Existence of Quasiperiodic Solutions of Elliptic Equations on the Entire Space

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Abstract

We consider elliptic equations on $\mathbb{R}^{N+1}$ of the form

$$\Delta_x u + u_{yy} + g(x,u) = 0, \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}, \tag{1}$$

where $g(x,u)$ is a sufficiently regular function with $g(\cdot,0) \equiv 0$. We give sufficient conditions for the existence of solutions of (1) which are quasiperiodic in y and decaying as $|x| \to \infty$ uniformly in y. Such solutions are found using a center manifold reduction and results from the KAM theory. We discuss several classes of nonlinearities g to which our results apply.

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Chapter 1

Introduction

In this dissertation, we consider elliptic equations of the form

$$\Delta u + u_{yy} + g(x,u) = 0, \quad (x,y) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.1}$$

where $(x,y) \in \mathbb{R}^N \times \mathbb{R}$, Δ is the Laplacian in x, and $g : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth function satisfying $g(\cdot,0) \equiv 0$. We investigate solutions of (1.1) which decay to 0 as $|x| \to \infty$, uniformly in y. Our concern is the behavior of such solutions in the remaining variable y; specifically, we are interested in the existence of solutions which are quasiperiodic in y. The purpose of this dissertation is twofold. First, we build a general framework for studying solutions of (1.1) using tools from dynamical systems, such as the center manifold theorem and the Kolmogorov-Arnold-Moser (KAM) theory. Then we show how these techniques yield quasiperiodic solutions in some specific classes of equations.

Geometric properties of solutions of (1.1) have been extensively studied by many authors. Best understood are positive solutions which decay to 0 in all variables. If g satisfies suitable assumptions, involving in particular symmetry and monotonicity conditions with respect to x, then a classical result of [31] establishes reflectional symmetry of such solutions, or even the radial symmetry about some origin in $\mathbb{R}^{N+1}$ if g is independent of x (see also [11, 12, 13, 26, 44, 45] or the surveys [10, 52, 56] for related symmetry results and additional references). It is very likely, and has already been proved in some situations, that, under similar hypotheses on g, bounded positive solutions which decay as $|x| \to \infty$ uniformly in y, but do not necessarily decay in y, enjoy the symmetry in x (see [34] for results of this form). Several authors have also exposed complexities of various solutions which do not decay at infinity. Examples, with $g = g(u)$, include multi-bump solutions decaying along all but finitely many rays [46], saddle shaped solutions and general multiple-end solutions [23, 24, 41], as well as solutions having both fronts (transitions) and bumps [63].

Solutions of the form considered in the present dissertation (that is, solutions decaying in x uniformly in y) were examined by Dancer in [19]. Considering homogeneous nonlinearities $g = g(u)$ of a certain type, with special focus on the nonlinearities $g(u) = u^p - u$ with a subcritical p, he proved the existence of solutions periodic (and nonconstant) in y. With the existence of periodic solutions established, one wonders if solutions with more complicated behavior in y may occur. The existence of quasiperiodic solutions then becomes one of the most immediate compelling problems. Looking for tools to address this problem, one thinks of the KAM theory quite naturally.

Since its inception [6, 40, 50], the KAM theory has been employed by many authors in proving the existence of invariant tori filled with quasiperiodic solutions for finite dimensional Hamiltonian systems (see, for example, [16, 20] for an overview of results and techniques, or [25] for a more detailed historical account and references). Extensions of the classical KAM results to infinite dimensional Hamiltonian systems generated by partial differential equations (PDEs) have been made by several authors (see, for example, [8, 15, 18, 30, 42, 43, 71] and references therein). In a recent paper [22], de la Llave and Sire took an *a posteriori* (cp. [29]) approach to applying KAM techniques in PDEs. This approach consists in finding approximate quasiperiodic solutions, and then proving the existence of true quasiperiodic solutions nearby. The procedure does not rely on the well-posedness of the initial value problem for the equation in question and is therefore applicable to some ill-posed equations (this is illustrated by the Boussinesq equation in [22]).

Potentially, their approach could give a way to deal with problems similar to ours if the nonlinearity is analytic. We take a different route, however. We examine (1.1) by its "spatial dynamics," formally viewing it as an evolution equation with the variable "y" taking the role of time. Invoking a center manifold theorem, we find a finite-dimensional Hamiltonian system to which classical KAM results can be applied.

Spatial dynamics, as a technique to study elliptic equations with an unbounded variable, was first used by Kirchgässner [39] and developed by Mielke [47, 48, 49] and others (see, for example, [17, 28, 33, 35, 53, 54, 70]). The main idea underlying this technique is that although the equation has an ill-posed initial value problem, a large class of its solutions is often described by a finite dimensional reduction – an ordinary differential equation with a well defined flow, which can be studied using tools from dynamical systems.

An application of KAM theorems via spatial dynamics has also appeared in the literature: in [69], Valls proves the existence of quasiperiodic solutions of semilinear elliptic equations on a strip. Applying a center manifold reduction and taking the Birkhoff normal form of the Hamiltonian of the reduced equation to a sufficiently large order, she writes the reduced equation as the sum of an integrable system and a (locally) small perturbation. This puts the problem in the form suitable for the KAM theory, although, because of the lack of analyticity of the center manifold reduction, KAM results for systems with finite degree of smoothness have to be used. Semilinear elliptic equations on a strip were also considered in an earlier work of Scheurle [65]. Similarly as in his paper [64] on analytic reversible ODEs, he designs a Newton iteration scheme to find families of quasiperiodic solutions bifurcating from an equilibrium. It is noteworthy that resolvent estimates typically used in the center manifold reduction are involved in [65], although the center manifold theorem is not invoked there. Working in the analytic setting (and not losing it in a center manifold reduction), while restrictive, has the advantage of leading to a finer description of the solutions, such as the analyticity of the solution branches. We also mention related results based on a variational approach to elliptic equations. In an extension of the Aubry-Mather theory to PDEs, as developed by Moser [51] and Bangert [9] (see also [27, 59, 67] and references therein), one considers integer-periodic elliptic equations (such as equation (1.1), where g is 1-periodic in the variables $x_1, \dots, x_N$, and u) as Euler-Lagrange equations of an associated functional and shows the existence of local minimizers whose graphs are within a bounded distance from a given hyperplane and obey a certain "no self-intersection" property. The behavior of such solutions depends on the orthogonal vector to the hyperplane, or the "rotation vector." For rationally independent rotation vectors one obtains solutions with a quasiperiodicity property relative to the integer translation. Note, however, that this class of solutions is quite different from those studied in [65, 69] or in this dissertation; in particular, they are all unbounded.

On a general level, our approach to constructing quasiperiodic solutions is similar to that of [69]. However, applying these techniques to (1.1) poses significant difficulties. The first one is that in our case the "cross-section" of the domain $\mathbb{R}^N \times \mathbb{R}$ is $\mathbb{R}^N$. Thus, the Schrödinger operator appearing in the evolution formulation of (1.1), namely, the operator $-\Delta - a_1(x)$ with $a_1(x) = g_u(x, 0)$, has a nonempty essential spectrum. For the center manifold reduction to apply, we need the essential spectrum to be away from and to the right of the origin on the real axis. On the other hand, the KAM theory calls for some eigenvalues of an underlying matrix operator to lie on the imaginary axis, and this in turn requires the Schrödinger operator to have a number of negative eigenvalues. Whether such eigenvalues exist, simultaneously with the essential spectrum contained in the positive half-line, depends on the specific problem and it takes some work to verify that they do for some equations of a given structure. The unboundedness of the cross-section complicates matters in other ways as well. One is the lack of the Fourier eigenfunction expansion, which is often useful for explicit computations when the cross-section is an interval or a rectangle (cp. [29, 69, 71]).

There is also a difficulty coming from the nonlinearity itself, since we allow the expansion of the function g at $u = 0$ to involve a nontrivial quadratic term. If the quadratic nonlinear term is absent, the analysis becomes simpler when it comes to the verification of certain nondegeneracy conditions needed in the KAM-type results [69, 71]. For example, in the approach of [69], when the nonlinearity is odd—in particular, the quadratic terms are absent—neither the reduction function (from the center manifold theorem) nor the change of coordinates from the Darboux theorem (to bring the symplectic structure to the standard one) enter the expansion of the reduced Hamiltonian up to order four. Since the Kolmogorov nondegeneracy condition involves terms of order at most four, verifying it amounts to an explicit computation. Including quadratic terms in the nonlinearity complicates matters, but it is necessary for some applications of our results to problems with a specific structure (for more on this, see Remark 2.2(v) below). On the other hand, in some situations, which we explore, the presence of a quadratic term satisfying some conditions can be used for verifying the Arnold nondegeneracy condition, which also yields the existence of quasiperiodic solutions.

Our main theorems give sufficient conditions for the existence of solutions of (1.1) which are quasiperiodic in y with n frequencies, where $n > 1$ is a given integer. As usual in KAM-type results, for equations satisfying the sufficient conditions, one automatically gets uncountably many quasiperiodic solutions whose frequency vectors form a set of positive measure in $\mathbb{R}^n$. As indicated above, we are mainly interested in y-quasiperiodic solutions which decay to zero as $|x| \to \infty$, but our general results are flexible enough to deal with other types of solutions, such as solutions which decay in some of the x-variables and are periodic in the others (see Remark 2.2(iv) below). Our sufficient conditions are formulated explicitly in terms of eigenvalues and eigenfunctions of the operator $-\Delta - a_1(x)$ and the third derivative $a_3(x) := g_{uuu}(x, 0)$ of the nonlinearity. In the case $n = 2$, we also formulate a condition involving the second derivative $a_2(x) := g_{uu}(x, 0)$. It is not difficult to show that the conditions are robust: if they hold for some a_1, a_3 (or a_2), then they continue to hold if a_1, a_3 (or a_2) are perturbed slightly. However, proving that they hold for some a_1, a_2, a_3 is not always so easy and may become increasingly difficult when one starts imposing structural assumptions on equation (1.1). Naturally, the more restrictive the structure, the less freedom one has to choose the functions so that the given conditions are satisfied. We verify that the conditions do hold for some radially symmetric a_1, a_3 (and all small, possibly nonradial, perturbations thereof).

The remainder of this dissertation is organized as follows. Our main results and an informal overview of the proofs are given in Chapter 2. We also show there examples of functions satisfying our hypotheses. In Chapter 3, we apply a center manifold reduction to an abstract form of (1.1). In Chapters 4 and 5, we employ the Hamiltonian structure of the reduced equation: using a Birkhoff normal form procedure, we write the Hamiltonian in a form suitable for the KAM theory. This yields, under certain hypotheses, quasiperiodic solutions and completes the proofs of two of our main theorems. Chapter 6 is devoted to using a different nondegeneracy condition to apply the KAM theory and derive the existence of quasiperiodic solutions, allowing us to prove the last of our theorems in Chapter 7. In Appendix A, we verify some of the technical hypotheses needed for the center manifold theorem, including the smoothness of Nemytskii operators acting on Sobolev spaces on $\mathbb{R}^N$.

Remark. This version of the dissertation is slightly modified from the original submitted to the University of Minnesota to address some minor issues in Chapter 7, namely, the construction of the center manifold reduction needs to be modified to take into account the fact that the linear part of the abstract equation depends on the parameter. The author is grateful to Professor Peter Poláčik for his help in finding and addressing these issues.

Chapter 2

Main results

In this chapter, we introduce some terminology and give precise statements of our main results. We also verify our hypotheses for some equations of the form (1.1) and outline the proofs of the main theorems.

Throughout this dissertation, $\mathscr{C}_{\mathrm{b}}(\mathbb{R}^N)$ is the space of continuous bounded (real valued) functions on $\mathbb{R}^N$ and $\mathscr{C}^k_{\mathrm{b}}(\mathbb{R}^N)$ for the space of functions on $\mathbb{R}^N$ with continuous bounded derivatives up to order k, $k \in \mathbb{N} := \{0, 1, 2, \dots\}$. When needed, we assume that these spaces are equipped with the usual norms. The space $\mathscr{C}_{\mathrm{rad}}(\mathbb{R}^N)$ (resp. $\mathscr{C}^k_{\mathrm{rad}}(\mathbb{R}^N)$) is the subspace of $\mathscr{C}_{\mathrm{b}}(\mathbb{R}^N)$ (resp. $\mathscr{C}^k_{\mathrm{b}}(\mathbb{R}^N)$) consisting of radially symmetric functions around 0. In a slight abuse of notation, a function $g \in \mathscr{C}_{\mathrm{rad}}(\mathbb{R}^N)$ will be seen either as a function $g(x)$ of $x \in \mathbb{R}^N$ or as a function $g(r)$ of $r \geq 0$. We also denote by $L^p_{\mathrm{rad}}(\mathbb{R}^N)$ and $H^k_{\mathrm{rad}}(\mathbb{R}^N)$ the subspaces of $L^p(\mathbb{R}^N)$ and $H^k(\mathbb{R}^N)$, respectively, consisting of radially symmetric functions about 0. In the sequel, every radially symmetric function is assumed to be symmetric around 0.

