# 3Y03 - Probability and Statistics for Engineering

WS19 Lecture 8

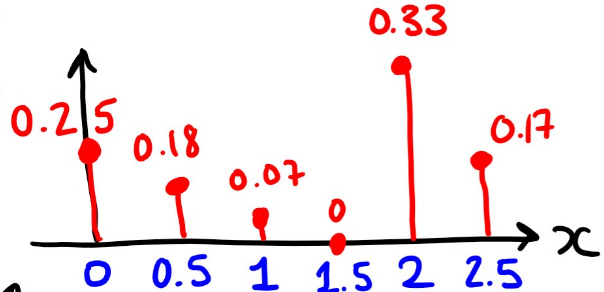
Yesterday: Probability Mass Functions (p.m.f.) $f(x)$
& Cumulative Distribution Functions (c.d.f.) $F(x)$

For discrete random variables.

pmf $f(x) = P(X = x)$ ; $\sum_x f(x) = 1$ & $f(x) \geq 0$.

cdf $F(x) = P(X \leq x)$

e.g.




Example

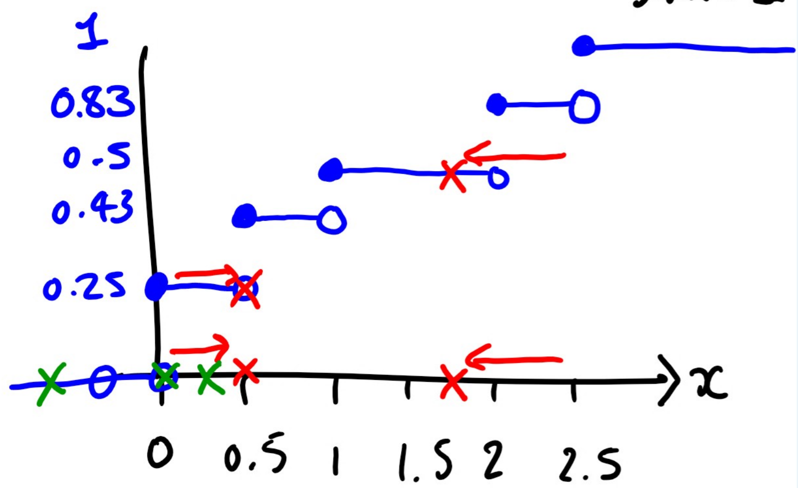
p.m.f. $f(x)$ given by:

| $x$ | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.25 | 0.18 | 0.07 | 0 | 0.33 | 0.17 |

Sum to 1

c.d.f.

$$F(x) = \begin{cases} 0 & x < 0 \\ 0.25 & 0 \leq x < 0.5 \\ 0.43 & 0.5 \leq x < 1 \\ 0.5 & 1 \leq x < 2 \\ 0.83 & 2 \leq x < 2.5 \\ 1 & x \geq 2.5 \end{cases}$$




Notice: F(x) starts at 0 & ends at 1
F(x) is non-decreasing ($x_1 \leq x_2 \Rightarrow f(x_1) \leq f(x_2)$)

F(x) is

Right continuous everywhere

Jumps at $x$ are equal to p.m.f. at $x$

So from any function $F(x)$ with this list of properties we can describe corresponding pmf $f(x)$ of a discrete r.v. :

← jumps described formally like this

$$\underbrace{f(a)}_{P(X=a)} = F(a) - \lim_{x \to a^-} F(x)$$

In example above $f(0) = 0.25 - 0 = 0.25$

For $a < b$ $\quad P(a \leq X \leq b) = F(b) - \lim_{x \to a^-} F(x)$.

<u>Example</u> Given $F(x)$ as follows, find $P(X=1)$, $P(1 \leq X \leq 2)$ & $P(X > 2)$.

| $x$ | $x < 0$ | $0 \leq x < 1$ | $1 \leq x < 2$ | $2 \leq x < 3$ | $x \geq 3$ |
|---|---|---|---|---|---|
| $F(x)$ | 0 | 0.1 | 0.4 | 0.65 | 1 |

(a) $P(X=1) = F(1) - \lim_{x \to 1^-} F(x) = F(1) - F(0)$
$= 0.4 - 0.1$
$= 0.3$

(b) $P(1 \leq X \leq 2) = F(2) - \lim_{x \to 1^-} F(x) = F(2) - F(0)$
$= 0.65 - 0.1$
$= 0.55$

(c) $P(X > 2) = 1 - P(X \leq 2) = 1 - F(2)$
$= 1 - 0.65 = 0.35$.

# 3-4 Mean & Variance of a Discrete Random Variable

Weighted average

Spread

→ summarise the prob. distribution

**Mean** "centre" "balance point"

↳ where prob. masses would balance

"weighted average"
with probabilities as weights.
↳ $f(x)$ values

Denoted $\mu = E(X) = \sum_x x f(x)$

expected value of X

(We expect $x$ to occur "in the long run" a fraction $f(x)$ of the time)

**Variance** "spread" "dispersion"

Denoted $\sigma^2 = V(X) = E\left((X-\mu)^2\right)$

$= \sum_x (x-\mu)^2 f(x)$

**Standard Deviation** $\sigma = +\sqrt{V(X)}$

← units match those of $X$ & $\mu$

Alternate formula for variance

$$\sigma^2 = \sum_x (x-\mu)^2 f(x)$$

$$= \sum_x \left(x^2 - 2x\mu + \mu^2\right) f(x)$$

$$= \sum_x x^2 f(x) - \sum_x 2x\mu f(x) + \sum_x \mu^2 f(x)$$

$$= \sum_x x^2 f(x) - 2\mu \underbrace{\sum_x x f(x)}_{\mu} + \mu^2 \underbrace{\sum_x f(x)}_{1}$$

$$= \sum_x x^2 f(x) \cancel{-2\mu^2 + \mu^2}$$

$$-\mu^2$$

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<u>Warning</u> 2 prob. distributions may have same $\mu$ & $\sigma^2$ (so cannot recover p.m.f. $f(x)$ from $\mu$ & $\sigma^2$)

See figure 3-6 p. 74 of text.

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<u>Example</u> p.m.f. $f(x)$; $X$ represents change in price in \$

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|
| $f(x)$ | 0.1 | 0.05 | 0.13 | 0.6 | 0.1 | 0.02 |

Calculate $\mu$, $\sigma^2$ and $\sigma$.

$$\mu = E(X) = \sum_x x \cdot f(x) = (-2)(0.1) + (-1)(0.05) + 0(0.13) + (1)(0.6) + 2(0.1) + 3(0.02)$$

$$= \$0.61$$

$$\sigma^2 = V(X) = \sum_x x^2 f(x) - \mu^2$$

$$= (-2)^2(0.1) + (-1)^2(0.05) + 0^2(0.13) + 1^2(0.6) + 2^2(0.1) + 3^2(0.02) - (0.61)^2$$

$$= 4(0.1) + 0.05 + 0 + 0.6 + 4(0.1) + 9(0.02) - (0.61)^2$$

$$= 1.2579 \ \$^2$$

$$\sigma = +\sqrt{1.2579} = \$1.12.$$

This was added after class due to a technological failure preventing the example from being completed in class time.