Portfolio optimization under Value-at-Risk constraint

Traian A Pirvu
Department of Mathematics
The University of British Columbia
Vancouver, BC, V6T1Z2
tpirvu@math.ubc.ca

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Abstract

In this paper, we analyze the effects arising from imposing a Value-at-Risk constraint in an agent’s portfolio selection problem. The financial market is incomplete and consists of multiple risky assets (stocks) plus a risk free asset. The stocks are modelled as exponential Brownian motions with random drift and volatility. The risk of the trading portfolio is reevaluated dynamically, hence the agent must satisfy the Value-at-Risk constraint continuously. We derive the optimal consumption and portfolio allocation policy in closed form for the case of logarithmic utility. The non-logarithmic CRRA utilities are considered as well, when the randomness of market coefficients is independent of the Brownian motion driving the stocks. The portfolio selection, a stochastic control problem is reduced, in this context, to a deterministic control one, which is analyzed and a numerical treatment is proposed.

1 Introduction

Managers limit the riskiness of their traders by imposing limits on the risk of their portfolios. Lately, the Value-at-Risk (VaR) risk measure has became a tool used to accomplish this purpose. The increased popularity of this risk measure is due to the fact that VaR is easily understood. It is the maximum loss of a portfolio over a given horizon, at a given confidence level. The Basle Committee on Banking Supervision requires U.S. banks to use VaR in determining the minimum capital required for their trading portfolios.

In what follows we give a brief description of the existing literature. Basak and Shapiro (2001) analyze the optimal dynamic portfolio and wealth-consumption policies of utility maximizing

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investors who must manage risk exposure using VaR. They find that VaR risk managers pick a larger exposure to risky assets than non-risk managers, and consequently incur larger losses when losses occur. In order to fix this deficiency they choose another risk measure based on the risk-neutral expectation of a loss. They called this risk measure *Limited Expected Loss (LEL)*. One drawback of their model is that the portfolios VaR is never reevaluated after the initial date, making the problem a static one. In a similar setup, Berkelaar, Cumperayot and Kouwenberg (2002) show that in equilibrium VaR reduces market volatility, but in some cases raises the probability of extreme losses. Emmer, Klüppelberg and Korn (2001) consider a dynamic model with *Capital-at-Risk* (a version of VaR) limits. However, they assume that portfolio proportions are held fixed during the whole investment horizon and thus the problem becomes a static one as well.

Cuoco, He and Issaenko (2001) develop a more realistic dynamically-consistent model of the optimal behavior of a trader subject to risk constraints. They assume that the risk of the trading portfolio is reevaluated dynamically by using the conditioning information, and hence the trader must satisfy the risk limit continuously. Another assumption they make is that when assessing the risk of a portfolio, the the proportions of different assets held in the portfolio are kept unchanged. Besides VaR risk measure, they consider a coherent risk measure *Tail value at Risk (TVaR)*, and establish that it is possible to identify a dynamic VaR risk limit equivalent to a dynamic TVaR limit. Another of their findings is that the risk exposure of a trader subject to VaR and TVaR limits is always lower than that of an unconstrained trader.

In Pirvu (2005) we start with the model of Cuoco, He and Issaenko (2001). We find the optimal growth portfolio subject to these risk measures. The main finding is that the optimal policy is a projection of the optimal portfolio of an unconstrained log agent (the Merton proportion) onto the constraint set with respect to the inner product induced by the volatility matrix of the risky assets. Closed-form solutions are derived even when the constraint set depends on the current wealth level.

Cuoco and Liu (2001) study the dynamic investment and reporting problem of a financial institution subject to capital requirements based on self-reported VaR estimates. They show that optimal portfolios display a local three-fund property. Leippold, Trojani and Vanini (2002) analyze VaR-based regulation rules and their possible distortion effects on financial markets. In partial equilibrium the effectiveness of VaR regulation is closely linked to the *leverage effect*, the tendency of volatility to increase when the prices decline.

Vidovic, Lavassani, Li and Ware (2003) considered a model with time dependent parameters but the risk constraints were imposed in a static fashion.

Yiu K.F.C. (2004) looks at the optimal portfolio problem, when an economic agent is maximizing utility of her intertemporal consumption over a period of time under a dynamic VaR constraint. A numerical method is proposed to solve the corresponding HJB-equation. They find that investment in risky assets are optimally reduced by the VaR constraint. Atkinson and Papakokkinou (2005) derive the solution to the optimal portfolio and consumption subject to CaR and VaR constraints by using stochastic dynamic programming.

This paper extends Pirvu (2005) by allowing for intertemporal consumption. We address an issue raised in Yiu K.F.C. (2004) and Atkinson and Papakokkinou (2005) by considering a
market with random coefficients. It was also suggested as a new research direction by Cuoco, He
and Issaenko (2001). By the best of our knowledge this is the first work in the portfolio choice
theory with CRRA type preferences, time dependent market coefficients, incomplete financial
market and dynamically consistent risk constraints.

Our Contribution. We propose a new approach with a potential to numerical applications. The
main idea is to consider on every path an auxiliary deterministic control problem which we
analyze. The existence of a solution of the deterministic control problem does not follow from
classical results. We establish it and we also show that first order necessary conditions are also
sufficient for optimality. We prove that a solution of this deterministic control problem is the
optimal portfolio policy for a given path. The advantage of our method over the classical ones
is that allows for a better numerical treatment.

The reminder of this paper is organized as follows. Section 2 describes the model, including
the definition of Value-at-Risk constraint. Section 3 formulates the objective and it shows
the limitations of the stochastic dynamic programming approach in this context. Section 4
treats the special case of logarithmic utility. The problem of maximizing expected logarithmic
utility of intertemporal consumption is solved in closed form. This is done by reducing it to a
nonlinear program, which is solved pathwise. One finding is that at the final time the agent
invests the least proportion of her wealth in stocks. The optimal policy is a projection of the
optimal portfolio and consumption of an unconstrained agent onto the constraint set. Section
5 treats the case of power utility, in the totally unhedgeble market coefficients paradigm (see Example 7.4, page 305 in [14]). The stochastic control portfolio problem is transformed into a
deterministic control one. The solutions is characterized by Pontryagin maximum principle (first
order necessary conditions). Section 6 contains an appropriate discretization of the deterministic
control problem. It leads to a nonlinear program which can be solved by standard methods. It
turns out that the necessary conditions of the discretized problem converges to the necessary
conditions of the continuous problem. For numerical experiments one can use NPSOL, a software
package for solving constrained optimization problems which employs a sequential quadratic
programming (SQP) algorithm. We end this section with some numerical experiments. The
conclusions are summarized in Section 7. We conclude the paper with an Appendix containing
the proofs of Lemmas.

2 Model Description

2.1 The Financial Market

Our model of a financial market, based on a filtered probability space \((\Omega, \{\mathcal{F}_t\}_{0 \leq t \leq \infty}, \mathcal{F}, \mathbb{P})\)
satisfying the usual conditions, consists of \(m + 1\) assets. The first one, \(\{S_0(t)\}_{t \in [0, \infty)}\), is a
riskless bond with a positive interest rate \(r\), i.e, \(dS_0(t) = S_0(t) r \, dt\). The remaining \(m\) are stocks
and evolve according to the following stochastic differential equation

\[
dS_i(t) = S_i(t) \left[ \alpha_i(t) \, dt + \sum_{j=1}^{n} \sigma_{ij}(t) \, dW_j(t) \right], \quad 0 \leq t \leq \infty, \quad i = 1, \ldots, m, \tag{2.1}
\]
where the process $\{W(t)\}_{t \in [0, \infty)} = \{(W_j(t))_{j=1, \ldots, n}\}_{t \in [0, \infty)}$ is an $n$-dimensional standard Brownian motion. Here $\{\alpha(t)\}_{t \in [0, \infty)} = \{\alpha_i(t)\}_{i=1, \ldots, m}\}_{t \in [0, \infty)}$ is an $\mathbb{R}^m$ valued mean rate of return process, and $\{\sigma(t)\}_{t \in [0, \infty)} = \{\sigma_{ij}(t)\}_{i=1, \ldots, m, j=1, \ldots, n}\}_{t \in [0, \infty)}$ is an $m \times n$-matrix valued variance-covariance process. We impose the following regularity assumptions on the coefficient processes $\alpha(t)$ and $\sigma(t)$.