Fix a positive integer N. The main equation we consider is

$$\Delta u + u_{yy} + a_1(x)u + f(x, u; s, b) = 0 \quad \text{for } (x, y) \in \mathbb{R}^N \times \mathbb{R} = \mathbb{R}^{N+1}, \qquad (2.1)$$

where $a_1 \in \mathscr{C}_{\mathrm{b}}(\mathbb{R}^N)$, $b \neq 0$ and $s \in \mathbb{R}$ are real parameters, and f is a sufficiently regular function on $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^2$. We will formulate regularity and other hypotheses on a_1 and f shortly. Structurally, we will assume f to have the form

$$f(x,u;s,b) = b\left(sa_2(x)u^2 + a_3(x)u^3\right) + u^4 f_1(x,u;s,b), \tag{2.2}$$

where $a_2, a_3 \in \mathscr{C}_{\mathrm{b}}(\mathbb{R}^N)$ and $f_1 : \mathbb{R}^{N+1} \times \mathbb{R}^2 \to \mathbb{R}$ are sufficiently smooth functions.

For our last result, we consider the equation

$$\Delta u + u_{yy} + a_1(r;s)u + a_2(r;s)u^2 + u^3 g(r,u;s) \quad r \geq 0,\ y \in \mathbb{R}, \tag{2.3}$$

where $s \approx 0$ is a parameter, g is a sufficiently regular function on $\mathbb{R}^N \times \mathbb{R}$, radially symmetric in the first argument, and we will formulate regularity and other hypotheses on a_1, a_2 and g in the next section.

Remark. Notice that, unlike equation (2.1), the cubic term can be missing in (2.3), in particular, one can take $g \equiv 0$.

2.1 Hypotheses

Given integers $n \geq 2$, $k \geq 1$, a vector $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ is said to be *nonresonant up to order k* if

$$\omega \cdot \alpha \neq 0 \text{ for all } \alpha \in \mathbb{Z}^n \setminus \{0\} \text{ such that } \ |\alpha| \leq k. \tag{2.4}$$

(Here $|\alpha| = |\alpha_1| + \cdots + |\alpha_n|$, and $\omega \cdot \alpha$ is the usual dot product.) If (2.4) holds for all $k = 1, 2, \dots$, we say that ω is *nonresonant*, or, equivalently, that the numbers $\omega_1, \dots, \omega_n$ are *rationally independent.* A special class of nonresonant vectors which will play a role later on is the class of *Diophantine* vectors, see Chapter 5.

Assuming $a_1 \in \mathscr{C}_{\mathrm{b}}(\mathbb{R}^N)$, consider the Schrödinger operator $A_1 = -\Delta - a_1(x)$, viewed as an unbounded self-adjoint operator on $L^2(\mathbb{R}^N)$ with domain $D(A_1) = H^2(\mathbb{R}^N)$. Fixing an integer $n \geq 2$, we make the following assumptions on a_1:

(A1)(a) $L := \limsup_{|x| \to \infty} a_1(x) < 0.$

(A1)(b) A_1 has exactly n negative eigenvalues $\mu_1 < \cdots < \mu_n$, all of which are simple, and 0 is not an eigenvalue of A_1.

Sometimes, we collectively refer to assumptions (A1)(a) and (A1)(b) as (A1).

Remark 2.1. If one is specifically interested in problems with radial symmetry; that is, when the functions a_1, f, and the sought-after solutions are required to be radially symmetric in x, then one can adapt this hypothesis to the new situation: rather than considering the Schrödinger operator $A_1 = -\Delta - a_1$ on the full space $L^2(\mathbb{R}^N)$, one can take its restriction to the subspace $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ (the domain of A_1 is then $H^2_{\mathrm{rad}}(\mathbb{R}^N)$). This implies that the eigenvalues are automatically simple, which is not guaranteed when A_1 is considered in the full space.

In our next hypotheses, K and m are integers satisfying

$$K \geq 6(n+1), \quad m > \frac{N}{2}. \tag{2.5}$$

We assume the following smoothness and nonresonance conditions on a_1:

(S1) $a_1 \in \mathscr{C}^{m+1}_{\mathrm{b}}(\mathbb{R}^N)$.

(NR) Denoting $\omega_j := \sqrt{|\mu_j|}$, $j = 1, \dots, n$, the vector $\omega = (\omega_1, \dots, \omega_n)$ is nonresonant up to order K.

Our smoothness requirement on the functions in (2.2) are as follows:

(S2) $a_2,\ a_3 \in \mathscr{C}^{m+1}_{\mathrm{b}}(\mathbb{R}^N)$; $f_1 \in \mathscr{C}^{K+m+4}(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^2)$ and for all $\vartheta > 0$, $\rho_0 > 0$, the function f_1 is bounded on $\mathbb{R}^N \times [-\vartheta, \vartheta] \times [-\rho_0, \rho_0]^2$ together with all its partial derivatives up to order $K + m + 4$.

Hypotheses (A1), (NR), (S1), (S2) are our standing hypotheses throughout Chapters 3 to 6. In addition, we will assume one of the following two hypotheses. The first one, (A2), involves the function a_3 from (2.2) and eigenfunctions of A_1; thus, in effect, it is a hypothesis on f and a_1. The other hypothesis, (A3), concerns a_1 only.

Let $\varphi_1, \dots, \varphi_n$ be eigenfunctions of A_1 corresponding to the eigenvalues μ_1, …, μ_n, respectively, normalized in the L^2-norm (they are determined uniquely up to signs).

(A2) The $n \times n$ matrix M_1 with entries

$$(M_1)_{ij} = (2 - \delta_{ij}) \int_{\mathbb{R}^N} a_3(x)\varphi_i^2(x)\varphi_j^2(x)\ dx \quad (i, j = 1, \ldots, n),$$

where δ_{ij} is the Kronecker delta, is nonsingular.

(A3) The eigenfunctions $\varphi_1, \ldots, \varphi_n$ have the following *quartic independence* property: the set of functions $\{\varphi_i^2\varphi_j^2 : 1 \leq i \leq j \leq n\}$ is linearly independent in some nonempty open subset $U \subset \mathbb{R}^N$, that is, the coefficients of any linear combination of these functions which vanishes identically in U are necessarily equal to 0.

We make some comments on the hypotheses made here.

Remark 2.2. (i) The sole role of hypothesis (A1)(a) is to guarantee that the essential spectrum $\sigma_{ess}(A_1)$ of the operator A_1 is contained in $(-L, \infty)$ [60]. The condition $\sigma_{ess}(A_1) \subset (-L, \infty)$, or any explicit condition which implies this inclusion, can safely be used as a hypothesis in place of (A1)(a). Note that, since $\sigma(A_1)\backslash\sigma_{ess}(A_1)$ consists of isolated eigenvalues, conditions (A1)(a), (A1)(b) imply in particular that there is $\gamma > 0$ such that $\sigma(A_1) \cap (-\gamma, \gamma) = \emptyset$. Also, it is well known that, as eigenfunctions corresponding to isolated simple eigenvalues, the functions $\varphi_j(x)$ have exponential decay as $|x| \to \infty$ [3, 4, 58]. In particular, the integrals in (A2) exist.

(ii) The regularity of f is needed mainly for two reasons. An application of the KAM theory forces us to a assume a sufficiently high smoothness of $f(x, u)$ with respect to u. The smoothness of a_1 and f with respect to x has more to do with our choice to set up a formulation of (2.1) in the spaces $H^m(\mathbb{R}^N)$ with a large enough m, rather than in the spaces $W^{2,p}(\mathbb{R}^N)$ with a sufficiently large p. Working in a Hilbert space setting simplifies some considerations, at the expense of the regularity requirements.

(iii) In our main results, Theorems 2.4 and 2.6 below, the smoothness of the function f_1 with respect to the parameters s, b is not relevant, only what happens at the quadratic and cubic terms of f is important (see Remark 4.12 for an explanation of this). However, in other theorems, such as the reduction to the center manifold and the Darboux change of coordinates, it is of interest to know how the smoothness of f with respect to the parameters reflects in the conclusions of those theorems.

(iv) The formulation of our hypotheses reflects our main objective to find y-quasiperiodic solutions which decay to zero as $|x| \to \infty$. To search for other types of y-quasiperiodic solutions, one would need to modify the hypotheses suitably. Suppose, for example, that $a_1(x)$ and $f(x,u)$ are even and periodic in x_N with period $2p > 0$, and one wants to find y-quasiperiodic solutions which decay in $x' = (x_1, \ldots, x_{N-1})$ and are even and $2p$-periodic in x_N. The operator $-\Delta - a_1$ is then to be considered as a self-adjoint operator, with natural domain, on the space of functions on $\mathbb{R}^N$ which are even and $2p$-periodic in x_N and whose restrictions to $\mathbb{R}^{N-1} \times (-p,p)$ are in $L^2(\mathbb{R}^{N-1} \times (-p,p))$. Hypothesis (A1)(a) has to be replaced by the condition $\sigma_{ess}(A_1) \subset (-L, \infty)$ (or an explicit sufficient condition), and the integrals in (A2) are taken over $\mathbb{R}^{N-1} \times (-p,p)$, rather than over $\mathbb{R}^N$. The remaining hypotheses can be kept intact. The evenness requirement can be dropped in this example, although in some specific situations the simplicity of the eigenvalues, as required in (A1)(b), may not be satisfied without it.

(v) Note that if (A1) is to be satisfied, a_1 cannot be a constant function. This is consequential for applications of our results to some specific equations, such as spatially homogeneous equations (1.1). Indeed, if $g = g(u)$ in (1.1) or, more generally, if the derivative $g_u(x,0)$ is constant, then in (2.1), (2.2) one cannot simply take the coefficients a_j from the Taylor expansion of g at the trivial solution. Instead, the Taylor expansion has to be taken at a nontrivial solution $\varphi = \varphi(x)$. Such an expansion will typically involve quadratic terms in u, regardless of any assumptions on the derivatives of g at 0. Mainly for this reason we insist on including the quadratic term in (2.2).

When dealing with equation (2.3), we will modify our hypotheses slightly. For

$\delta > 0$ sufficiently small and $s \in [0, \delta]$, we now consider the Schrödinger operator $A_1(s) := -\Delta - a_1(r; s)$ acting on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ with domain $H^2_{\mathrm{rad}}(\mathbb{R}^N)$, and assume the following hypotheses:

(A1')(a) There exists $L < 0$ such that $\limsup_{r\to\infty} a_1(r; s) \le L$ for all $s \in [0, \delta]$.

(A1')(b) For all $s \in (0, \delta]$, $A_1(s)$ has exactly two negative eigenvalues $\mu_1(s) < \mu_2(s)$, and 0 is not an eigenvalue of $A_1(s)$. For $s = 0$, $A_1(0)$ has exactly one negative eigenvalue $\mu_1(0)$, and $\mu_2(0) = 0$ is an eigenvalue of $A_1(0)$.

(S1') $a_1(\cdot; s) \in \mathscr{C}^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ for each s, and the map $s \in [0, \delta] \mapsto a_1(\cdot; s) \in \mathscr{C}^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ is of class $\mathscr{C}^K$ (with K as in (2.5)).

(S2') $a_2(\cdot; s) \in \mathscr{C}^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ for each $s \in [0, \delta]$, the map $s \in [0, \delta] \mapsto a_2(\cdot; s) \in \mathscr{C}^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$ is of class $\mathscr{C}^K$, $g \in \mathscr{C}^{K+m+4}(\mathbb{R}^N \times \mathbb{R} \times [0, \delta])$, and for all $\vartheta > 0$, the function g is bounded on $\mathbb{R}^N \times [-\vartheta, \vartheta] \times [0, \delta]$ together with all its partial derivatives up to order $K + m + 4$. Also, $g = g(x, u; s)$ is radially symmetric in $x \in \mathbb{R}^N$.

(A4) Denoting $\varphi_j(\cdot; s)$, $j = 1, 2$, the eigenfunction of $A_1(s)$ associated to $\mu_j(s)$, normalized in the L^2-norm, and satisfying $\varphi_j(0; s) > 0$, one has

$$\int_{\mathbb{R}^N} a_2(x; 0)\varphi_2^3(x; 0)dx \neq 0.$$

(NR') Denoting $\omega_j(s) := \sqrt{|\mu_j(s)|}$, $j = 1, 2$, the vector $\omega(s) = (\omega_1(s), \omega_2(s))$ is nonresonant up to order K for all $s \in (0, \delta]$.

These hypotheses will be assumed to hold throughout Chapter 7.

Remark 2.3. (i) Since the operator $A_1(s)$ is restricted to radially symmetric functions, its eigenvalues are automatically simple. Moreover, hypothesis (S1') implies that the eigenvalues $\mu_1(s)$ and $\mu_2(s)$ of $A_1(s)$ depend continuously on s, see, e.g., [38].

(ii) As long as the dependence of a_2 and g on s is sufficiently regular (that is, of class $\mathscr{C}^K$), no further information on how these functions depend on s is required; in particular, one can consider the case when both functions are independent of s.

