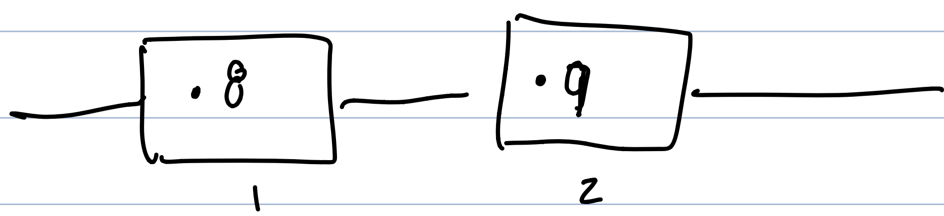
$= P(A_1) + P(A_2) - P(A_1) \cdot P(A_2)$

$= .95 + .95 - (.95)^2$

$= 0.9975$

used for redundancy

Series Circuit



reliability $=$ P[both devices work]

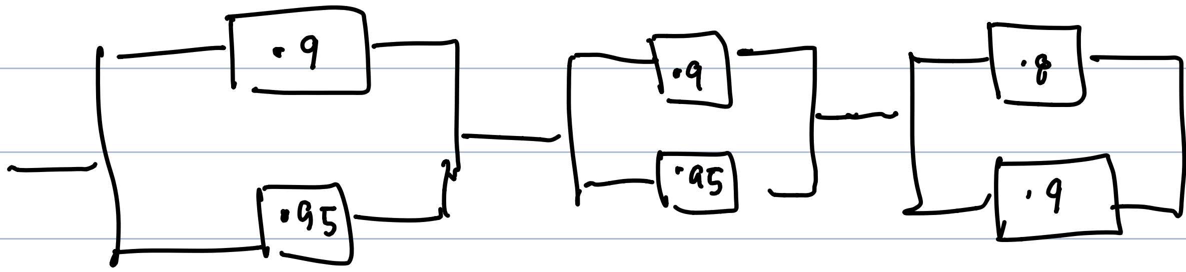
$= P(A_1 \cap A_2)$

$= P(A_1) P(A_2)$

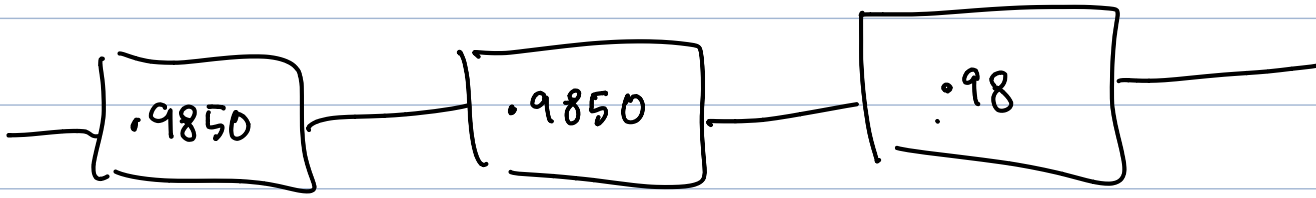
$= .8 (.9) = .72$

Mixed Circuits → Simplify to equivalent circuits

Exercise 2-157



Equivalent circuit for reliability question

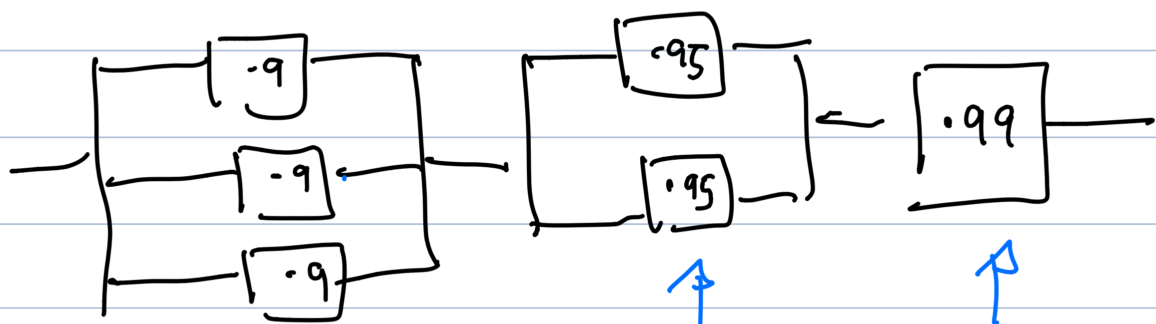


$.9 + .95 - .9(.95)$
$= .9850$

$.8 + .9 - .8(.9) = .9800$

series $\quad .9850(.9850)(.9800) = 0.9702$

Example



as above

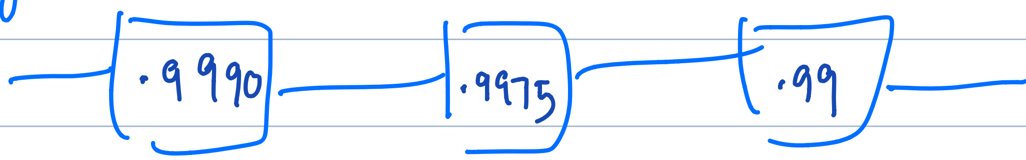
"Recall" $P(A \cup B \cup C)$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

reliability of extreme left circuit is

$$= 3(.9) - 3(.9)^2 + (.9)^3 = 0.9990$$

equivalent circuit



reliability of middle circuit

$$.95 + .95 - (.95)^2 = 0.9975$$

Reliability of entire circuit

$$= .9990(.9975)(.99)$$
$$= 0.9865$$