- All the components of the process $\{\alpha(t)\}_{t \in [0, \infty)}$ are assumed positive, continuous and $\{\mathcal{F}_t\}$-adapted.
- The matrix-valued volatility process $\{\sigma(t)\}_{t \in [0, \infty)}$ is assumed continuous, $\{\mathcal{F}_t\}$-adapted and with linearly independent rows for all $t \in [0, \infty)$, a.s.

The last assumption precludes the existence of a redundant asset and arbitrage opportunities. The rates of excess return is the $\mathbb{R}^m$ valued process $\{\mu(t)\}_{t \in [0, \infty)} = \{\mu_i(t)\}_{i=1, \ldots, m}\}_{t \in [0, \infty)}$, with $\mu_i(t) = \alpha_i(t) - r$, which is assumed positive. This also covers the case of an incomplete market if $n > m$ (more sources of randomness than stocks).

### 2.2 Consumption, Trading Strategies and Wealth

In this model the agent is allowed to consume. The intermediate consumption process denoted $\{C(t)\}_{t \in [0, \infty)}$ is assumed positive, and $\{\mathcal{F}_t\}$-progressively measurable. Let $\{(\zeta(t), c(t))\}_{t \in [0, \infty)} = \{\zeta_i(t), c(t)\}_{t \in [0, \infty)}$ be an $\mathbb{R}^{m+1}$ valued portfolio-proportion process. At time $t$ its components are the proportions of the agent’s wealth invested in stocks, $\zeta(t)$ and her consumption rate, $c(t)$. An $\mathbb{R}^{m+1}$ valued portfolio-proportion process is called admissible if it is $\{\mathcal{F}_t\}$-progressively measurable and satisfies

$$\int_0^t |\zeta^T(u)\mu(u)| \, du + \int_0^t ||c^T(u)\sigma(u)||^2 \, du + \int_0^t c(u) \, du < \infty, \text{ a.s., for all } t \in [0, \infty),$$  

where $|| \cdot ||$ is the standard Euclidean norm in $\mathbb{R}^m$. Given $\{(\zeta(t), c(t))\}_{t \in [0, \infty)}$ a portfolio proportion process, the leftover wealth $X^{\zeta, c}(t)(1 - \sum_{i=1}^m \zeta_i(t))$ is invested in the riskless bond $S_0(t)$. It may happen that this quantity is negative in which case we are borrowing at rate $r > 0$. The dynamics of the wealth process $\{X^{\zeta, c}(t)\}_{t \in [0, \infty)}$ of an agent using the portfolio proportion process $\{(\zeta(t), c(t))\}_{t \in [0, \infty)}$ is given by the following stochastic differential equation

$$dX^{\zeta, c}(t) = X^{\zeta, c}(t) \left( (\zeta^T(t)\alpha(t) - c(t)) \, dt + \zeta^T(t)\sigma(t) \, dW(t) \right) + \left( 1 - \sum_{i=1}^m \zeta_i(t) \right) X^{\zeta, c}(t) r \, dt$$

$$= X^{\zeta, c}(t) \left( (r - c + \zeta^T(t)\mu(t)) \, dt + \zeta^T(t)\sigma(t) \, dW(t) \right).$$

Let us define the $p$-quadratic correction to the saving rate $r$:

$$Q_p(t, \zeta, c) \triangleq r - c + \zeta^T \mu(t) + \frac{p-1}{2} ||\zeta^T \sigma(t)||^2.$$  

(2.3)
The above stochastic differential equation has a unique strong solution if (2.2) is satisfied and is given by the explicit expression

\[
X^{\zeta,c}(t) = X(0) \exp \left\{ \int_0^t Q_0(u, \zeta(u), c(u)) \, du + \int_0^t \zeta^T(u) \sigma(u) \, dW(u) \right\}. \quad (2.4)
\]

The initial wealth \(X^{\zeta,c}(0) = X(0)\) takes values in \((0, \infty)\), and is exogenously given.

### 2.3 Value-At-Risk Limits

For the purposes of risk measurement, one can use an approximation of the distribution of the investor’s wealth at a future date. A detailed explanation of why this practice should be employed can be found on p. 8 in [6] (see also p. 18 in [12]). Given a fixed time-instance \(t \geq 0\), and a length \(\tau > 0\) of the measurement horizon \([t, t + \tau]\), the projected distribution of the wealth from trading and consumption is usually calculated under the assumptions that

1. the portfolio proportion process \(\{(\zeta(u), c(u))\}_{u \in [t,t+\tau]}\), as well as
2. the market coefficients \(\{(\alpha(u))_{u\in[t,t+\tau]}\}\) and \(\{(\sigma(u))_{u\in[t,t+\tau]}\}\),

will stay constant and equal their present value throughout \([t, t + \tau]\). If \(\tau\) is small, for example \(\tau = 1\) trading day, the market coefficients will not change much and this comes to support assumption 2. The wealth’s dynamics equation yields the projected wealth at \(t + \tau\)

\[
X^{\zeta,c}(t + \tau) = X^{\zeta,c}(t) \exp \left\{ Q_0(t, \zeta(t), c(t)) \tau + \zeta^T(t) \sigma(t) (W(t + \tau) - W(t)) \right\},
\]

whence the projected wealth loss on the time interval \([t, t + \tau]\) is

\[
X^{\zeta,c}(t) - X^{\zeta,c}(t + \tau) = X^{\zeta,c}(t) \left[ 1 - \exp \left\{ Q_0(t, \zeta(t), c(t)) \tau + \zeta^T(t) \sigma(t) (W(t + \tau) - W(t)) \right\} \right].
\]

The random variable \(\zeta^T(t) \sigma(t) (W(t + \tau) - W(t))\) is, conditionally on \(\mathcal{F}_t\), normally distributed with mean zero and standard deviation \(\|\zeta^T(t) \sigma(t)\|\sqrt{\tau}\). Let the confidence parameter \(\alpha \in (0, \frac{1}{2}]\) be exogenously specified. The \(\alpha\)-percentile of the projected loss \(X^{\zeta,c}(t) - X^{\zeta,c}(t + \tau)\) conditionally on \(\mathcal{F}_t\) is

\[
X^{\zeta,c}(t) \left[ 1 - \exp \left\{ Q_0(t, \zeta(t), c(t)) \tau + N^{-1}(\alpha) \|\zeta^T(t) \sigma(t)\|\sqrt{\tau} \right\} \right],
\]

where \(N(\cdot)\) denotes the standard cumulative normal distribution function. This prompts the Value-at-Risk (VaR) of projected loss

\[
\text{VaR}(t, \zeta, c, x) \triangleq x \left[ 1 - \exp \left\{ Q_0(t, \zeta, c) \tau + N^{-1}(\alpha) \|\zeta^T \sigma(t)\|\sqrt{\tau} \right\} \right]^+.
\]

Let \(a_V \in (0,1)\) be an exogenous risk limit. The Value-at-Risk constraint is that the agent at every time-instant \(t \geq 0\), must choose a portfolio proportion \((\zeta(t), c(t))\) which would result in a relative VaR of the projected loss on \([t, t + \tau]\) less than \(a_V\). Strictly speaking is the set of all admissible portfolios which, for all \(t \geq 0\) belong to \(\mathcal{F}_V(t)\) defined by

\[
\mathcal{F}_V(t) \triangleq \left\{ (\zeta, c) \in \mathbb{R}^m \times [0, \infty); \frac{\text{VaR}(t, \zeta, c, x)}{x} \leq a_V \right\}. \quad (2.5)
\]
The fraction $\frac{VaR}{x}$ rather than $VaR$ is employed, whence the name relative $VaR$. If one imposes $VaR$ in absolute terms, the constraint set depends on the current wealth level and this makes the analysis more involved (see [6] and [17]). For a given path $\omega$ let us denote $\omega^{(t)} = (\omega_s)_{s \leq t}$ the projection up to time $t$ of its trajectory. One can see that for a fixed $\omega^{(t)}$, the set $F_V(t)$ is compact and convex, being the level set of a convex, unbounded function $f_V(t, \zeta, c)$,

$$F_V(t) = \left\{ (\zeta, c) \in \mathbb{R}^m \times [0, \infty); f_V(t, \zeta, c) \leq \log \frac{1}{1 - a_V} \right\},$$

where

$$f_V(t, \zeta, c) \triangleq -Q_0(t, \zeta, c)\tau - N^{-1}(\alpha)||\zeta^T \sigma(t)||\sqrt{T}. \quad (2.6)$$

The function $f_V$ although quadratic in $\zeta$ and linear in $c$ may still fail to be convex in $(\zeta, c)$ if $\alpha \geq \frac{1}{2}$, thus $F_V(t)$ may not be a convex set (see Fig. 1, p. 10 in [6]). However the choice of $\alpha \in (0, \frac{1}{2}]$ makes $N^{-1}(\alpha) \leq 0$ and this yield convexity.