(iii) Note that our framework includes the case $g \equiv 0$; in other words, the nondegeneracy condition required to apply the KAM theory can be derived from the quadratic terms of (2.3).

2.2 Existence of quasiperiodic solutions

A function $u : (x, y) \mapsto u(x, y) : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is said to be *quasiperiodic* in y if there exist an integer $n \geq 2$, a nonresonant vector $\omega^* = (\omega_1^*, \dots, \omega_n^*) \in \mathbb{R}^n$, and an injective function U defined on $\mathbb{T}^n$ (the n-dimensional torus) with values in the space of real-valued functions on $\mathbb{R}^N$ such that

$$u(x, y) = U(\omega_1^* y, \dots, \omega_n^* y)(x). \quad (x \in \mathbb{R}^N,\ y \in \mathbb{R}). \tag{2.6}$$

The vector ω^* is called a *frequency vector* of u.

We emphasize that the nonresonance of the frequency vector is a part of our definition. In particular, a quasiperiodic function is not periodic and, if it has some regularity properties, its image is dense in an n-dimensional manifold diffeomorphic to $\mathbb{T}^n$.

In our first theorem, we consider one of the following two settings:

(a) $b \in \mathbb{R} \setminus \{0\}$ is fixed and $|s| \geq 0$ is sufficiently small,

(b) $s \in \mathbb{R}$ is fixed and $|b| > 0$ is sufficiently small.

We refer to the above assumptions on the smallness of one of the parameters (with the other parameter fixed) as Case (a) and Case (b). It is understood here that how small a parameter has to be depends on the other parameter (and the other given data: the functions a_1 and f).

Theorem 2.4. *Suppose that hypotheses* (A1)*,* (NR)*,* (S1)*,* (S2) *(with K, m as in* (2.5)*), and* (A2) *are satisfied. In both Cases* (a) *and* (b)*, the following conclusion holds. There exists a solution $u(x,y)$ of equation* (2.1) *(with f as in* (2.2)*) such that $u(x,y) \to 0$ as $|x| \to \infty$ uniformly in y, and $u(x,y)$ is quasiperiodic in y. In fact, there is an uncountable family of such quasiperiodic solutions, their frequency vectors forming a set of positive measure in $\mathbb{R}^n$ (n is as in* (A1)(b)*).*

In Case (b), Theorem 2.4 is a perturbative result, where the quadratic and cubic terms in f become small at the same rate, as $b \to 0$. Case (a) is partly a perturbative result as well, requiring the quadratic term to be small relative to the cubic term. Note, however, that $s = 0$ with any fixed $b > 0$ is allowed in Case (a). Thus, in the class of functions with no quadratic term, in particular, in the class of functions which are odd in u, there is no smallness requirement and Theorem 2.4 is not a perturbative result.

Remark 2.5. The statement of Theorem 2.4 can be strengthened as follows. For an arbitrary $\rho_0 > 0$, if $b \in [-\rho_0, \rho_0] \setminus \{0\}$ is fixed, then the conclusion of Theorem 2.4 holds for all $s \in \{0\} \cup ([-\rho_0, \rho_0] \setminus D_1)$, where $D_1 \subset \mathbb{R}$ is a finite set; if $s \in [-\rho_0, \rho_0] \setminus \{0\}$ is fixed, then the conclusion of Theorem 2.4 holds for all $b \in [-\rho_0, \rho_0] \setminus D_2$ where $D_2 \subset \mathbb{R}$ is a finite set containing 0. This is explained in detail in Remark 5.5 and Lemma 5.2, where we also give a general sufficient condition for the validity of the conclusion of Theorem 2.4. The condition is formulated in terms of the functions a_2, a_3, but it is rather implicit and hard to verify for specific choices of these functions (with the parameters s and b fixed), unless $a_2 = 0$. On the other hand, Remark 5.5 shows that the condition is satisfied for all s, save for isolated values (with $b \neq 0$ fixed), if it is satisfied for some s; and, likewise, it is satisfied for all b, save for isolated values, if it is satisfied for some b (with s fixed).

In our next theorem, both parameters $s \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$ are fixed and neither is required to be small.

Theorem 2.6. *Let a_2 and f_1 be as in* (S2)*, and a_1 as in* (S1)*, where K, m are constants satisfying* (2.5)*. Suppose that conditions* (A1)*,* (NR)*, and* (A3) *are satisfied and let $s \in \mathbb{R}$, $b \in \mathbb{R} \setminus \{0\}$ be arbitrary. Then there is an open and dense set $\mathcal{B}$ in $\mathscr{C}_{\mathrm{b}}^{m+1}(\mathbb{R}^N)$ such that the conclusion of Theorem 2.4 holds for each $a_3 \in \mathcal{B}$.*

We remark that, although it is easy to show that if a_1 satisfies (A3), then the set of functions a_3 satisfying (A2) is open and dense, Theorem 2.6 does not follow from Theorem 2.4. Indeed, Theorem 2.4 states that (A2) is a sufficient condition for the validity of the conclusion if one of the parameters s, b is small, which is not assumed in Theorem 2.6.

Remark 2.7. If the functions a_1, a_2 are radial, Theorem 2.6 remains valid if the space $\mathscr{C}_{\mathrm{b}}^{m+1}(\mathbb{R}^N)$ is replaced by $\mathscr{C}_{\mathrm{rad}}^{m+1}(\mathbb{R}^N)$ (cp. Remark 5.6 below).

Our last theorem concerns equation (2.3).

Theorem 2.8. *Suppose that hypotheses* (A1')*,* (S1')*,* (S2')*,* (NR') *(with K, m as in* (2.5) *and $n = 2$) and* (A4) *are satisfied. If $\delta > 0$ is sufficiently small, then for each $s \in (0, \delta]$ the following holds. There exists a solution $u(x, y)$ of equation* (2.3) *such that $u(x, y)$ is radially symmetric in x, $u(x, y) \to 0$ as $|x| \to \infty$ uniformly in y, and $u(x, y)$ is quasiperiodic in y. In fact, there is an uncountable family of such quasiperiodic solutions, their frequency vectors forming an uncountable subset of $\mathbb{R}^2$.*

We remark that Theorem 6.3, below, contains a more general sufficient condition for the existence of quasiperiodic solutions of (2.3), which allows for quasiperiodic solutions with any $n > 1$ number of frequencies. Nevertheless, this condition is quite difficult to verify for a specific choice of a_1, a_2 and g, even if $g \equiv 0$.

2.3 Validity of the hypotheses

In this section, we give examples of functions a_1, a_3 which satisfy the hypotheses of Theorems 2.4 and 2.6. First of all, we show the robustness of the hypotheses.

Proposition 2.9. *Let $k \geq 0$ be an integer.*

(i) *The set of all functions $(a_1, a_3) \in \mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N) \times \mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N)$ such that conditions* (A1)*,* (NR)*, and* (A2) *are satisfied is open in $\mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N) \times \mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N)$.*

(ii) *The set of all functions $a_1 \in \mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N)$ such that conditions* (A1)*,* (NR) *are satisfied is open in $\mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N)$, and so is the set set of all functions $a_1 \in \mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N)$ such that all three conditions* (A1)*,* (NR)*, and* (A3) *are satisfied.*

Proof. The results are consequences of standard perturbation results [38]. Suppose first that (A1), (NR), are satisfied for some $a_1 \in \mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N)$. The upper semicontinuity of the spectrum, and the continuity of simple eigenvalues imply that (A1)(b), (NR) remain valid if a_1 is perturbed slightly in $\mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N)$. The same is obviously true of (A1)(a). The simplicity of the eigenvalues implies that the normalized eigenfunctions $\varphi_1, \dots, \varphi_n$ can be chosen such that they depend continuously on a_1 (in a small neighborhood of the unperturbed function) as $H^2(\mathbb{R}^N)$-valued functions. Standard elliptic regularity estimates allow us to bootstrap this continuity to eventually show that $\varphi_1, \dots, \varphi_n$ depend continuously on a_1 as $W^{2,p}(\mathbb{R}^N)$-valued functions for any $p \in (1, \infty)$, and, in particular, as $L^4(\mathbb{R}^N)$-valued functions. This implies that if now $a_3 \in \mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N)$ is such that (A2) holds, then (A2) will continue to hold if a_1 and a_3 are perturbed slightly in $\mathscr{C}_{\mathrm{b}}^k(\mathbb{R}^N)$. Statement (i) is thus proved.

For statement (ii), we just need to observe, in addition, that the linear independence of the functions $\varphi_i^2 \varphi_j^2$, $1 \leq i \leq j \leq n$, is preserved because of the continuous dependence of $\varphi_1, \dots, \varphi_n$ on a_1 (in a small neighborhood of the unperturbed function a_1) as $L^p(\mathbb{R}^N)$-valued functions for any $p \in (1, \infty)$: a simple way to see this is by considering a suitable Gram matrix of the functions $\varphi_i^2 \varphi_j^2$. □

To find examples of functions a_1, a_3 satisfying our hypotheses, we start with the following statement concerning hypothesis (A1).

Proposition 2.10. *There exists a radially symmetric function $a_1 \in \mathscr{C}_{\mathrm{b}}^\infty(\mathbb{R}^N)$ such that* (A1) *holds.*

Proof. If $N = 1$, take $c \geq 0$ and consider an even function $a_1 \in \mathscr{C}^\infty(\mathbb{R})$ such that $a_1(x) \equiv -1$ for $|x| > 2$, $a_1 \equiv c \in \mathbb{R}$ for $|x| < 3/2$, and the rest of the values of a_1 are between c and -1. If c is sufficiently large, then the operator $-\Delta - a_1(x)$ has at least n negative eigenvalues. All these eigenvalues are automatically simple. If $c = 0$, then $a_1 \leq 0$ and $-\Delta - a_1(x)$ has no eigenvalues in $(-\infty, 0]$. Consequently, for suitable intermediate values of c, $-\Delta - a_1(x)$ has exactly n negative eigenvalues and 0 is not an eigenvalue.

Let now $N \geq 2$. A similar continuity argument as above yields a radial potential such that (A1) holds for the restriction of the operator $A_1 = -\Delta - a_1(x)$ to $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ (cp. Remark 2.2(iv)), but not necessarily in the full space $L^2(\mathbb{R}^N)$. To show that (A1) holds without the restriction to $L^2_{\mathrm{rad}}(\mathbb{R}^N)$, one has to make sure that A_1, in addition to having n negative eigenvalues with radial eigenfunctions, has no negative eigenvalue with a nonradial eigenfunction (such an eigenvalue is never simple for a radial potential). This has been done in [55]. More precisely, Lemmas 2.2 and 2.3 of [55] show that there is a smooth radial function $a_1(x)$, identical to -1 outside a sufficiently large ball, with the following property. The operator A_1 has at least n negative eigenvalues with radially symmetric eigenfunctions (all these eigenvalues are simple) and, at the same time, 0 is the minimal eigenvalue having a nonradial eigenfunction. We now replace a_1 by $a_1 - d$, where d is a positive constant. This has the effect of shifting the spectrum $\sigma(A_1)$ to $\sigma(A_1) + d$. Obviously, choosing d suitably, we achieve that exactly n eigenvalues remain in $(-\infty, 0)$, while all the other eigenvalues are contained in $(0, \infty)$. The resulting operator then has all the desired properties. □

Next, we deal with the nonresonance condition.

Lemma 2.11. *For any integer $K > 1$ and any set of negative numbers $\mu_1 < \cdots < \mu_n$, the set of all $\epsilon > 0$ such that the vector $\omega(\epsilon) = (\sqrt{|\mu_1| + \epsilon}, \ldots, \sqrt{|\mu_n| + \epsilon})$ is nonresonant up to order K is open and dense in $(0, \infty)$. Consequently, the set of all $\epsilon > 0$ such that the vector $\omega(\epsilon) := (\sqrt{|\mu_1| + \epsilon}, \ldots, \sqrt{|\mu_n| + \epsilon})$ is nonresonant is residual, hence dense, in $(0, \infty)$.*

Proof. Obviously, it is sufficient to prove that for any fixed $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n \setminus \{0\}$, the function $\epsilon \mapsto \omega(\epsilon) \cdot \alpha$ has only isolated zeros. This follows, since the function is analytic in $[0, \infty)$, if we prove that it has a nonzero derivative of some order at $\epsilon = 0$. Suppose that, to the contrary, all the derivatives at $\epsilon = 0$ vanish. This implies that for all odd positive integers ℓ one has

$$\frac{\alpha_1}{|\mu_1|^{\frac{\ell}{2}}} + \cdots + \frac{\alpha_n}{|\mu_n|^{\frac{\ell}{2}}} = 0.$$

Since the $|\mu_j|$ are mutually distinct, we conclude from this that $\alpha = 0$, a contradiction. □

Corollary 2.12. *Let a_1 be as in Proposition 2.10. Then there is $\epsilon > 0$ such that after replacing a_1 by $a_1 + \epsilon$, hypothesis* (A1) *is satisfied and the vector* $(\sqrt{|\mu_1|}, \dots, \sqrt{|\mu_n|})$ *is nonresonant. In particular,* (NR) *holds for any K.*

Proof. When a_1 is replaced by $a_1 + \epsilon$, the eigenvalues $\mu_1, \dots, \mu_n$ of $-\Delta - a_1$ get replaced by $\mu_1 - \epsilon, \dots, \mu_n - \epsilon$. The result now follows from Lemma 2.11 (we choose ϵ sufficient small, so that (A1) remains valid after the replacement). □

We can now easily give examples of functions a_1, a_3 satisfying hypotheses (A1), (NR), (A2).