3 Objective

Let a finite time horizon $T$, and a discount factor $\delta$ (the agent’s impatient factor) be primitives of the model. Given $X(0)$ the agent seeks to choose an admissible portfolio proportion process such that $(\zeta(t), c(t)) \in F_V(t)$ for all $0 \leq t \leq T$, and the expected value of her CRRA utility of intertemporal consumption and final wealth

$$\int_0^T e^{-\delta t} U_p(C(t)) dt + e^{-\delta T} U_p(X^{\zeta, c}(T)),$$

is maximized over all admissible portfolios processes satisfying the same constraint. Here $U_p(x) \triangleq \frac{x^p}{p}$, with $p < 1$ the coefficient of relative risk aversion (CRRA). Let us assume for the moment that the market coefficients are constants. In this case the constraint set $F_V(t)$ does not change over time and we denote it $F_V$. Then one can use the dynamic programming techniques to characterize the optimal portfolio and consumption policy. The problem is to find a solution to the HJB-equation. Define the optimal value function as

$$V(x, t) = \max_{(\zeta, c) \in F_V} \mathbb{E}_t \left[ \int_t^T e^{-\delta u} U_p(C(u)) du + e^{-\delta T} U_p(X^{\zeta, c}(T)) \right],$$

where $\mathbb{E}_t$ is the conditional operator given the information known up to time $t$ and $X^{\zeta, c}(t) = x$. The HJB equation is

$$\max_{(\zeta, c) \in F_V} J(t, x, \zeta, c) = 0,$$

where

$$J(t, x, c, \zeta) \triangleq e^{-\delta t} U_p(cx) + \frac{\partial V}{\partial t} + x(r - c + \zeta^T \mu) \frac{\partial V}{\partial x} + \frac{||\zeta^T \sigma||^2 x^2}{2} \frac{\partial^2 V}{\partial x^2},$$

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with the boundary condition \( V(x, T) = e^{-\delta T} U_p(x) \). The value function \( V \) inherits the concavity from the utility functions \( U_p \). Being jointly concave in \((\zeta, c)\), the function \( J \) is maximized over the set \( F_V \) at a unique point \((\zeta, \tau)\). Moreover this point should lie on the boundary of \( F_V \) and one can derive first order optimality conditions by means of Lagrange multipliers. Together with the HJB equation yield a highly PDE which is hard to solve numerically (a numerical scheme is proposed in [21] but no convergence result is shown). In what follows we approach the portfolio optimization problem by reducing it to a deterministic control one. We are able to obtain explicit solutions for logarithmic utility.

4 Logarithmic utility

Let us consider the case in which the agent is deriving utility from intermediate consumption only. It is straightforward how to extend it to encompass the utility of the final wealth as well. In the light of (2.4)

\[
\log X^{\zeta, c}(t) = \log X(0) + \int_0^t Q_0(s, \zeta(s), c(s)) ds + \int_0^t \zeta^T(s) \sigma(s) dW(s). \tag{4.1}
\]

What facilitates the analysis is the decomposition of the utility from intertemporal consumption into signal, a Lebesque integral and noise which comes at a \( \text{Itô} \) integral rate. The decomposition is additive and the expectation operator cancel the noise. Indeed

\[
\int_0^T e^{-\delta t} \log C(t) dt = \int_0^T e^{-\delta t} \log (c(t) X^{\zeta, c}(t)) dt = \frac{1 - e^{-\delta T}}{\delta} \log X(0) + \int_0^T e^{-\delta t} \log c(t) dt
\]

\[
+ \int_0^T \int_0^t e^{-\delta s} Q_0(s, \zeta(s), c(s)) ds dt + \int_0^T e^{-\delta t} \int_0^t \zeta^T(s) \sigma(s) dW(s) dt.
\]

By Fubini’s Theorem

\[
\int_0^T \int_0^t e^{-\delta t} Q_0(s, \zeta(s), c(s)) ds dt = \int_0^T \left( \int_0^T e^{-\delta t} Q_0(s, \zeta(s), c(s)) dt \right) ds
\]

\[
= \int_0^T \frac{e^{-\delta t} - e^{-\delta T}}{\delta} Q_0(t, \zeta(t), c(t)) dt,
\]

hence

\[
\int_0^T e^{-\delta t} \log C(t) dt = \frac{1 - e^{-\delta T}}{\delta} \log X(0) + \int_0^T e^{-\delta t} \left( \log c(t) + \frac{1}{\delta} (1 - e^{-\delta(T-t)}) Q_0(t, \zeta(t), c(t)) \right) dt
\]

\[
+ \int_0^T e^{-\delta t} \int_0^t \zeta^T(s) \sigma(s) dW(s) dt. \tag{4.2}
\]

The linearly independent rows assumption on matrix-valued volatility process yields the existence of the inverse \((\sigma(t)\sigma^T(t))^{-1}\) and so the equation

\[
\sigma(t)\sigma^T(t)\zeta_M(t) = \mu(t), \tag{4.3}
\]
uniquely defines a stochastic process \( \{ \zeta_M(t) \}_{t \in [0, \infty)} = \{ (\sigma(t)(\sigma^T(t))^{-1}\mu(t) \}_{t \in [0, \infty)}, \) called the Merton-ratio process. It has the pleasant property that it maximizes (in the absence of portfolios constraints), the rate-of-growth, and the log-optimizing investor would invest exactly using the components of \( \zeta_M(t) \) as portfolios proportions (see Section 3.10 in [13]). By (4.3)

\[
\| \zeta_M^T(t)\sigma(t) \|^2 = \zeta_M^T(t)\mu(t) = \mu^T(t)(\sigma(t)\sigma^T(t))^{-1}\mu(t).
\]

The following integrability assumption is rather technical but it guarantees that a local martingale (Itô integral) is a (true) martingale (see p. 130 in [13]). Let us assume that

\[
\mathbb{E} \int_0^T \| \zeta_M^T(u)\sigma(u) \|^2 du < \infty.
\] (4.5)

This requirement although imposed on the market coefficients (see (4.4)), is also inherited for all the portfolios satisfying the Value-at-Risk constraint.

**Lemma 4.1** For every \((\zeta(t), c(t)) \in F_V(t)\) the process \( \int_0^t \zeta^T(s)\sigma(s) dW(s), t \in [0, T] \) is a martingale, hence \( \mathbb{E} \int_0^t \zeta^T(s)\sigma(s) dW(s) = 0. \)

**Proof:** See the Appendix.

In the light of this Lemma, the expectation of the noise vanish, i.e.,

\[
\mathbb{E} \int_0^T e^{-\delta t} \int_0^t \zeta^T(s)\sigma(s) dW(s) dt = 0,
\] after interchanging the order of integration. Thus taking expectation in the additive utility decomposition (4.2)

\[
\mathbb{E} \int_0^T e^{-\delta t} \log C(t) dt = \frac{1 - e^{-\delta T}}{\delta} \log X(0) + \mathbb{E} \int_0^T e^{-\delta t} \left( \log c(t) + \frac{1}{\delta}(1 - e^{-\delta(T-t)})Q_0(t, \zeta(t), c(t)) \right) dt.
\] (4.6)

Therefore to maximize

\[
\mathbb{E} \int_0^T e^{-\delta t} \log C(t) dt,
\]

over the constraint set it suffices to maximize

\[
g(t, \zeta, c) \triangleq \log c(t) + \frac{1}{\delta}(1 - e^{-\delta(T-t)})Q_0(t, \zeta(t), c(t))
\] (4.7)
pathwise over the constraint set. For a fixed path $\omega$ and a time-instance $t$, we need to solve

\[(P1) \quad \text{maximize } g(t, \zeta, c)\]

subject to \[f_V(t, \zeta, c) \triangleq -Q_0(t, \zeta, c)\tau - N^{-1}(\alpha)\|\zeta^T \sigma(t)\|\sqrt{\tau} \leq \log \frac{1}{1-aV}.\]

The optimal policy for an agent maximizing her logarithmic utility of intertemporal consumption without the risk constraint is to hold the proportion \[{(\zeta_M(t), c_M(t))}_{t \in [0, T)}, \text{where } c_M(t) \triangleq \frac{\delta}{1-e^{-a(1-T)}} \text{ (the optimum of } (P1) \text{ without the constraint is } (\zeta_M(t), c_M(t)).}\]

**Lemma 4.2** The solution of (P1) is given by

\[\tilde{\zeta}(t) = (1 \land (\beta(t) \lor 0))\zeta_M(t),\]

\[\tilde{c}(t) = u(t, (1 \land \beta(t)))c_M(t)1_{\{\beta(t) > 0\}} + \left(r + \frac{1}{r} \log \frac{1}{1-aV}\right)1_{\{\beta(t) \leq 0\}},\]

where $\beta(t)$ is the root of the equation

\[f_V(t, z\zeta_M(t), u(t, z)c_M(t)) = \log \frac{1}{1-aV},\]

in the variable $z$, with

\[u(t, z) \triangleq 1 + \frac{\sqrt{\tau}||\zeta_M^T(t)\sigma(t)||}{N^{-1}(\alpha)}(1-z).\]

**Proof:** See the Appendix.

\[\Box\]

**Theorem 4.3** For maximizing the logarithmic utility of intertemporal consumption,

\[\mathbb{E} \int_0^T e^{-\delta t} \log C(t) dt,\]

over processes $((\zeta(t), c(t)) \in F_V(t), 0 \leq t \leq T$, the optimal portfolio is \[{(\tilde{\zeta}(t), \tilde{c}(t))}_{t \in [0, T]}.\]

**Proof:** It is a direct consequence of (4.6) and Lemma 4.2.

\[\Diamond\]

**Remark 4.4** Since at the final time $c_M(T) = \infty$ and $\tilde{c}(t)$ is bounded we must have $\beta(T) \leq 0$, so $\tilde{\zeta}(T) = 0$, and it means that at the final time the agent invests the least proportion (in absolute terms) of her wealth in stocks. By (4.8) and (4.9) it follows that $\tilde{\zeta}(t) \leq \zeta_M(t)$, and $\tilde{c}(t) \leq c_M(t)$, for any $0 \leq t \leq T$, which means that the constrained agent is consuming and investing less in the risky assets than the unconstrained agent. Let $T_1$ and $T_2$ two final time horizons, $T_1 > T_2$. Because $c_M(t, T_1) < c_M(t, T_2)$, from equations (4.8) and (4.9) we conclude that $\beta(t, T_1) > \beta(t, T_2)$, hence $\tilde{\zeta}(t, T_1) > \tilde{\zeta}(t, T_2)$, and $\tilde{c}(t, T_1) > \tilde{c}(t, T_2)$. Therefore long-term agents can afford to invest more in the stock market and consume more than short term agents (in terms of proportions).
5 Non-logarithmic utility

Let us recall that we want to maximize expected CRRA utility \(U_p(x) = \frac{x^p}{p}, p \neq 0\) from intertemporal consumption and terminal wealth:

\[
E \int_0^T e^{-\delta t} U_p(C(t)) dt + e^{-\delta T} U_p(X^{\zeta,c}(T)),
\]

(5.1)

over portfolio proportion processes satisfying the Value-at-Risk constraint, i.e, \((\zeta(t), c(t)) \in F_V(t), 0 \leq t \leq T\). One cannot obtain an additive decomposition into signal and noise as in the case of logarithmic utility. However a multiplicative decomposition can be performed. By (2.4)

\[
U_p(X^{\zeta,c}(t)) = \frac{X^p(0)}{p} \exp \left( \int_0^t pQ_0(s, \zeta(s), c(s)) ds + \int_0^t p\zeta^T(s) \sigma(s) dW(s) \right)
\]

\[
= \frac{X^p(0)}{p} \exp \left( \int_0^t \left( pQ_0(s, \zeta(s), c(s)) - \frac{1}{2} p^2 \|\zeta^T(s) \sigma(s)\|^2 \right) ds + \int_0^t p\zeta^T(s) \sigma(s) dW(s) \right)
\]

\[
= \frac{X^p(0)}{p} N^{\zeta,c}(t) Z^\zeta(t),
\]

where

\[
N^{\zeta,c}(t) \triangleq \exp \left( \int_0^t pQ_0(s, \zeta(s), c(s)) ds \right),
\]

(5.2)

\[
Z^\zeta(t) \triangleq \exp \left( -\frac{1}{2} \int_0^t p^2 \|\zeta^T(s) \sigma(s)\|^2 ds + \int_0^t p\zeta^T(s) \sigma(s) dW(s) \right),
\]

(5.3)

with \(Q_0\) defined in (2.3). By taking expectation

\[
E U_p(X^{\zeta,c}(t)) = \frac{X^p(0)}{p} E(N^{\zeta,c}(t) Z^\zeta(t))
\]

(5.4)

The process \(N^{\zeta,c}(t)\) is the signal and \(Z^\zeta(t)\), a stochastic exponential is the noise. Stochastic exponentials are local martingales, but if we impose the following assumption

\[
E \left( \exp \frac{p^2}{2} \int_0^T \|\zeta_M^T(u) \sigma(u)\|^2 du \right) < \infty,
\]

(5.5)

on market coefficients (see (4.4)), the process \(Z^\zeta(t)\) is a (true) martingale for all portfolio processes satisfying the constraint, as the next Lemma shows.

**Lemma 5.1** For every \((\zeta(t), c(t)) \in F_V(t)\) the process \(Z^\zeta(t), t \in [0,T]\) is a martingale, hence \(E Z^\zeta(t) = 1\).

**Proof:** See the Appendix.
As for utility from intertemporal consumption
\[
\int_0^T e^{-\delta t} U_p(C(t)) dt = \int_0^T e^{-\delta t} U_p(X^{\zeta, c}(t)) c^p(t) dt = \frac{X^p(0)}{p} \int_0^T e^{-\delta t} c^p(t) N^{\zeta, c}(t) Z^\zeta(t) dt. \tag{5.6}
\]
We claim that
\[
\mathbb{E}\left( \int_0^T e^{-\delta t} c^p(t) N^{\zeta, c}(t)(Z^\zeta(t) - Z^\zeta(T)) dt \right) = 0.
\]
Indeed by conditioning and Lemma 5.1 we get
\[
\mathbb{E}\left( e^p(t) N^{\zeta, c}(t)(Z^\zeta(t) - Z^\zeta(T)) \right) = \mathbb{E}(\mathbb{E}[e^p(t) N^{\zeta, c}(t)(Z^\zeta(t) - Z^\zeta(T)) | F_t]) = \mathbb{E}(e^p(t) N^{\zeta, c}(t) \mathbb{E}[(Z^\zeta(t) - Z^\zeta(T)) | F_t]) = 0,
\]
and Fubini’s Theorem proves the claim. Hence combined with (5.6) yields
\[
\mathbb{E}\int_0^T e^{-\delta t} U_p(C(t)) dt = \frac{X^p(0)}{p} \mathbb{E}\left( Z^\zeta(T) \int_0^T e^{-\delta t} c^p(t) N^{\zeta, c}(t) dt \right). \tag{5.7}
\]
The decomposition for the total expected utility ((5.4) and (5.7)) is
\[
\mathbb{E}\int_0^T e^{-\delta t} U_p(C(t)) dt + \mathbb{E}e^{-\delta T} U_p(X^{\zeta, c}(T)) = \frac{X^p(0)}{p} \mathbb{E}(Z^\zeta(T) Y^{\zeta, c}(T)), \tag{5.8}
\]
where the signal \( Y^{\zeta, c}(T) \) is given by
\[
Y^{\zeta, c}(T) = \int_0^T e^{-\delta t} c^p(t) N^{\zeta, c}(t) dt + e^{-\delta T} N^{\zeta, c}(T), \tag{5.9}
\]
with \( N^{\zeta, c}(t) \) defined in (5.2). It appears naturally at this point to maximize \( Y^{\zeta, c}(T) \) pathwise over the constraint set. For a given path \( \omega \), the existence of an optimizer \( \{ (\zeta(t, \omega), c(t, \omega)) \}_{t \in [0, T]} \) is given by Lemma 5.2. Notice that \( N^{\zeta, c}(t, \omega) \) depends on the trajectory of \( (\zeta(\cdot, \omega), c(\cdot, \omega)) \) on \( [0, T] \) so one is faced with a deterministic control problem. From now on to keep the notations simple we drop \( \omega \). In the language of deterministic control we can write (5.9) as a cost functional
\[
I[x, u] = g(x(T)) + \int_0^T f_0(t, x(t), u(t)) dt, \quad g(x) \triangleq e^{-\delta T} x, \tag{5.10}
\]
where \( u = (\zeta, c) \) is the control, \( x \) is the state variable, and the function
\[
f_0(t, x, u) \triangleq e^{-\delta t} c^p x, \tag{5.11}
\]
is defined on the set