Example 2.13. Proposition 2.10 and Corollary 2.12 yield a smooth radial function a_1 satisfying (A1) and (NR) (for any K). Let a_3 be a smooth bounded function on $\mathbb{R}^N$ which is sufficiently close, as a distribution, to δ_z (Dirac delta), where $z \in \mathbb{R}^N$ is not a zero of any of the eigenfunctions φ_j, $j = 1, \dots, n$ (the set of such z is open and dense in $\mathbb{R}^N$). Then (A2) holds. Alternatively, one can take a smooth radial function a_3 sufficiently close to the "radial δ-function" δ_ρ, where $\rho > 0$ is not a zero of any of the eigenfunctions φ_j, $j = 1, \dots, n$, viewed as functions of $r = |x|$ (in this view, the zeros of the eigenfunctions are isolated).

To justify these statements, note that for $a_3 \approx \delta_z$, the matrix M_1 in (A2) is close to the matrix with entries

$$(2 - \delta_{ij})\varphi_i^2(z)\varphi_j^2(z) \quad (i, j = 1, \dots, n).$$

It is sufficient to show that this matrix has nonzero determinant. This follows, since $\varphi_i^2(z) \neq 0$ for $i = 1, \ldots, n$, from the fact that the matrix whose diagonal entries are all equal to 1 and the off-diagonal entries are all equal to 2 is nonsingular. (One can verify this by replacing the first row by the sum of all the rows and then carrying out an elimination.) The radial case can be dealt with similarly.

Finally, we include hypothesis (A3) into consideration.

Proposition 2.14. *For any positive integer K, there exists a radially symmetric function $a_1 \in \mathscr{C}^\infty_{\mathrm{rad}}(\mathbb{R}^N)$ such that hypotheses* (A1)*,* (NR)*, and* (A3) *are satisfied.*

Proof. Without loss of generality, we may assume that $K \geq 8$. Fix any such K.

As in Example 2.13, we first use Proposition 2.10 and Corollary 2.12 to find a smooth radial function a_1 satisfying (A1) and (NR). By (A1)(b), a_1 has to be positive somewhere, hence, by (A2)(a), a_1 vanishes somewhere. Thus, there is R_0 such that $a_1(x) = 0$ for $|x| = R_0$. We now introduce a radial perturbation of a_1, modifying it near $\{x : |x| = R_0\}$ only, such that the perturbed function vanishes identically in $\{x : R_1 < |x| < R_2\}$ for some $R_1 < R_2$ near R_0. This can be done in such a way that the perturbation is small, as small as one wishes in the supremum norm, but the perturbed function is smooth. By Proposition 2.9(ii), (A1) and (NR) are unaffected by small perturbations.

Thus, we may proceed by assuming that a_1 is a smooth radial function such that $a_1 \equiv 0$ on $\{x : R_1 < |x| < R_2\}$, for some $R_2 > R_1 > 0$, and (A1), (NR) hold. We show that in this situation (A3) is satisfied without any further perturbations of a_1.

Assume first that $N \geq 2$. For $j = 1, \ldots, n$, the eigenfunction φ_j satisfies

$$\Delta\varphi_j + a_1(x)\varphi_j + \mu_j\varphi_j = 0 \quad \text{in } \mathbb{R}^N. \tag{2.7}$$

In the radial variable $r = |x|$, this equation reads as follows:

$$\varphi_j'' + \frac{N-1}{r}\varphi_j' + (a_1(r) + \mu_j)\varphi_j = 0, \quad r > 0$$

Here $\varphi_j' = d\varphi_j/dr$, and we are abusing the notation slightly by writing $a_1 = a_1(r)$, $\varphi_j = \varphi_j(r)$ (and viewing them as functions of $r \geq 0$). On the interval (R_1, R_2) the equation simplifies, due to $a_1 \equiv 0$:

$$\varphi_j'' + \frac{N-1}{r}\varphi_j' + \mu_j \varphi_j = 0. \tag{2.8}$$

Since $\mu_j < 0$, the general solution of this equation, and therefore also the solution φ_j on (R_1, R_2), can be expressed in terms of modified Bessel functions rescaled by $\omega_j := \sqrt{|\mu_j|}$. More specifically, for some constants C_{j1}, C_{j2} one has $\varphi_j \equiv \tilde{\varphi}_j$ on (R_1, R_2), where

$$\tilde{\varphi}_j(r) := C_{j1} r^{1-N/2} I_{N/2-1}(\omega_j r) + C_{j2} r^{1-N/2} K_{N/2-1}(\omega_j r). \tag{2.9}$$

Here $I_{N/2-1}$ and $K_{N/2-1}$ are modified Bessel functions of the first and second kind, respectively. Note that these functions are defined for all $r \in (0, \infty)$ and are analytic in this interval (of course, the eigenfunctions φ_j themselves may not be analytic outside (R_1, R_2)). The constants C_{j1}, C_{j2} cannot be both equal to zero: otherwise, $\varphi_j \equiv 0$ on $[R_1, R_2]$, hence φ_j, as a solution of a second order equation, vanishes identically on $[0, \infty)$, which is impossible for an eigenfunction.

We now recall the asymptotics of the modified Bessel functions as $r \to \infty$. For $j = 1, \ldots, n$, we have:

$$\begin{aligned} I_{N/2-1}(\omega_j r) &= C_j e^{\omega_j r} r^{-1/2}(1 + \mathcal{O}(1/r)), \\ K_{N/2-1}(\omega_j r) &= C_j e^{-\omega_j r} r^{-1/2}(1 + \mathcal{O}(1/r)), \end{aligned} \tag{2.10}$$

with some nonzero constants C_j.

For $1 \leq j \leq \ell \leq n$ (we call such indices j, ℓ admissible), define

$$b(j, \ell) = \begin{cases} 2\omega_j + 2\omega_\ell & \text{if } C_{j1} \neq 0,\ C_{\ell 1} \neq 0, \\ -2\omega_j + 2\omega_\ell & \text{if } C_{j1} = 0,\ C_{\ell 1} \neq 0, \\ 2\omega_j - 2\omega_\ell & \text{if } C_{j1} \neq 0,\ C_{\ell 1} = 0, \\ -2\omega_j - 2\omega_\ell & \text{if } C_{j1} = 0,\ C_{\ell 1} = 0. \end{cases}$$

Note that, as $r \to \infty$, we have, by (2.9), (2.10),

$$\tilde{\varphi}_j^2(r)\tilde{\varphi}_\ell^2(r) \sim r^{2-2N} e^{b(j,\ell)r}. \tag{2.11}$$

Since $(\omega_1, \ldots, \omega_n)$ is nonresonant up to order 8, it follows that $b(j,\ell) \neq b(j',\ell')$ for all admissible $(j,\ell) \neq (j',\ell')$. We can thus arrange all the admissible indices in a finite sequence $(j(k), \ell(k))$, $k = 1, \ldots, n(n+1)/2$, such that $b(j(k), \ell(k)) > b(j(k'), \ell(k'))$ if $k < k'$.

We now conclude the proof of the proposition by showing that, on (R_1, R_2), the functions $\varphi_j^2\varphi_\ell^2 \equiv \tilde{\varphi}_j^2\tilde{\varphi}_\ell^2$, $1 \leq j \leq \ell \leq n$, are linearly independent. For that aim, let $c_{j\ell}$, $1 \leq j \leq \ell \leq n$, be constants such that

$$\sum_{\ell=1}^{n}\sum_{j=1}^{\ell} c_{j\ell}\tilde{\varphi}_j^2(r)\tilde{\varphi}_\ell^2(r) = 0 \tag{2.12}$$

for all $r \in (R_1, R_2)$. By the analyticity of $\tilde{\varphi}_j$, (2.12) then holds for all $r > 0$. We rewrite (2.12) as

$$\sum_{k=1}^{n(n+1)/2} c_{j(k)\ell(k)}\tilde{\varphi}_{j(k)}^2(r)\tilde{\varphi}_{\ell(k)}^2(r) = 0, \tag{2.13}$$

where $j(k)$ and $\ell(k)$ are as above. Dividing this identity by $r^{2-2N}e^{b(j(1),\ell(1))r}$, we obtain

$$\sum_{k=1}^{n(n+1)/2} c_{j(k)\ell(k)}\frac{\tilde{\varphi}_{j(k)}^2(r)\tilde{\varphi}_{\ell(k)}^2(r)}{r^{2-2N}e^{b(j(1),\ell(1))r}} = 0. \tag{2.14}$$

Since $b(j(1), \ell(1)) > b(j(k), \ell(k))$ for all $k \in \{2, \ldots, n(n+1)/2\}$, using (2.11) we obtain

$$\lim_{r\to\infty}\frac{\tilde{\varphi}_{j(k)}^2(r)\tilde{\varphi}_{\ell(k)}^2(r)}{r^{2-2N}e^{b(j(1),\ell(1))r}} \begin{cases} = 0 & \text{for } k \in \{2, \ldots, n(n+1)/2\}, \\ \neq 0 & \text{for } k = 1. \end{cases}$$

Thus, taking $r \to \infty$ in (2.14), we deduce that $c_{j(1),\ell(1)} = 0$. We then successively divide by $r^{2-2N}e^{b(j(k),\ell(k))r}$, $k = 2, \ldots, n(n+1)/2$, and take $r \to \infty$ to conclude that $c_{j(k),\ell(k)} = 0$ for $k = 1, \ldots, n(n+1)/2$. Hence, all the coefficients in (2.12) must vanish, which proves the desired linear independence.

The case $N = 1$ can be treated similarly. This time, for $r \in (R_1, R_2)$ the eigenfunctions φ_j, $j = 1, \ldots, n$, satisfy

$$\varphi_j'' + \mu_j \varphi_j = 0.$$

Letting again $\omega_j = \sqrt{|\mu_j|} \neq 0$, it follows that, on (R_1, R_2), one has $\varphi_j \equiv \tilde{\varphi}_j$, where

$$\tilde{\varphi}_j(r) = C_{j1} e^{\omega_j r} + C_{j2} e^{-\omega_j r}$$

with C_{j1}, C_{j2} not both equal to 0. Using an argument based on the analyticity, very similar to the one used above, our assertion follows. □

Remark. The results in this section can be easily adapted to address the robustness of hypotheses (A1'), (NR'), (S1'), (S2'), and (A4). For (A1'), let a_1 be the function from Proposition 2.10, with $n = 2$ negative eigenvalues. Replacing a_1 by $a_1 - (d - s)$, with d a suitable positive constant, yields a radially symmetric function satisfying (A1'). Also, given K as in (2.5), if $\delta > 0$ is sufficiently small, then, using that the eigenvalues of $A_1(s)$ are isolated, there exists a constant $C > 0$ such that $\mu_1(s) < C < 0$ and $C > 1/(K\mu_2(s))$ for all $s \in [0, \delta]$. Hypothesis (NR') easily follows from this fact.

2.4 An outline of the proofs of the main theorems

In the first step of the proof of Theorems 2.4 and 2.6, we write (2.1) as a system

$$\begin{cases} \dfrac{du_1}{dy} = u_2, \\ \dfrac{du_2}{dy} = A_1 u_1 - \tilde{f}(u_1). \end{cases} \tag{2.15}$$

Here, for any fixed (s, b), $\tilde{f}(u)(x) = f(x, u(x); s, b)$ is the Nemytskii operator associated to f, and A_1 is the Schrödinger operator $-\Delta - a_1(x)$; they are considered on suitable Hilbert spaces. Under our hypotheses, the linear operator $A(u_1, u_2) =$

$(u_2, A_1 u_1)$ has n pairs of complex conjugate eigenvalues on the imaginary axis, and the rest of it spectrum does not intersect the strip $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| < \gamma\}$, where $\gamma > 0$. Applying a center manifold theorem, we obtain a system of $2n$ ordinary differential equations (the "reduced equation"):

$$\begin{cases} \dot{\xi} = h_1(\xi, \eta), \\ \dot{\eta} = h_2(\xi, \eta), \end{cases} \tag{2.16}$$

whose solutions are in one-to-one correspondence with a class of solutions of (2.15). Our goal is to find quasiperiodic solutions of the reduced equation near the origin (which is an equilibrium of (2.15)).