$$A = \{(t, x, u) | (t, x) \in [0, T] \times (0, K), u(t) \in F_\nu(t)\} \subset \mathbb{R}^{m+3}. \quad (5.12)$$

The dynamics of the state variable is given by the differential equation

$$\frac{dx}{dt} = f(t, x(t), u(t)), \quad 0 \leq t \leq T,$$

with the boundary condition $x(0) = 1$, where

$$f(t, x, u) \triangleq x \left( pr - pc + p_\xi^T \mu(t) + \frac{p(p-1)}{2} \| \zeta^T \sigma(t) \|_2 \right). \quad (5.14)$$

The constraints are $(t, x(t)) \in [0, T] \times (0, K]$ and $u(t) \in F_\nu(t)$. Due to the compactness of the set $F_\nu(t)$, $0 \leq t \leq T$ it follows that $K < \infty$. A pair $(x, u)$ satisfying the above conditions is called admissible. The problem of finding the maxima of $I[x, u]$ within all admissible pairs $(x, u)$ is called the Bolza control problem. The classical existence theory for deterministic control does not apply to the present situation and we proceed with a direct proof of existence.

**Lemma 5.2** There exists a solution $\{\ddot{u}(t)\}_{0 \leq t \leq T} = \{(\ddot{\zeta}(t), \ddot{\epsilon}(t))\}_{0 \leq t \leq T}$ for the Bolza control problem defined above.

**Proof:** See the Appendix. \qed

An optimal solution $\{\ddot{u}(t)\}_{0 \leq t \leq T} = \{(\ddot{\zeta}(t), \ddot{\epsilon}(t))\}_{0 \leq t \leq T}$ is characterized by a system of forward backward equations (also known as Pontryagin maximum principle). Let $\lambda = (\lambda_0, \lambda_1)$ be the adjoint variable and

$$H(t, x, u, \dot{\lambda}) = \lambda_0 f_0(t, x, u) + \lambda_1 f(t, x, u),$$

the Hamiltonian function. The necessary conditions for Bolza control problem (Pontryagin maximum principle) can be found in [4] (Theorem 5.1.i). In general they are not sufficient for optimality. We prove that in our context the necessary conditions are also sufficient as the next Lemma shows.

**Lemma 5.3** A pair $\ddot{x}(t), \ddot{u}(t) = (\ddot{\zeta}(t), \ddot{\epsilon}(t)) \in F_\nu(t), \quad 0 \leq t \leq T$ is optimal, i.e., gives the maximum for the functional $I[x, u]$ if and only if there is an absolutely continuous nonzero vector function of Lagrange multipliers $\ddot{\lambda} = (\lambda_0, \lambda_1), \quad 0 \leq t \leq T$, with $\lambda_0$ a constant, $\lambda_0 \geq 0$ such that the function $M(t) \triangleq H(t, \ddot{x}(t), \ddot{u}(t), \ddot{\lambda}(t))$ is absolutely continuous and one has:

1. Adjoint equations:

$$\frac{dM}{dt} = H_x(t, \ddot{x}(t), \ddot{u}(t), \ddot{\lambda}(t)) \text{ a.e.}, \quad (5.15)$$

$$\frac{d\lambda_1}{dt} = -H_x(t, \ddot{x}(t), \ddot{u}(t), \ddot{\lambda}(t)) \text{ a.e.}, \quad (5.16)$$
2. **Maximum condition:**

\[ \hat{u}(t) \in \arg \max_{v \in F_V(t)} H(t, \bar{x}(t), v, \bar{\lambda}(t)) \text{ a.e.,} \]

(5.17)

3. **Transversality:**

\[ \lambda_1(T) = \lambda_0 g'(\bar{x}(T)). \]

(5.18)

**Proof:** See the Appendix.

The following technical requirement on the market coefficients is sufficient to make \( \{ (\bar{\zeta}(t), \bar{c}(t)) \}_{t \in [0,T]} \) an optimal portfolio process for maximizing the CRRA utility under Value-at-Risk constraint, as Theorem 5.4 shows. We assume that market coefficients are totally unhedgeable, i.e., the mean rate of return process \( \alpha(t) \) and the matrix-valued volatility process \( \sigma(t) \) are adapted to a filtration \( \{ \mathcal{F}_t \}_{0 \leq t \leq \infty} \) generated by a Brownian motion independent of the Brownian motion driving the stocks (see (2.1)).

**Theorem 5.4** A solution for maximizing

\[ \mathbb{E} \int_0^T e^{-\delta t} U_p(C(t)) dt + \mathbb{E} e^{-\delta T} U_p(X(T)), \]

over processes \( (\zeta(t), c(t)) \in F_V(t), 0 \leq t \leq T \) is a process, \( \{ (\bar{\zeta}(t), \bar{c}(t)) \}_{t \in [0,T]} \), which on every \( \omega \) solves (5.15), (5.16), (5.17) and (5.18).

**Proof:** Lemma 5.2 gives the existence on every \( \omega \) of \( \{ (\bar{\zeta}(t), \bar{c}(t)) \}_{t \in [0,T]} \), optimal for Bolza control problem, i.e., it maximizes \( Y^{\bar{\zeta}, \bar{c}}(T) \) defined in (5.9) over \( F_V(t), 0 \leq t \leq T \). According to Lemma 5.3 it should solve (5.15), (5.16), (5.17) and (5.18) pathwise and this equations are sufficient for optimality. Let \( (\zeta(t), c(t)) \in F_V(t) \), be another control. Let \( Z^{\bar{\zeta}}(t), Z^{\zeta}(t), \) and \( Y^{\bar{\zeta}, \bar{c}}(T), Y^{\zeta, c}(T) \) as in (5.3) and (5.9). The processes \( \{ Z^{\bar{\zeta}}(t) \}_{0 \leq t \leq T}, \{ Z^{\zeta}(t) \}_{0 \leq t \leq T} \), are martingales by Lemma 5.1. Moreover the independence of \( \{ \mathcal{F}_t \}_{0 \leq t \leq \infty} \) and \( \{ \mathcal{F}_t \}_{0 \leq t \leq \infty} \) implies

\[ \mathbb{E}[Z^{\bar{\zeta}}(T) | \mathcal{F}_T] = \mathbb{E}[Z^{\bar{\zeta}}(T) | \mathcal{F}_T] = 1. \]