The second step is to write the reduced equation as a Hamiltonian system in $\mathbb{R}^{2n}$ with respect to a suitable symplectic form. The Darboux theorem then allows us to choose local coordinates in which the system is Hamiltonian with respect to the standard symplectic structure on $\mathbb{R}^{2n}$. It is well known by abstract results [48] that all this can be done; but it is important for us to have the Hamiltonian of the transformed system in as explicit a form as possible, at least up to the fourth-order terms in its Taylor expansion. We rely here on known procedures to compute the expansion for the center manifold, from which we obtain the expansion for the first symplectic form and, subsequently, for the Darboux transformation.

In the third step, we write the Hamiltonian as the sum of an integrable Hamiltonian H^0 and a perturbation H^1, which is small in a class of finitely differentiable functions. This is achieved by bringing the Hamiltonian to its Birkhoff normal form to a sufficiently high order; the Birkhoff normal form provides the integrable part, thanks to the nonresonance condition (NR). In the perturbation H^1, we include terms of high order of vanishing in (ξ, η). Again, it is important to have some understanding of the second and fourth order terms in the expansion of H^0 (the third order terms all vanish in the normal form), and, specifically, how the functions a_1, a_2, a_3 from the original PDE enter into these terms.

The final step consists in verifying that the integrable part H^0 satisfies the hypotheses of a suitable KAM-type theorem (we use a theorem by Pöschel [57]).

Having computed the expansion of the Hamiltonian carefully when going through the above transformations, we can easily translate a key nondegeneracy condition from the KAM theorem to a condition on the functions a_1, a_2, a_3. In the proof of Theorem 2.4, where one of the parameters is small, the nondegeneracy condition follows from our hypothesis (A2). In the proof of Theorem 2.6, we verify that the nondegeneracy condition is satisfied for an open and dense set of functions a_3. The KAM theorem yields quasiperiodic solutions to the reduced equation (2.16), and these correspond to y-quasiperiodic solutions of the original equation (2.1).

For the proof of Theorem 2.8, we show that if Φ is the Hamiltonian of the reduced equation in Birkhoff normal form, under some assumptions it is possible to find quasiperiodic solutions for the Hamiltonian system corresponding to $\Phi + \Phi^2$, from which we find quasiperiodic solutions of the original equation (2.3). The nondegeneracy condition required to apply the aforementioned KAM-type theorem will be a consequence of hypothesis (A4).

Chapter 3

The center manifold reduction

In this chapter, we first state an abstract center manifold theorem, based on the exposition in [35, 70] (see also [21, 48]). Then we write equation (2.1) in a form fitting the abstract setting, so that the hypotheses of the center manifold theorem can be verified.

3.1 An abstract center manifold theorem

Let X and Z be Hilbert spaces such that $Z \hookrightarrow X$ (continuous imbedding). Consider the following abstract equation with a parameter τ:

$$\frac{du}{dt} = Au + R(u;\tau), \quad (t \in \mathcal{I}). \tag{3.1}$$

Here $A \in \mathscr{L}(Z,X)$, $R : Z \times \mathbb{R}^d \to Z$, and $\mathcal{I} \subset \mathbb{R}$ is an interval. We are primarily interested in the case $\mathcal{I} = \mathbb{R}$, and we consider classical solutions of (3.1), that is, functions $u \in \mathscr{C}^1(\mathcal{I},X) \cap \mathscr{C}(\mathcal{I},Z)$ satisfying (3.1). At this point, the dimension $d \geq 0$ of the parameter space $\mathbb{R}^d$ is arbitrary ($d=0$ corresponds to the equation with no parameters), but in our specific problems we will take either $\tau = (s,b) \in \mathbb{R}^2$ or $\tau = s \in \mathbb{R}$. We also fix an open and bounded set $\mathscr{P} \subset \mathbb{R}^d$ and make the following assumptions on R:

(H1) There is a neighborhood V of $0 \in Z$ such that $R \in \mathscr{C}^k(V \times \mathbb{R}^d, Z)$ for some $k \geq 2$, and

$$R(0;\tau) = 0, \quad D_u R(0;\tau) = 0 \quad (\tau \in \mathscr{P}). \tag{3.2}$$

In the following hypotheses concerning the spectral properties of the operator A, we view it as an unbounded operator in X with domain $D(A) = Z \subset X$. While we assume that Z and X are real spaces, for the spectral properties we consider, as usual, the complexifications of Z, X, and A.

(H2) $\sigma(A) = \sigma_c \cup \sigma_h$, where $\sigma_h \subset \{z \in \mathbb{C} : |\operatorname{Re} z| > \gamma\}$ for some $\gamma > 0$ and σ_c consists of finitely many purely imaginary eigenvalues with finite algebraic multiplicities.

Hypothesis (H2) implies that the resolvent set of A is nonempty; moreover, A is a closed operator whose graph norm is equivalent to the norm of Z. To the decomposition $\sigma(A) = \sigma_c \cup \sigma_h$, there corresponds the spectral projection $P_c \in \mathscr{L}(X)$, characterized uniquely by the properties that it commutes with A and that its range $X_c := P_c X$ is spanned by the set of all generalized eigenvectors of A corresponding to the eigenvalues in σ_c (see [38]). Clearly, $X_c \subset Z$. Letting $P_h := 1 - P_c$, we note further that P_c and P_h restrict to bounded operators on Z. In particular, $P_h Z$ is a closed subspace of Z. When needed, we consider $P_h Z$ as a Banach space with the norm induced from Z.

The third hypothesis concerns the resolvent of A:

(H3) There exist $\hat{\omega}_0 > 0$ and $c > 0$ such that for all $\hat{\omega} \in \mathbb{R} \setminus (-\hat{\omega}_0, \hat{\omega}_0)$ we have:

(a) $i\hat{\omega}$ is in the resolvent set of A.

(b) $\|(i\hat{\omega} - A)^{-1}\|_{\mathscr{L}(X)} \leq \dfrac{c}{|\hat{\omega}|}$.

Theorem 3.1. *Assume that hypotheses* (H1)–(H3) *are satisfied. Then there exist a map* $\sigma \in \mathscr{C}^k(X_c \times \bar{\mathscr{P}}, P_h Z)$ *and a neighborhood* $\mathscr{N}$ *of* 0 *in* Z *such that*

$$\sigma(0;\tau) = 0, \; D_u\sigma(0;\tau) = 0 \quad (\tau \in \mathscr{P}) \tag{3.3}$$

and for each $\tau \in \bar{\mathscr{P}}$ the manifold

$$W_c(\tau) = \{u_0 + \sigma(u_0;\tau) : u_0 \in X_c\} \subset Z$$

has the following properties:

(a) *If $u(t)$ is a solution of* (3.1) *on $\mathcal{I} = \mathbb{R}$ and $u(t) \in \mathscr{N}$ for all $t \in \mathbb{R}$, then $u(t) \in W_c(\tau)$ for all $t \in \mathbb{R}$; that is, $W_c(\tau)$ contains the orbit of each solution of* (3.1) *which stays in $\mathscr{N}$ for all $t \in \mathbb{R}$.*

(b) *If $z : \mathbb{R} \to X_c$ is a solution of the equation*

$$\frac{dz}{dt} = A\big|_{X_c} z + P_c R(z + \sigma(z;\tau);\tau) \tag{3.4}$$

on some interval $\mathcal{I}$, and $u(t) := z(t) + \sigma(z(t);\tau) \in \mathscr{N}$ for all $t \in \mathcal{I}$, then $u : \mathcal{I} \to Z$ is a solution of (3.1) *on $\mathcal{I}$.*

Moreover, σ satisfies the following relations:

(i) *$\sigma(\cdot;\tau) \equiv 0$ whenever $\tau \in \mathscr{P}$ is such that $R(\cdot;\tau) \equiv 0$;*

(ii) *if $2 \le \ell \le k-1$ is an integer, then $\sigma(u;\tau) = \mathcal{O}(\|u\|^{\ell+1})$ as $u \to 0$ whenever $\tau \in \mathscr{P}$ is such that $R(u;\tau) = \mathcal{O}(\|u\|^{\ell+1})$ as $u \to 0$.*

Remark 3.2. Since $\ell \le k-1$, the notation $\sigma(u;\tau) = \mathcal{O}(\|u\|^{\ell+1})$ as $u \to 0$ in (ii) simply means that the derivatives of $\sigma(\cdot;\tau)$ up to order ℓ vanish at $u = 0$. If this is true for all $\tau \in \bar{\mathscr{P}}$, then, in view of compactness of $\bar{\mathscr{P}}$, we have $\sigma(u;\tau) = \mathcal{O}(\|u\|^{\ell+1})$, as $u \to 0$, uniformly for $\tau \in \bar{\mathscr{P}}$, simply because the derivative of order $\ell+1$ is bounded uniformly for u a neighborhood of $0 \in X_c$ and $\tau \in \bar{\mathscr{P}}$. This simple observation will be used below for other sufficiently smooth functions depending on parameters.

With the exception of statements (i), (ii), the proof of the theorem can be found in [35, 70], although a comment on the parameter dependence is necessary here. In our formulation the manifold $W_c(\tau)$ is defined for all parameters $\tau \in \bar{\mathscr{P}}$.

It is more common to just take τ in a small neighborhood of some point τ_0 (such a local-parameter version of the theorem follows from a version without parameters, cp. [35, Section 2.3.1], for example). If the center manifold were unique—which is not the case in general—then, due to (3.2) and the compactness of $\bar{\mathscr{P}}$, the global-parameter version would be a consequence of the local-parameter version. Nonetheless, such a compactness argument can be made if we recall how the center manifold theorem is proved, that is, how the function σ is found. This is done by first modifying the nonlinearity outside a small neighborhood $\mathscr{N} \ni 0$ using a suitable cutoff function, so that the new nonlinearity is globally Lipschitz in u with a small Lipschitz constant. For the modified nonlinearity, one finds a *unique* global center manifold, which then serves as local center manifold for the original equation in the sense that statements (a) and (b) are satisfied. Our point is that, under hypothesis (H1), the modification of the nonlinearity can be done once—with one cut-off function—for all parameters in a neighborhood of the compact set $\bar{\mathscr{P}}$. One then gets a function σ with the stated regularity properties and a fixed neighborhood $\mathscr{N}$ such that (3.3) and statements (a), (b) hold.

The uniqueness of the global center manifold for the modified nonlinearity implies that statement (i) holds: in fact, the center space X_c itself is the center manifold whenever the modified nonlinearity vanishes identically, which is the case when $R(\cdot;\tau)$ vanishes identically.

Statement (ii) follows from a recursive computation of the Taylor expansion of σ up to order k (although there is nonuniqueness of σ stemming from the choice of the cutoff function, the Taylor expansion is uniquely determined). The procedure is described in [36, Section 6] and [48, Section 2] and it goes as follows. The starting point is the following identity for σ:

$$D_u\sigma(u;\tau)[A\big|_{X_c}u + P_cR(u+\sigma(u;\tau);\tau)] = A\big|_{X_h}\sigma(u;\tau) + P_hR(u+\sigma(u;\tau);\tau) \quad (3.5)$$

(cp. equation (2.10) in [48]). Now expand σ as

$$\sigma(u;\tau) = \sigma^2(u;\tau) + \cdots + \sigma^\ell(u;\tau) + \sigma'(u;\tau),$$

where, for $j \in \{2, \dots, \ell\}$, σ^j is a homogeneous $P_h Z$-valued polynomial in u of degree j (with τ-dependent coefficients) and $\|\sigma'(u;\tau)\|_Z = \mathcal{O}(\|u\|^{\ell+1})$ as $u \to 0$, uniformly for $\tau \in \bar{\mathscr{P}}$. Substituting in (3.5) and equating terms of the same order one finds an equation for $\sigma^j(\cdot;\tau)$, for each $\tau \in \bar{\mathscr{P}}$:

$$D_u \sigma^j(u;\tau) A\big|_{X_c} u - A\big|_{X_h} \sigma^j(u;\tau) = r^j(u;\tau), \tag{3.6}$$

where $r^j(\cdot;\tau)$ is determined by the Taylor expansion of $R(\cdot;\tau)$ at 0 of order j and the terms $\sigma^2(\cdot;\tau), \dots, \sigma^{j-1}(\cdot;\tau)$ (if $j = 2$, r^2 is determined by $P_h D_u^2 R(0;\tau)$ alone). This equation determines the polynomial $\sigma^j(\cdot;\tau)$ uniquely (see [36, 48] for explicit forms of the solution). An induction argument then allows one to conclude that $R(u;\tau) = \mathcal{O}(\|u\|^{\ell+1})$ as $u \to 0$ implies $\sigma^2(\cdot;\tau) = \cdots = \sigma^\ell(\cdot;\tau) = 0$, which gives the conclusion in (ii).

In the sequel, the function σ is called the *reduction function*, X_c the *center space*, W_c the *center manifold*, and equation (3.4) is the *reduced equation*.

For us, the most important conclusion of Theorem 3.1 is statement (b): if we can find a "small" solution of the reduced equation (3.4) (that is, $\|z(t)\|_Z$ is sufficiently small for all t), then we have a solution of the original equation via the reduction function. Our goal is to find quasiperiodic solutions this way. Note also that the reduced equation is an ordinary differential equation: the space X_c is finite-dimensional due to hypothesis (H2).

3.2 Center manifold for equation (2.1)

We now verify that (2.1) can be rewritten as a system of the form (3.1), with operators A and R, and spaces X and Z chosen in such a way that hypotheses (H1)–(H3) hold with $k = K + 1$, K as in (2.5) if condition (A1), (S1), and (S2) are satisfied. The center manifold for equation (2.3) can be obtained similarly, with some minor changes which will be discussed in Chapter 7.

Fixing an integer $m > N/2$, as in (2.5), we set $X = H^{m+1}(\mathbb{R}^N) \times H^m(\mathbb{R}^N)$, $V = Z = H^{m+2}(\mathbb{R}^N) \times H^{m+1}(\mathbb{R}^N)$. Note that the relation $m > N/2$ implies that $H^m(\mathbb{R}^N)$ is continuously imbedded in a space of bounded Hölder continuous functions on $\mathbb{R}^N$.

Further, we fix any finite $\rho_0 > 0$ and set $\mathscr{P} := (-\rho_0, \rho_0)^2 \subset \mathbb{R}^2$.

Consider the $H^m(\mathbb{R}^N)$-realization of the Schrödinger operator $-\Delta - a_1(x)$, that is, the operator $u \mapsto -\Delta u - a_1 u$ defined on $H^{m+2}(\mathbb{R}^N)$. We will view it, as appropriate for the context, either as a bounded operator in $\mathscr{L}(H^{m+2}(\mathbb{R}^N), H^m(\mathbb{R}^N))$ (which is justified when $a \in \mathscr{C}_{\mathrm{b}}^m(\mathbb{R}^N)$) or as an unbounded operator on $H^m(\mathbb{R}^N)$ with domain $H^{m+2}(\mathbb{R}^N)$. Without fearing confusion, we use the same symbol A_1 as in Section 2.1 for this operator, noting that, by elliptic regularity estimates, the spectrum, the eigenvalues and their multiplicity, as well as the eigenfunctions do not change if instead of the $L^2(\mathbb{R}^N)$-realization we take the $H^m(\mathbb{R}^N)$-realization.

The abstract form of (2.1) is given by

$$\begin{cases} \dfrac{du_1}{dy} = u_2, \\ \dfrac{du_2}{dy} = A_1 u_1 - \tilde{f}(u_1; s, b), \end{cases} \tag{3.7}$$

where A_1 is the H^m-realization of $-\Delta - a_1(x)$, as above, and $\tilde{f} : H^{m+2}(\mathbb{R}^N) \times \mathbb{R}^2 \to H^{m+1}(\mathbb{R}^N)$ is the Nemytskii operator of f, that is, $\tilde{f}(u; s, b)(x) = f(x, u(x); s, b)$. In Appendix A.1, we verify that this operator is well defined.

System (3.7) can be written in the form (3.1) by defining the operator A on X, with domain $D(A) = Z$, and $R : Z \times \mathbb{R}^2 \to Z$ as

$$\begin{aligned} A(u_1, u_2) &= (u_2, A_1 u_1)^T, \\ R(u_1, u_2; s, b) &= (0, \tilde{f}(u_1; s, b))^T. \end{aligned} \tag{3.8}$$

The smoothness of the operator R is inherited from the smoothness of $\tilde{f}$, which is shown in Appendix A.1 (see Theorem A.1 and Lemma A.3). More precisely, if f satisfies (S2), then the map $\tilde{f} : H^{m+2}(\mathbb{R}^N) \times \mathbb{R}^2 \to H^{m+1}(\mathbb{R}^N)$ is of class $\mathscr{C}^{K+1}$ and so

$$R \in \mathscr{C}^{K+1}(V \times \mathbb{R}^2, Z). \tag{3.9}$$

In addition, relation (2.2) implies that $R(0; s, b) = 0$, $D_u R(0; s, b) = 0$ for all $(s, b) \in \mathbb{R}^2$.

In order to find the spectrum of A, viewed as an unbounded operator on X, consider the problem

$$A\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} - \lambda \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \tag{3.10}$$

where $(g_1, g_2) \in X$. Equivalently, (3.10) reads

$$\begin{aligned} v_2 - \lambda v_1 &= g_1, \\ -\Delta v_1 - a_1(x)v_1 - \lambda v_2 &= g_2, \end{aligned}$$

and eliminating v_2 we obtain

$$-\Delta v_1 - a_1(x)v_1 - \lambda^2 v_1 = g_2 + \lambda g_1, \tag{3.11}$$

where $g_2 + \lambda g_1 \in H^m(\mathbb{R}^N)$. From (3.11) we deduce that

$$\sigma(A) = \{\pm\sqrt{\lambda} : \lambda \in \sigma(A_1)\}.$$

We know that, by (A1), $\sigma(A_1)$ contains exactly n negative eigenvalues μ_j, $j = 1, \ldots, n$ and the rest of the spectrum is contained in (γ^2, ∞), for some $\gamma > 0$ (see Remark 2.2(i)). We conclude that the spectrum of A contains $2n$ (purely) imaginary eigenvalues $\pm i\sqrt{|\mu_j|}$, with simple multiplicities, and the rest of the spectrum is contained in $\{\lambda \in \mathbb{C} : |\operatorname{Re}\lambda| > \gamma\}$. So we can write

$$\sigma(A) = \sigma_c \cup \sigma_h,$$

with $\sigma_c = \{\pm i\sqrt{|\mu_j|} : j = 1, \ldots, n\}$ and $\sigma_h = \sigma(A) \setminus \sigma_c$. The bound on the resolvent of A (hypothesis (H3)(b)) is verified in Appendix A.2. We have thus verified all the hypotheses of Theorem 3.1.

Hence, Theorem 3.1 with $k = K + 1$ applies in our problem. Moreover, fixing $s = 0$ and applying statement (ii) (with just one parameter b), we obtain that, as $u \to 0$,

$$\sigma(u; 0, b) = \mathcal{O}(\|u\|^3) \quad (b \in (-\rho_0, \rho_0)). \tag{3.12}$$

We now write the reduced equation in suitable coordinates. Denote

$$\omega_j := \sqrt{|\mu_j|}, \quad j = 1, \ldots, n.$$

The eigenfunction of A associated to $\pm i\omega_j$ is, up to a constant multiple, $(\varphi_j, \pm i\omega_j\varphi_j)^T$. (As in Section 2.1, $\varphi_1, \ldots, \varphi_n$ are the eigenfunctions of A_1 corresponding to the eigenvalues $\mu_1, \ldots, \mu_n$, respectively, normalized in the L^2-norm). Taking real and imaginary part, we obtain the center space:

$$X_c = \{(g, \tilde{g})^T : g, \tilde{g} \in \operatorname{span}\{\varphi_1, \ldots, \varphi_n\}\}.$$

The spectral projection $P_c : X \to X_c$ corresponding to the imaginary eigenvalues of A is given by

$$P_c \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \Pi v_1 \\ \Pi v_2 \end{pmatrix}, \tag{3.13}$$

where Π is the orthogonal projection of $L^2(\mathbb{R}^N)$ onto $\operatorname{span}\{\varphi_1, \ldots, \varphi_n\}$. Indeed, Π (or, more precisely, its restriction to $H^m(\mathbb{R}^N)$) is the spectral projection of A_1 associated with the spectral set $\{\mu_1, \ldots, \mu_n\}$. Using this, one shows easily that P_c, as defined in (3.13), commutes with A. It is obviously a projection: $P_c^2 = P_c$. Finally, its range is clearly the space X_c, thus P_c is the spectral projection, as claimed.

Setting $X_h = (1 - P_c)X$, we have $H^{m+1} \times H^m = X_c \oplus X_h$ and, additionally, the spaces X_c and X_h are orthogonal with respect to the $(L^2(\mathbb{R}^N))^2$-inner product.

For $j = 1, \ldots, n$, let $\psi_j = (\varphi_j, 0)^T$, $\zeta_j = (0, \varphi_j)^T$, so

$$\mathscr{B} = \{\psi_1, \ldots, \psi_n, \zeta_1, \ldots, \zeta_n\}$$

is a basis of X_c. If

$$\begin{aligned} \xi &= (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \\ \eta &= (\eta_1, \ldots, \eta_n) \in \mathbb{R}^n, \\ \psi &:= (\psi_1, \ldots, \psi_n) : \mathbb{R}^N \to \mathbb{R}^{2n}, \\ \zeta &:= (\zeta_1, \ldots, \zeta_n) : \mathbb{R}^N \to \mathbb{R}^{2n}, \end{aligned}$$

we can write the center space as

$$X_c = \{\xi \cdot \psi + \eta \cdot \zeta : \xi, \eta \in \mathbb{R}^n\},$$

where $\xi \cdot \psi = \xi_1\psi_1 + \cdots + \xi_n\psi_n$, and similarly for $\eta \cdot \zeta$.

We use $(\xi, \eta) \in \mathbb{R}^{2n}$ as coordinates on the center manifold. Let $\hat{\sigma} : X_c \times \bar{\mathscr{P}} \to P_h Z$ be the reduction function, as in Theorem 3.1. If $(g, \tilde{g}) \in X_c$, then there exists a unique $(\xi, \eta) \in \mathbb{R}^{2n}$ such that

$$(g, \tilde{g}) = \xi \cdot \psi + \eta \cdot \zeta,$$

so

$$\hat{\sigma}(g, \tilde{g}; s, b) = \hat{\sigma}(\xi \cdot \psi + \eta \cdot \zeta; s, b).$$

Thus, we can define $\sigma : \mathbb{R}^{2n} \times \bar{\mathscr{P}} \to P_h Z$ by

$$\sigma(\xi, \eta; s, b) = \hat{\sigma}(\xi \cdot \psi + \eta \cdot \zeta; s, b). \tag{3.14}$$

Defining further a function $\Lambda : \mathbb{R}^{2n} \times \bar{\mathscr{P}} \to P_h Z$ as

$$\Lambda(\xi, \eta; s, b) = \xi \cdot \psi + \eta \cdot \zeta + \sigma(\xi, \eta; s, b), \tag{3.15}$$

the center manifold can be written as

$$W_c(s, b) = \{\Lambda(\xi, \eta; s, b) : \xi, \eta \in \mathbb{R}^n\}.$$

We next find the matrix of $A\big|_{X_c}$ with respect to the basis $\mathscr{B}$. Denoting $\varphi := (\varphi_1, \ldots, \varphi_n)$, for any $(\xi, \eta) \in \mathbb{R}^{2n}$ we have

$$A(\xi \cdot \psi + \eta \cdot \zeta) = A \begin{pmatrix} \xi \cdot \varphi \\ \eta \cdot \varphi \end{pmatrix} = \begin{pmatrix} \eta \cdot \varphi \\ A_1(\xi \cdot \varphi) \end{pmatrix} = \begin{pmatrix} \eta \cdot \varphi \\ (M_0\xi) \cdot \varphi \end{pmatrix} = \eta \cdot \psi + (M_0\xi) \cdot \zeta,$$

where $M_0 = \mathrm{diag}(\mu_1, \ldots, \mu_n)$. Therefore, setting

$$M_A = \begin{bmatrix} 0 & 1 \\ M_0 & 0 \end{bmatrix},$$

we find

$$A(\xi \cdot \psi + \eta \cdot \zeta) = M_A \begin{pmatrix} \xi^T \\ \eta^T \end{pmatrix} \cdot \begin{pmatrix} \psi^T \\ \zeta^T \end{pmatrix}.$$

To write the reduced equation (3.4) in the coordinates (ξ, η), we use y for the time variable and view ξ, η as functions of y: (3.4) becomes

$$\frac{d}{dy}(\xi \cdot \psi + \eta \cdot \zeta) = M_A \begin{pmatrix} \xi^T \\ \eta^T \end{pmatrix} \cdot \begin{pmatrix} \psi^T \\ \zeta^T \end{pmatrix} + P_c \begin{pmatrix} 0 \\ \tilde{f}(\Lambda(\xi, \eta; s, b); s, b) \end{pmatrix}.$$

Equivalently, this equation can be written as

$$\begin{cases} \dot{\xi} = h_1(\xi, \eta; s, b), \\ \dot{\eta} = h_2(\xi, \eta; s, b), \end{cases} \tag{3.16}$$

where $\dot{\xi} = d\xi/dy$, $\dot{\eta} = d\eta/dy$, and

$$h(\xi, \eta; s, b) = \begin{pmatrix} h_1(\xi, \eta; s, b) \\ h_2(\xi, \eta; s, b) \end{pmatrix} = M_A \begin{pmatrix} \xi^T \\ \eta^T \end{pmatrix} + \left\{ \begin{pmatrix} 0 \\ \Pi \tilde{f}(\Lambda(\xi, \eta; s, b)); s, b) \end{pmatrix} \right\}_{\mathscr{B}},$$

where Π is as in (3.13) and $\{\cdot\}_{\mathscr{B}}$ denotes the coordinates of the argument with respect to the basis $\mathscr{B}$.