Lemma 5.3 shows that the process \( Y^{\bar{\zeta}, \bar{c}}(T) \) is measurable with respect to \( \mathcal{F}_T \). Therefore by (5.8) and iterated conditioning

\[ \mathbb{E} \int_0^T e^{-\delta t} U_p(C(t)) dt + \mathbb{E} e^{-\delta T} U_p(X^{\bar{\zeta}, \bar{c}}(T)) = \frac{X^p(0)}{p} \mathbb{E}(Z^{\bar{\zeta}}(T)Y^{\bar{\zeta}, \bar{c}}(T)) \]

\[ = \frac{X^p(0)}{p} \mathbb{E}(Z^{\bar{\zeta}}(T)Y^{\bar{\zeta}, \bar{c}}(T) | \mathcal{F}_T)) \]

\[ = \frac{X^p(0)}{p} \mathbb{E}(Y^{\bar{\zeta}, \bar{c}}(T)E[Z^{\bar{\zeta}}(T) | \mathcal{F}_T]) \]

\[ = \frac{X^p(0)}{p} \mathbb{E}Y^{\bar{\zeta}, \bar{c}}(T). \]
Since \((\zeta(t), \bar{c}(t))\) maximizes \(Y^{\zeta,c}(T)\) over the constraint set
\[
\mathbb{E} \int_0^T e^{-\delta t} U_p(C(t)) dt + \mathbb{E} e^{-\delta T} U_p(X^{\zeta,c}(T)) = \frac{X^p(0)}{p} \mathbb{E} (Z^\zeta(T) Y^{\zeta,e}(T))
\leq \frac{X^p(0)}{p} \mathbb{E} (Z^\zeta(T) Y^{\zeta,e}(T))
= \frac{X^p(0)}{p} \mathbb{E} (Z^\zeta(T) Y^{\zeta,e}(T) | \mathcal{F}(T))
= \frac{X^p(0)}{p} \mathbb{E} (Y^{\zeta,e}(T) E[Z^\zeta(T) | \mathcal{F}(T)])
= \frac{X^p(0)}{p} \mathbb{E} Y^{\zeta,e}(T).
\]

**Remark 5.5** Let the interest rate and the discount factor be stochastic processes. In the formulae of \(f_V(t, \zeta, c)\) and \(Q_p(t, \zeta, c)\) the constant \(r\) gets replaced by \(r(t)\). All the results remain true if we assume that \(\{r(t)\}_{0 \leq t \leq \infty}\) and \(\{\delta(t)\}_{0 \leq t \leq \infty}\) are nonnegative uniformly bounded continuous processes adapted to \(\mathcal{F}(t)\). We considered the case of Value-at-Risk (VaR) in defining the risk constraint. The same methodology applies if one considers other measures of risk, as long as the corresponding constraint set is convex and compact.

## 6 Numerical Solution

Theorem 5.4 shows that the solution for every path of a Bolza control problem yields the optimal portfolio proportion for a VaR constrained agent. The solution exists and is characterized by a system of forward backward equations which are also sufficient for optimality. In this section by an appropriate discretization of control and state variables, the Bolza control problem is transformed into a finite dimensional nonlinear program which can be solved by standard sequential quadratic programming (SQP) methods. The first step is to transform the Bolza problem into a Mayer control problem by introducing a new state variable \(x^0\), with the boundary condition \(x^0(0) = 0\) and an additional differential equation,
\[
\frac{dx^0}{dt} = f_0(t, x(t), u(t)).
\]
The cost functional is then \(I[x, u] = x^0(T) + g(x(T))\) (see 5.10). Let us denote \(y = (x^0, x)\) the vector of states variables which satisfy the differential equation
\[
\frac{dy}{dt} = \tilde{f}(t, x(t), u(t)),
\]
with \(\tilde{f} = (f_0, f)\) (see (5.11) and (5.13)). The following discretization scheme is taken from [18]. The novel feature of it is that the necessary first order conditions of the discretized problem converge to the necessary first order conditions of the continuous problem.
A partition of the time interval 

\[ 0 = t_1 < t_2 < \cdots < t_N = T, \]

is chosen. The parameters \(Y\) of the nonlinear program are the values of control and state variables at the grid points \(t_j, j = 1, \cdots, N\) and the final time \(t_N = T\),

\[ Y = (u(t_1), \cdots, u(t_N), y(t_1), \cdots, y(t_N), t_N) \in \mathbb{R}^{4N+1}. \]

The controls are chosen as piecewise linear interpolating functions between \(u(t_j)\) and \(u(t_{j+1})\), for \(t_j \leq t < t_{j+1}\),

\[ u_{app}(t) = u(t_j) + \frac{t - t_j}{t_{j+1} - t_j} (u(t_{j+1}) - u(t_j)). \]

The states are chosen as continuously differentiable functions and piecewise cubic Hermite polynomials between \(y(t_j)\) and \(y(t_{j+1})\), with \(\dot{y}_{app}(s) = \dot{f}(x(s), u(s), s)\) at \(s = t_j, t_{j+1}\),

\[ y_{app}(t) = \sum_{k=0}^{3} d_k^j \left( \frac{t - t_j}{h_j} \right)^k, \quad t_j \leq t < t_{j+1} \quad j = 1, \cdots, N - 1, \tag{6.1} \]

\[ d_0^j = y(t_j), \quad d_1^j = h_j \ddot{f}_j, \tag{6.2} \]

\[ d_2^j = -3y(t_j) - 2h_j \ddot{f}_j + 3y(t_{j+1}) - h_j \ddot{f}_{j+1}, \tag{6.3} \]

\[ d_3^j = 2y(t_{j+1}) + h_j \ddot{f}_j - 2y(t_{j+1}) + h_j \ddot{f}_{j+1}, \tag{6.4} \]

where \(\ddot{f}_j \triangleq \ddot{f}(x(t_j), u(t_j), t_j)\), \(h_j \triangleq t_{j+1} - t_j\).

The reader can learn more about this in common textbooks of Numerical Analysis such as e.g. [19]. This way of discretizing has two advantages. The number of parameters of nonlinear program is reduced because \(\dot{y}_{app}(t_j), j = 1, \cdots, N\) are not parameters and the number of constraints is reduced because the constraints \(\dot{y}_{app}(t_j) = \dot{f}(x(t_j), u(t_j), t_j), j = 1, \cdots, N\), are already fulfilled.

We impose the Value-at-Risk constraint (see (2.6)) at the grid points

\[ f_V(t_j, u(t_j)) \leq \log \frac{1}{1 - a_V}, \quad u = (\zeta, c), \quad j = 1, \cdots, N, \tag{6.5} \]

Another constraint imposed is the so called collocation constraint

\[ \ddot{f}(x(\tilde{t}_j), u(\tilde{t}_j), \tilde{t}_j) = \dot{y}_{app}(\tilde{t}_j) \quad j = 1, \cdots, N, \]

or componentwise

\[ f_0(x(\tilde{t}_j), u(\tilde{t}_j), \tilde{t}_j) = \dot{x}_{app}^0(\tilde{t}_j) \quad j = 1, \cdots, N, \tag{6.6} \]

and

\[ f(x(\tilde{t}_j), u(\tilde{t}_j), \tilde{t}_j) = \dot{x}_{app}(\tilde{t}_j) \quad j = 1, \cdots, N, \tag{6.7} \]
where \( \tilde{t}_j \triangleq \frac{t_j + t_{j+1}}{2} \), and the boundary condition \( y(0) = (0, 1) \). The nonlinear program is to maximize \( I[y_N, t_N] \) subject to constraints (6.5), (6.6) and (6.7). It can be solved using NPSOL, a set of Fortran subroutines developed in [10]. NPSOL uses a sequential quadratic programming (SQP) algorithm, in which the search directions is the solution of a quadratic programming (QP) subproblem. The Lagrangian of the nonlinear program is

\[
L(Y, \phi, v) = I[y_N, t_N] + \sum_{j=1}^{N} v_j \left( f_V(t_j, u(t_j)) - \log \frac{1}{1 - a_V} + \phi_j^0 (f_0(x(t_j), u(t_j), \tilde{t}_j) - \hat{x}_{\text{app}}(\tilde{t}_j)) \right) + \sum_{j=1}^{N-1} \phi_j^1 (f(x(t_j), u(t_j), \tilde{t}_j) - \hat{x}_{\text{app}}(\tilde{t}_j)),
\]

where \( v = (v_1, \ldots, v_N) \in R^N \), \( \phi^0 = (\phi^0_1, \ldots, \phi^0_{N-1}) \in R^{N-1} \), and \( \phi^1 = (\phi^1_1, \ldots, \phi^1_{N-1}) \in R^{N-1} \) are the shadow prices. Let us denote \( \zeta(t_i) \triangleq \zeta_i \), \( c(t_i) \triangleq c_i \) and \( x(t_i) \triangleq x_i \). A solution of the nonlinear program satisfies the necessary first order conditions of Karush, Kuhn, and Tucker:

\[
\frac{\partial L}{\partial \zeta_i} = 0, \quad \frac{\partial L}{\partial c_i} = 0, \quad \frac{\partial L}{\partial x_i} = 0, \quad i = 1, \ldots, N.
\]