We remark that system (3.7) is reversible (specifically, if $(u_1(x, y), u_2(x, y))$ a solution, so is $(u_1(x, -y), -u_2(x, -y)))$. As a consequence, one can show a reversibility property of the reduced equation [35, 48], but we do not employ this additional structure.

The specific form of the nonlinearity, see (2.2), implies the following properties of the reduction function σ.

Lemma 3.3. *One has*

$$\sigma(\xi, \eta; s, b) = sb\sigma^2(\xi, \eta) + \tilde{\sigma}(\xi, \eta; s, b), \tag{3.17}$$

where σ^2 is a $P_h Z$-valued homogeneous polynomial in (ξ, η) of degree 2 and $\tilde{\sigma}$ is a $\mathscr{C}^{K+1}$ function on $\mathbb{R}^{2n} \times \bar{\mathscr{P}}$ of order $\mathcal{O}(|(\xi, \eta)|^3)$ as $(\xi, \eta) \to (0, 0)$, uniformly for $(s, b) \in \bar{\mathscr{P}}$.

Proof. Recall that $\sigma(\xi, \eta; s, b) = \hat{\sigma}(\xi \cdot \psi + \eta \cdot \zeta; s, b)$ (cp. (3.14)), and the quadratic term in the expansion of $\hat{\sigma}(\cdot; s, b)$ is determined uniquely from (3.6) with $j = 2$

(take $\hat{\sigma}$ in place of σ there). For $j = 2$, the right hand side of (3.6) is given by $P_h D_u^2 R(0;\tau)[u,u]/2$. In our specific case,

$$D_u^2 R(0;\tau)[u,u]/2 = (0, bsa_2 u_1^2))^T \quad (u = (u_1, u_2) \in Z)$$

(cp. (3.8), (2.2)). Using this, the uniqueness of the solution of (3.6), and the fact that the left-hand side of (3.6) is linear in σ^2, we obtain (3.17), with $\tilde{\sigma}(\xi, \eta; s, b) = \mathcal{O}(|(\xi,\eta)|^3)$ as $(\xi,\eta) \to (0,0)$ for each (s,b). Relation (3.17) implies that $\tilde{\sigma}$ is of class $\mathscr{C}^{K+1}$, which also gives the uniformity in (s,b) as stated in the lemma (cp. Remark 3.2). □

Remark. For the sake of notational simplicity, in the sequel, we sometimes omit the argument (s,b) from R, σ, Λ, W_c, h, and other similar functions when there is no need to emphasize the dependence on the parameters.

The following simple lemma will be useful in Chapter 4:

Lemma 3.4. *Let $D\Lambda(\xi,\eta)$ denote the derivative of Λ with respect to (ξ,η). Then, in a neighborhood of the origin,*

$$D\Lambda(\xi,\eta)h(\xi,\eta) = A\Lambda(\xi,\eta) + R(\Lambda(\xi,\eta)).$$

Proof. Fix (ξ_0, η_0) close to the origin, and let $(\xi(y), \eta(y))$ be the solution of (3.16) with $(\xi(0), \eta(0)) = (\xi_0, \eta_0)$. Substituting $\Lambda(\xi,\eta)$ in (3.1), and using Theorem 3.1(b), we obtain

$$\begin{aligned} A\Lambda(\xi_0,\eta_0) + R(\Lambda(\xi_0,\eta_0)) &= \frac{d}{dy}\Lambda(\xi,\eta)\Big|_{y=0} \\ &= D\Lambda(\xi,\eta)(\dot{\xi},\dot{\eta})\big|_{y=0} \\ &= D\Lambda(\xi_0,\eta_0)(h_1(\xi_0,\eta_0), h_2(\xi_0,\eta_0)) \\ &= D\Lambda(\xi_0,\eta_0)h(\xi_0,\eta_0), \end{aligned}$$

where we used (3.16) to derive the second to last equality. □

Chapter 4

The reduced Hamiltonian

In this chapter, we write the reduced equation (3.16) as a Hamiltonian system with respect to a certain symplectic structure on $\mathbb{R}^{2n}$. Using the Darboux theorem, we then transform it locally to a Hamiltonian system with respect to the standard symplectic form. Finally, employing the Birkhoff normal form, we write the Hamiltonian as the sum of of an integrable Hamiltonian and a small perturbation. We compute the expansion of the integrable part explicitly up to order four; this will later allow us to verify a nondegeneracy condition from a KAM theorem.

Throughout this chapter, we assume the standing hypotheses (A1), (S1), (NR), and (S2) to be satisfied. We use the notation introduced in Chapter 3. In particular, we use the coordinates (ξ, η) as in Chapter 3.2 and view the reduction function σ as a function of (ξ, η) (and the parameters (s, b)) with values in $P_h Z$, $Z = H^{m+2}(\mathbb{R}^N) \times H^{m+1}(\mathbb{R}^N)$, see (3.14).

4.1 The Hamiltonian and the symplectic structure

Define

$$F(x, u; s, b) = \int_0^u f(x, \vartheta; s, b)\, d\vartheta.$$

For $(u,v) \in Z$, and any fixed $(s,b) \in \bar{\mathscr{P}}$, let

$$H(u,v) = \int_{\mathbb{R}^N} \frac{-1}{2}|\nabla u(x)|^2 + \frac{1}{2}a_1(x)u^2(x) + F(x,u(x);s,b) + \frac{1}{2}v^2(x)\ dx. \quad (4.1)$$

An integration by parts shows that

$$DH(u,v)(\bar{u},\bar{v}) = \int_{\mathbb{R}^N} \big(\Delta u(x) + a_1(x)u(x) + f(x,u(x);s,b)\big)\bar{u}(x)\,dx + \int_{\mathbb{R}^N} v(x)\bar{v}(x)\,dx.$$

In other words, $(\Delta u + a_1 u + f(\cdot,u(\cdot);s,b),v)$ is the gradient, $\nabla H(u,v)$, of $H(u,v)$ with respect to the $(L^2(\mathbb{R}^N))^2$ inner product.

Denoting by $\mathbb{J}_{L^2}$ the operator on $(L^2(\mathbb{R}^N))^2$ given by

$$\mathbb{J}_{L^2} = \begin{bmatrix} 0 & \mathbb{I}_{L^2} \\ -\mathbb{I}_{L^2} & 0 \end{bmatrix},$$

$\mathbb{I}_{L^2}$ being the identity operator on $L^2(\mathbb{R}^N)$, we can write equation (3.7) as

$$\frac{d}{dy}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mathbb{J}_{L^2}\nabla H(u_1,u_2). \quad (4.2)$$

Written this way, (3.7) fits the context of abstract Hamiltonian systems considered in [48]. General results from [48] can then be used to show that the reduction of the equation to the center manifold is the Hamiltonian system with respect to the Hamiltonian H restricted to the center manifold and with respect to the symplectic form which is also the restriction of a symplectic form on the space Z to the center manifold. Lemmas 4.1 and 4.2 below are essentially an interpretation of these remarks in the coordinates (ξ,η), and they can certainly be derived from [48]. But it is simple enough to prove them instead by direct explicit computations, and we will do it that way. These explicit computations will also help us find the Taylor expansion of the Hamiltonian up to order four.

Let Λ be as in (3.15). Recalling that for $(\xi,\eta) \in \mathbb{R}^{2n}$, $\Lambda(\xi,\eta)$ and $\sigma(\xi,\eta)$ are elements of the product space $Z = H^{m+2}(\mathbb{R}^N) \times H^{m+1}(\mathbb{R}^N)$, we write them as

$\Lambda(\xi,\eta) = (\Lambda_1(\xi,\eta), \Lambda_2(\xi,\eta))$ and $\sigma(\xi,\eta) = (\sigma_1(\xi,\eta), \sigma_2(\xi,\eta))$. Define

$$\Phi(\xi,\eta) := H(\Lambda(\xi,\eta)) = H(u,v)\,\Big|_{u=\Lambda_1(\xi,\eta),\, v=\Lambda_2(\xi,\eta)} \qquad ((\xi,\eta)\in\mathbb{R}^{2n}). \tag{4.3}$$

The parameters $(s,b)\in\bar{\mathscr{P}}$ will not be specifically included the notation until they start playing a role again. For now they can be considered fixed.

In the next two lemmas, we show that the reduced equation (3.16) is the Hamiltonian system corresponding to the Hamiltonian Φ and the symplectic form ω defined on a neighborhood of the origin of $\mathbb{R}^{2n}$ by

$$\begin{aligned}\omega(\xi,\eta)\big((t_1,t_2),(\bar t_1,\bar t_2)\big) &= t_1\cdot\bar t_2 - t_2\cdot\bar t_1 + \int_{\mathbb{R}^N} D\sigma_1(\xi,\eta)(t_1,t_2)D\sigma_2(\xi,\eta)(\bar t_1,\bar t_2)\,dx\\ &- \int_{\mathbb{R}^N} D\sigma_2(\xi,\eta)(t_1,t_2)D\sigma_1(\xi,\eta)(\bar t_1,\bar t_2)\,dx \qquad \big((\xi,\eta),(t_1,t_2),(\bar t_1,\bar t_2)\in\mathbb{R}^{2n}\big),\end{aligned} \tag{4.4}$$

where D denotes the derivative with respect to (ξ,η). Note that for all (ξ,η) and $(t_1,t_2)\in\mathbb{R}^{2n}$ the values $\sigma_j(\xi,\eta)$ and $D\sigma_j(\xi,\eta)(t_1,t_2)$ are elements of $H^{m+1}(\mathbb{R}^N)$, hence they are functions of $x\in\mathbb{R}^N$. In the integrals above, and similar integrals below, we suppress the argument x for the sake of notational simplicity.

For $(\xi,\eta)\in\mathbb{R}^{2n}$, the (ξ,η)-dependent matrix of the bilinear map $\omega(\xi,\eta)$ defined by (4.4) is the block matrix:

$$\begin{aligned}S(\xi,\eta) &:= \begin{bmatrix} 0 & I\\ -I & 0\end{bmatrix}\\ &+ \int_{\mathbb{R}^N}\begin{bmatrix}\nabla_\xi\sigma_1(\xi,\eta)\big(\nabla_\xi\sigma_2(\xi,\eta)\big)^T & \nabla_\xi\sigma_1(\xi,\eta)\big(\nabla_\eta\sigma_2(\xi,\eta)\big)^T\\ \nabla_\eta\sigma_1(\xi,\eta)\big(\nabla_\xi\sigma_2(\xi,\eta)\big)^T & \nabla_\eta\sigma_1(\xi,\eta)\big(\nabla_\eta\sigma_2(\xi,\eta)\big)^T\end{bmatrix}dx\\ &- \int_{\mathbb{R}^N}\begin{bmatrix}\nabla_\xi\sigma_2(\xi,\eta)\big(\nabla_\xi\sigma_1(\xi,\eta)\big)^T & \nabla_\xi\sigma_2(\xi,\eta)\big(\nabla_\eta\sigma_1(\xi,\eta)\big)^T\\ \nabla_\eta\sigma_2(\xi,\eta)\big(\nabla_\xi\sigma_1(\xi,\eta)\big)^T & \nabla_\eta\sigma_2(\xi,\eta)\big(\nabla_\eta\sigma_1(\xi,\eta)\big)^T\end{bmatrix}dx,\end{aligned} \tag{4.5}$$

where I is the $n\times n$ identity matrix and ∇_ξ, ∇_η stand for the usual gradients written as columns (so the blocks are $n\times n$ matrices).