The necessary first order optimality conditions of the continuous problem are obtained in the limit from (6.8) as follows. Let \( h \triangleq \max\{h_j = t_{j+1} - t_j : j = 1, \ldots, N - 1\} \) be the norm of the partition. Letting \( h \to 0 \) after some calculations (see p. 5 in [18]) it is shown that at \( t = t_i \)

\[
\frac{\partial L}{\partial \zeta_i} \to \frac{3}{2} \phi^1_i \frac{\partial f(x(t_i), u(t_i), t_i)}{\partial \zeta} + \frac{3}{2} \phi^0_i \frac{\partial f_0(x(t_i), u(t_i), t_i)}{\partial \zeta} + v_i \frac{\partial f_V(u(t_i), t_i)}{\partial \zeta},
\]

\[
\frac{\partial L}{\partial c_i} \to \frac{3}{2} \phi^1_i \frac{\partial f(x(t_i), u(t_i), t_i)}{\partial c} + \frac{3}{2} \phi^0_i \frac{\partial f_0(x(t_i), u(t_i), t_i)}{\partial c} + v_i \frac{\partial f_V(u(t_i), t_i)}{\partial c},
\]

and

\[
\frac{\partial L}{\partial x_i} \to \frac{3}{2} \phi^1_i + \frac{3}{2} \phi^0_i \frac{\partial f(x(t_i), u(t_i), t_i)}{\partial x} + \frac{3}{2} \phi^0_i \frac{\partial f_0(x(t_i), u(t_i), t_i)}{\partial x}.
\]

Therefore the equations \( \frac{\partial L}{\partial x_i} = 0 \) and \( \frac{\partial L}{\partial c_i} = 0 \) converge to an equation equivalent to the maximum condition (5.17), and \( \frac{\partial L}{\partial x_i} = 0 \) converge to the adjoint equation (5.16). This discretization scheme gives good estimates for the adjoint variables.

In what follows we perform some numerical experiments. We consider one stock following a geometric Brownian motion with drift \( \alpha_1 = 0.12 \), volatility \( \sigma = 0.2 \). The choice of the horizon \( \tau \) and the confidence level \( \alpha \), are largely arbitrary, although the Basle Committee proposals of April 1995 prescribed that VaR computations for the purpose of assessing bank capital requirements should be based on a uniform horizon of 10 trading days (two calendar weeks) and a 99% confidence level (see [11]). We take \( \tau = \frac{1}{25}, \alpha = 0.01 \), the interest rate \( r = 0.05 \) and the discount factor \( \delta = 0.1 \).
Fig. 1. Asset allocations with and without VaR constraints, for the utility maximization of intertemporal consumption and final wealth. The graphs correspond to different values of CRRA, $p$. For $p = -1.5$ → Graph 1, $p = -1$ → Graph 2, $p = -0.5$ → Graph 3, and $p = 0.5$ → Graph 4. The $x$ axis represents the time and the $y$ axis the proportion of wealth invested in stocks. Let us notice the Merton line and, as time goes by, the portfolio value increases hence the VaR constraint becomes binding and reduces the investment in the risky asset. At the final time the agent is investing the least in stocks (in terms of proportions). When $p$ increases, i.e., when the agent becomes less risk averse the effect of VaR constraint becomes more significant.
7 Concluding Remarks

Let us summarize the results. This paper examines in a stochastic paradigm the portfolio choice problem under a risk constraint, which is applied dynamically consistent at every time instant. The classical stochastic control methods, Dynamic Programming and Martingale Method, are not very effective in this context. The last one works if the risk constraint is imposed in a static way. The Dynamic Programming approach (as shown in the objective section) leads to a highly nonlinear PDE. If the agent has CRRA preferences we proposed a new method which relies on a decomposition of the utility into signal and noise. We neglect the noise (the expectation operator takes care of it) and this leads to a deterministic control problem on every path. We have reported explicit analytical solutions for the case of logarithmic utility even if the market coefficients are random processes. In this case, on every path the deterministic control problem is just a time dependent constrained nonlinear program. The explicit solution shows that constrained agents consume and invest less in stocks than unconstrained agents, and the long-term agents invest and consume more than short-term agents. These effects come to support the use of dynamically consistent risk constraints. If the utility is non-logarithmic CRRA we have to analyze a Bolza control problem on every path. We still allow the market coefficients to be random but independent of the Brownian motions driving the stocks. Theorem 5.4 shows that a solution of the deterministic control problem is an optimal policy. Although the existence of an optimal policy is known if the constraint set is convex (see [5]), it does not necessarily yield existence for the Bolza problem. Standard existence theorems do not apply, but we managed to give a direct proof of existence in Lemma 5.2. The solution of Bolza problem must solve a system of forward backward equations (the first order necessary conditions) and this is also sufficient for optimality. In section 6 we suggest a numerical treatment of Bolza problem. The reduction of the stochastic control problem to a deterministic one relies on the structure of CRRA preferences. It would be interesting to be extended to other class of preferences as well, because it turns out very effective for the case of dynamically consistent risk constraints.

8 Appendix

Proof of Lemma 4.1: In order to prove the martingale property of \( \int_0^T \zeta^T(s) \sigma(s) \, dW(s) \) it suffices to show

\[
\mathbb{E} \int_0^T \| \zeta^T(u) \sigma(u) \|^2 \, du < \infty.
\]  

(8.1)

Let us notice that

\[
\| \zeta^T(t) \mu(t) \| = \| \zeta^T(t) \sigma(t) \sigma^T(t) \zeta_M(t) \| \leq \| \zeta^T(t) \sigma(t) \| \cdot \| \zeta_M^T(t) \sigma(t) \|. 
\]  

(8.2)

For \( (\zeta(t), c(t)) \in F_V(t) \), we have

\[
\left( r - c(t) + \zeta^T(t) \mu(t) - \frac{1}{2} \| \zeta^T(t) \sigma(t) \|^2 \right) \tau + N^{-1}(\alpha) \| \zeta^T(t) \sigma(t) \| \sqrt{\tau} \geq \log(1 - a_V).
\]  

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This combined with (8.2) yields
\[ ||\zeta^T(t)\sigma(t)|| \leq k_1 \vee ||\zeta^T_M(t)\sigma(t)||, \] (8.3)
for some positive constant \(k_1\), where as usual \(a \vee b = \max(a, b)\). In the light of the assumption (4.5) the inequality (8.1) follows.

\(\Box\)

Proof of Lemma 4.2:
The proof relies on the method of Lagrange multipliers. The concave function
\[ g(t, \zeta, c) \triangleq \log c + \frac{1}{\delta}(1 - e^{-\delta(T-t)})Q_0(t, \zeta, c), \] (8.4)
is maximized over \((\zeta, c) \in \mathbb{R}^m \times [0, \infty)\) by \((\zeta_M(t), c_M(t))\), where \(\zeta_M(t) \triangleq (\sigma(t)\sigma^T(t))^{-1}\mu(t)\), and \(c_M(t) \triangleq \frac{1 - e^{-\delta(T-t)}}{1 - e^{-\deltaT}}\). If this point is in the Value-at-Risk constraint set, then is the optimal solution of \((P1)\). Otherwise the concave function \(g\) is maximized over the compact, convex set \(F_V(t)\), at a unique point \((\tilde{\zeta}(t), \tilde{c}(t))\); moreover, this point must be on the boundary of \(F_V(t)\). Hence it solves the optimization problem
\[ (P2) \quad \text{maximize } g(t, \zeta, c) \]
subject to \(f_V(t, \zeta, c) \triangleq -Q_0(t, \zeta, c)\tau - N^{-1}(\alpha)||\zeta^T\sigma(t)||\sqrt{\tau} = \log \frac{1}{1-\alpha}. \)

The function \(f_V\) is not differentiable when \(\zeta\) is the zero vector. Let us assume that the optimal \(\tilde{\zeta}(t)\) is not the zero vector. According to the Lagrange multiplier theorem, either \(\nabla f_V(t, \tilde{\zeta}(t), \tilde{c}(t))\) is the zero vector or else there is a positive \(\lambda\) such that
\[ \nabla g(t, \tilde{\zeta}(t), c_V(t)) = \lambda \nabla f_V(t, \tilde{\zeta}(t), c_V(t)). \] (8.5)
The first case cannot happen and computations show that \(\tilde{\zeta}(t)\) is parallel to \(\zeta_M(t)\). This implies that the optimal solution \((\tilde{\zeta}(t), \tilde{c}(t)) = (\lambda_1\zeta_M(t), \lambda_2c_M(t))\), with \(\lambda_1, \lambda_2\) the solution of
\[ (P3) \quad \text{maximize } l(\lambda_1, \lambda_2) \]
subject to \(f_V(t, \lambda_1\zeta_M(t), \lambda_2c_M(t)) = \log \frac{1}{1-\alpha}; \]
where
\[ l(\lambda_1, \lambda_2) \triangleq g(t, \lambda_1\zeta_M(t), \lambda_2c_M(t)), \] (8.6)
The concave function \(l\) is maximized over \(\mathbb{R}^2\) at \((1, 1)\). We know this point is not in the constraint set (this is because we assumed \((\zeta_M(t), c_M(t)) \notin F_V(t)\)), hence \(\lambda_1 < 1, \lambda_2 < 1\), and either \(\nabla f_V(t, \lambda_1\zeta_M(t), \lambda_2c_M(t)) = 0\) or else \(\nabla l(\lambda_1, \lambda_2) = \gamma \nabla f_V(t, \lambda_1\zeta_M(t), \lambda_2c_M(t))\), for some positive Lagrange multiplier \(\gamma\). The first case cannot happen and by eliminating \(\gamma\) we get \(\lambda_2 = u(t, \lambda_1)\), where \(u\) was defined in (4.11). Henceforth \(\lambda_1\) is the unique root of the equation
\[ f_V(t, z\zeta_M(t), u(t, z)c_M(t)) = \log \frac{1}{1 - aV}, \] (8.7)
in the variable \(z\). It may happen that the root of this equation is negative in which case \((\tilde{\zeta}(t), \tilde{c}(t)) = (0_m, r + \frac{1}{T} \log \frac{1}{1-aV})\), where \(0_m\) is the \(m-\)dimensional vector of zeros.
Proof of Lemma 5.1:
The assumption (5.5) combined with (8.3) and The Novikov Condition (see [13], p. 199, Corollary 5.13), make the processes $Z^\xi(t)$ a martingale.

Proof of Lemma 5.2:
According to (5.14) and (5.13)
\[
x(t) = \exp \left( \int_0^t f_1(t, c(u), \zeta(u)) \, du \right),
\]
with
\[
f_1(t, c, \zeta) \triangleq pr - pc + p\zeta^T \mu(t) + \frac{p(p-1)}{2} ||\zeta^T \sigma(t)||^2.
\]
Let us recall that for $u(t) = (\zeta(t), c(t)) \in F_V(t)$,
\[
||\zeta^T(t) \sigma(t)|| \leq k_1 \vee ||\zeta^T_M(t) \sigma(t)||,
\]
(8.9) hence $||\zeta^T(t) \sigma(t)||$ is uniformly bounded on $[0,T]$ due to the continuity of market coefficients (see (4.4)). Moreover one can conclude that $||\zeta(t)|| \leq K_1$ and $c(t) \leq K_1$ for some constant $K_1$. Gronwall’s Lemma gives $x(t) \leq K_2$ and $|\dot{x}(t)| \leq K_2$ on $[0,T]$ (here $\dot{x}(t) = \frac{dx}{dt}$). Let $(x_n, u_n)$ be a maximizing sequence, i.e., $I[x_n, u_n] \to \sup I[x, u]$. The above arguments show that the sequence $x_n$ is uniformly bounded and equicontinuous, thus by Arzela-Ascoli theorem converges uniformly to some function $\bar{x}$. According to Komlos Lemma (see Lemma A1.1 in [8]) we can find some sequences of convex combinations $\bar{\zeta}_n \in \text{conv}(\zeta_n, \zeta_{n+1}, \ldots)$ and $\bar{c}_n \in \text{conv}(c_n, c_{n+1}, \ldots)$ which converges a.e. to some measurable functions $\bar{\zeta}$ and $\bar{c}$. Moreover $\bar{u}(t) \triangleq (\bar{\zeta}(t), \bar{c}(t)) \in F_V(t), 0 \leq t \leq T$, due to the convexity and compactness of the set $F_V(t)$. Let us denote $\bar{x}_n$ the sequence of state variables corresponding to these controls, i.e.,
\[
\bar{x}_n(t) = \exp \left( \int_0^t f_1(t, \bar{c}_n(s), \bar{\zeta}(s)) \, ds \right), \quad 0 \leq t \leq T,
\]
(see (8.8)). Let us assume $p > 0$, the case $p < 0$ can be treated similarly. Due to the concavity of the function $f_1$, $\ln \bar{x}_n(t) \geq \text{conv}(\ln x_n(t), \ln x_{n+1}(t), \ldots)$, where the convex combination is the one defining $\bar{\zeta}_n, \bar{c}_n$. If $\bar{y}_n \triangleq \exp(\text{conv}(\ln x_n(t), \ln x_{n+1}(t), \ldots))$, then $\bar{x}_n(t) \geq \bar{y}_n(t)$, and $\bar{y}_n(t) \to \bar{x}(t)$, i.e., $\bar{y}_n(t) - x_n(t) \to 0$ for $t \in [0,T]$. By dominated convergence theorem $\bar{x}_n(t) \to \bar{x}(t), 0 \leq t \leq T$, a.e., where
\[
\bar{x}(t) = \exp \left( \int_0^t f_1(t, \bar{c}(s), \bar{\zeta}(s)) \, ds \right), \quad 0 \leq t \leq T.
\]
By Fatou’s Lemma, the dominated convergence theorem and the concavity of the function $f_0$ in $u$ (see (5.11)) it follows that
\[
I[\bar{x}, \bar{u}] \geq \limsup I[\bar{x}_n, \bar{u}_n] \geq \limsup I[y_n, \bar{u}_n] = \limsup I[x_n, \bar{u}_n] = \sup I[x, u].
\]
Proof of Lemma 5.3:

By Theorem 5.1.i in [4] \{(\zeta(t), \bar{c}(t))\}_{t \in [0,T]} should solve (5.15), (5.16), (5.17) and (5.18). For sufficiency let us consider \( \lambda_0 = 1 \), and define the maximized Hamiltonian

\[
H^*(t, x, \bar{\lambda}) \triangleq \max_{v \in F_v(t)} H(t, x, v, \bar{\lambda}).
\]

Let \((x(t), u(t))\) be another admissible pair. Since the Hamiltonian is linear in \( x \), by the adjoint equation for \( \lambda_1 \) and maximum condition:

\[
H(t, \bar{x}(t), \bar{u}(t), \bar{\lambda}(t)) - H(t, x(t), u(t), \bar{\lambda}(t)) \geq H^*(t, \bar{x}(t), \bar{\lambda}(t)) - H^*(t, x, \bar{\lambda}(t))
\]

(8.10)

\[
= \lambda_1^*(t)(x(t) - \bar{x}(t)).
\]

One has

\[
I[\bar{x}, \bar{u}] - I[x, u] = \int_0^T (H(t, \bar{x}(t), \bar{u}(t), \bar{\lambda}(t)) - H(t, x(t), u(t), \bar{\lambda}(t))) dt
\]

\[
+ \int_0^T \lambda_1(t)(\dot{\bar{x}}(t) - \dot{x}(t)) dt - g(x(T)) + g(\bar{x}(T))
\]

The inequality (8.10) and the transversality condition (5.18) yield:

\[
I[\bar{x}, \bar{u}] - I[x, u] \geq \int_0^T \lambda_1^*(t)(x(t) - \bar{x}(t)) dt + \int_0^T \lambda_1(t)(\dot{x}(t) - \dot{\bar{x}}(t)) dt
\]

\[
- g(x(T)) + g(\bar{x}(T))
\]

\[
= \lambda_1(T)(x(T) - \bar{x}(T)) - g(x(T)) + g(\bar{x}(T))
\]

\[
= g'(x(T))(x(T) - \bar{x}(T)) - g(x(T)) + g(\bar{x}(T))
\]

\[
= g(x(T)) - g(\bar{x}(T)) - g(x(T)) + g(\bar{x}(T)) = 0,
\]

proving optimality of \((\bar{x}(t), \bar{u}(t))\).