Lemma 4.1. *Let $h=(h_1,h_2)$ be as in* (3.16) *and ω be as in* (4.4). *For all (ξ,η) in a neighborhood of $(0,0)$ and $(\bar\xi,\bar\eta)\in\mathbb{R}^{2n}$ we have*

$$D\Phi(\xi,\eta)(\bar\xi,\bar\eta) = \omega(\xi,\eta)\big(h(\xi,\eta),(\bar\xi,\bar\eta)\big). \tag{4.6}$$

Proof. Let $\langle\cdot,\cdot\rangle$ denote the inner product in $(L^2(\mathbb{R}^N))^2$. Differentiating Φ with respect to (ξ,η), we obtain, by (3.8), (4.2), and Lemma 3.4,

$$\begin{aligned} D\Phi(\xi,\eta)(\bar\xi,\bar\eta) &= DH(\Lambda(\xi,\eta))D\Lambda(\xi,\eta)(\bar\xi,\bar\eta)\\ &= \big\langle \mathbb{J}_{L^2}\big(A\Lambda(\xi,\eta)+R(\Lambda(\xi,\eta))\big), D\Lambda(\xi,\eta)(\bar\xi,\bar\eta)\big\rangle\\ &= \big\langle \mathbb{J}_{L^2}D\Lambda(\xi,\eta)h(\xi,\eta), D\Lambda(\xi,\eta)(\bar\xi,\bar\eta)\big\rangle\,. \end{aligned}$$

Here, writing $\varphi=(\varphi_1,\dots,\varphi_n)$ and $(a,b)\in\mathbb{R}^{2n}$,

$$D\Lambda(\xi,\eta)(a,b) = \begin{pmatrix} D\Lambda_1(\xi,\eta)(a,b)\\ D\Lambda_2(\xi,\eta)(a,b)\end{pmatrix} = \begin{pmatrix} a\cdot\varphi + D\sigma_1(\xi,\eta)(a,b)\\ b\cdot\varphi + D\sigma_2(\xi,\eta)(a,b)\end{pmatrix};$$

thus,

$$\begin{aligned} &D\Phi(\xi,\eta)(\bar\xi,\bar\eta) =\\ &= \int_{\mathbb{R}^N}\big(-h_2(\xi,\eta)\cdot\varphi - D\sigma_2(\xi,\eta)(h_1(\xi,\eta),h_2(\xi,\eta))\big)\big(\bar\xi\cdot\varphi + D\sigma_1(\xi,\eta)(\bar\xi,\bar\eta)\big)\,dx\\ &\quad+\int_{\mathbb{R}^N}\big(h_1(\xi,\eta)\cdot\varphi + D\sigma_1(\xi,\eta)(h_1(\xi,\eta),h_2(\xi,\eta))\big)\big(\bar\eta\cdot\varphi + D\sigma_2(\xi,\eta)(\bar\xi,\bar\eta)\big)\,dx. \end{aligned}$$

Since the eigenfunctions $\varphi_1,\dots,\varphi_n$ are $L^2(\mathbb{R}^N)$-orthonormal,

$$\int_{\mathbb{R}^N}(-h_2(\xi,\eta)\cdot\varphi)(\bar\xi\cdot\varphi)dx = -h_2(\xi,\eta)\cdot\bar\xi,$$

and

$$\int_{\mathbb{R}^N}(h_1(\xi,\eta)\cdot\varphi)(\bar\eta\cdot\varphi)dx = h_1(\xi,\eta)\cdot\bar\eta.$$

Now, σ takes values in X_h, which is $(L^2(\mathbb{R}^N))^2$-orthogonal to X_c (cp. (3.13)). It follows that

$$\int_{\mathbb{R}^N}(-h_2(\xi,\eta)\cdot\varphi)D\sigma_1(\xi,\eta)(\bar\xi,\bar\eta)\,dx = 0,$$

and similarly for the other integrals involving the product of a linear combination of the functions φ_j with $D\sigma_1$ or $D\sigma_2$. Thus,

$$\begin{aligned} D\Phi(\xi,\eta)(\bar{\xi},\bar{\eta}) &= h_1(\xi,\eta)\cdot\bar{\eta} - h_2(\xi,\eta)\cdot\bar{\xi} \\ &\quad + \int_{\mathbb{R}^N} D\sigma_1(\xi,\eta)(h_1(\xi,\eta),h_2(\xi,\eta))D\sigma_2(\xi,\eta)(\bar{\xi},\bar{\eta})\,dx \\ &\quad - \int_{\mathbb{R}^N} D\sigma_2(\xi,\eta)(h_1(\xi,\eta),h_2(\xi,\eta))D\sigma_1(\xi,\eta)(\bar{\xi},\bar{\eta})\,dx. \end{aligned}$$

Therefore, with ω as in (4.4), we have

$$D\Phi(\xi,\eta)(\bar{\xi},\bar{\eta}) = \omega(\xi,\eta)\big(h(\xi,\eta),(\bar{\xi},\bar{\eta})\big). \qquad \square$$

Below, α denotes the standard symplectic form on $\mathbb{R}^{2n}$, that is, the constant 2-form given by

$$\alpha(\xi,\eta) := (\xi,\eta)J(\xi,\eta)^T \quad (\xi,\eta\in\mathbb{R}^n),$$

with the matrix

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix},$$

where I is the identity matrix in $\mathbb{R}^n$.

Lemma 4.2. *Let ω be the 2-form defined in* (4.4)*. There is a neighborhood of* $(0,0)\in\mathbb{R}^{2n}$ *independent of the parameters* $(s,b)\in\bar{\mathscr{P}}$ *on which ω is a symplectic form of class $\mathscr{C}^K$.*

Proof. Since $\sigma = (\sigma_1,\sigma_2)$ is of class $\mathscr{C}^{K+1}$ as a Z-valued map (hence also as a $(L^2(\mathbb{R}^N))^2$-valued map), the matrix-valued function $(\xi,\eta)\to S(\xi,\eta)$, with $S(\xi,\eta)$ as in(4.5), is of class $\mathscr{C}^K$, that is, the form ω is of class $\mathscr{C}^K$.

Since $\sigma(\xi,\eta) = \mathcal{O}(|(\xi,\eta)|^2)$ as $(\xi,\eta)\to(0,0)$ (uniformly for $(s,b)\in\bar{\mathscr{P}}$), $(\omega-\alpha)(\xi,\eta) = \mathcal{O}(|(\xi,\eta)|^2)$ as well. This implies that there exists a neighborhood of $(0,0)\in\mathbb{R}^{2n}$ independent of the parameters $(s,b)\in\bar{\mathscr{P}}$ on which ω is nondegenerate. A straightforward computation, which we omit, shows that $d\omega = 0$, so ω is a closed form. Obviously, the matrix $S(\xi,\eta)$ is skew-symmetric. Thus ω is a symplectic form in the aforementioned neighborhood of $(0,0)\in\mathbb{R}^{2n}$ for all $(s,b)\in\bar{\mathscr{P}}$. $\square$

Remark 4.3. When the parameters are taken into account, $\sigma = (\sigma_1, \sigma_2)$ is of class $\mathscr{C}^{K+1}$ in $(\xi, \eta) \in \mathbb{R}^{2n}$ and $(s, b) \in \bar{\mathscr{P}}$, therefore the matrix-valued function (4.5) is of class $\mathscr{C}^K$ in $(\xi, \eta) \in \mathbb{R}^{2n}$ and $(s, b) \in \bar{\mathscr{P}}$.

We now specifically consider the dependence of ω on $(s, b) \in \bar{\mathscr{P}}$; we write $\omega(\xi, \eta; s, b)$ for the bilinear map defined in (4.4), stressing its dependence on $(s, b) \in \bar{\mathscr{P}}$ via σ. The following result is a direct consequence of Lemma 3.3.

Corollary 4.4. *One has*

$$\omega(\xi, \eta; s, b) = \alpha(\xi, \eta) + s^2 b^2 \omega^2(\xi, \eta) + \tilde{\omega}(\xi, \eta; s, b), \tag{4.7}$$

where ω^2 and $\tilde{\omega}(\cdot, \cdot; s, b)$ are 2-forms on a neighborhood of $(0,0)$, $\omega^2(\xi, \eta)$ is a homogeneous polynomial in (ξ, η) of degree 2 (taking values in the space of skew-symmetric bilinear maps), and $\tilde{\omega}(\xi, \eta; s, b)$ is of order $\mathcal{O}(|(\xi, \eta)|^3)$ as $(\xi, \eta) \to (0,0)$, uniformly for $(s, b) \in \bar{\mathscr{P}}$.

Using Lemma 4.1, we can write equation (3.16) as

$$\frac{d}{dy}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = X_\Phi(\xi, \eta), \tag{4.8}$$

where X_Φ is the Hamiltonian vector field associated to Φ on a neighborhood of $0 \in \mathbb{R}^{2n}$ endowed with the symplectic form ω.

4.2 Transforming to the standard symplectic form

We recall the Darboux theorem:

Theorem 4.5. *Let ω be a $\mathscr{C}^k$-symplectic form on a ball around $0 \in \mathbb{R}^{2n}$ and α be the standard symplectic form on $\mathbb{R}^{2n}$. Then there exists a near-identity $\mathscr{C}^k$-transformation ϕ such that*

$$\phi^* \omega = \alpha.$$

Here $\phi^*\omega$, the pull-back of ω, is the form obtained from ω by the change of coordinates $(\xi, \eta) = \phi(\xi', \eta')$. The effect of the change of coordinates from the Darboux theorem on Hamiltonian systems is well known: any Hamiltonian system with respect to the symplectic form ω transforms to a Hamiltonian system with respect to the standard symplectic form α (and the transformed Hamiltonian).

We want to apply this change of coordinates to the symplectic form in (4.4). It will be useful to choose the diffeomorphism ϕ—which is not unique—so that it satisfies additional estimates, as stated in the following lemma.

Lemma 4.6. *Let ω be the 2-form defined in* (4.4). *Then there exist a neighborhood $\mathscr{V}$ of $(0,0) \in \mathbb{R}^{2n}$ and a $\mathscr{C}^K$ map $\phi : \mathscr{V} \times \mathbb{R}^2 \to \mathbb{R}^{2n}$ such that $\phi^*(\cdot,\cdot; s, b)$ $\omega(\cdot,\cdot; s, b) = \alpha$, and one has*

$$\phi(\xi, \eta; s, b) = (\xi, \eta) + s^2 b^2 \phi^3(\xi, \eta) + \tilde{\phi}(\xi, \eta; s, b), \tag{4.9}$$

where $\phi^3 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a homogeneous polynomial of degree 3 *and $\tilde{\phi}$ is (a map of class $\mathscr{C}^K$ which is) of order $\mathcal{O}(|(\xi, \eta)|^4)$ as $(\xi, \eta) \to (0, 0)$, uniformly for $(s, b) \in \bar{\mathscr{P}}$.*

Proof. The statement holds if the map ϕ is constructed in a suitable way. We recall briefly how the Lie transform method of the proof of the Darboux theorem goes (see, e.g., [1, 37]).

For $t \in [0, 1]$, let

$$\omega_t = \alpha + t(\omega - \alpha),$$

so $\omega_0 = \alpha$ and $\omega_1 = \omega$. For each $(s, b) \in \bar{\mathscr{P}}$, we seek a family of diffeomorphisms ϕ^t satisfying $\phi^0 = \mathrm{Id}$ (the identity map in $\mathbb{R}^{2n}$), and

$$(\phi^t)^* \omega_t = \alpha,$$

so $\phi = \phi^1$ is the desired transformation. Such ϕ^t is found as the flow of a t-dependent vector field X_t; namely, ϕ^t has the desired property if

$$\omega_t(X_t, \cdot) = -\lambda, \tag{4.10}$$

where λ is a 1-form of class $\mathscr{C}^K$ on a neighborhood of $(0,0) \in \mathbb{R}^{2n}$ such that $d\lambda = \omega - \alpha$. The existence of such a 1-form is guaranteed by the Poincaré lemma (because $d\omega = 0$), but, again, because of nonuniqueness, some care is needed in selecting a "good" one. We claim that λ can be chosen such that

$$\lambda(\xi, \eta; s, b) = s^2 b^2 \lambda^3(\xi, \eta) + \tilde{\lambda}(\xi, \eta; s, b), \tag{4.11}$$

where λ^3 is a 1-form whose coefficients are homogeneous polynomials of degree 3 and $\tilde{\lambda}(\xi, \eta; s, b) = \mathcal{O}(|(\xi, \eta)|^4)$, as $(\xi, \eta) \to (0, 0)$, uniformly for $(s, b) \in \bar{\mathscr{P}}$. Indeed, this follows from Corollary 4.4 if one uses the Lie transform method in the proof of the Poincaré lemma, which amounts to taking integrals with respect to (ξ, η) of the coefficients of the 2-form $\omega - \alpha$ (see the proofs in [1, Theorem 6.4.14] or [61, Theorem 10.39]).

Now, $\omega_t - \alpha$ is of order $\mathcal{O}(|(\xi, \eta)|^2)$ as $(\xi, \eta) \to (0, 0)$ uniformly in $(s, b) \in \bar{\mathscr{P}}$, $t \in [0, 1]$; in particular, ω_t is nondegenerate near $(0, 0)$. Thus we can solve (4.10) for X_t uniquely; for this, we just need to invert the (ξ, η)-dependent matrix of the bilinear map ω_t and apply it to the coefficients of the 1-form on the left. This yields the following form of the vector field X_t:

$$X_t(\xi, \eta; s, b) = s^2 b^2 X^3(\xi, \eta) + \tilde{X}_t(\xi, \eta; s, b)$$

where X^3 is a homogeneous polynomial vector field of degree 3 and $\tilde{X}_t(\xi, \eta; s, b) = \mathcal{O}(|(\xi, \eta)|^4)$, as $(\xi, \eta) \to (0, 0)$, uniformly for $(s, b) \in \bar{\mathscr{P}}$ and $t \in [0, 1]$. Moreover, $\tilde{X}_t$ and X_t inherit the smoothness of α and ω: they are of class $\mathscr{C}^K$ in $(\xi, \eta) \approx 0$ and $(s, b) \in \bar{\mathscr{P}}$.

Finally, we take the flow ϕ^t of the vector field X_t. The vector field X_t vanishes at $(\xi, \eta) = (0, 0)$ together with its derivatives up to order 2. From this we obtain, first of all, that near the origin (and for all $(s, b) \in \bar{\mathscr{P}}$) the flow is defined up to $t = 1$. Computing the derivatives of ϕ^t with respect (ξ, η) by solving the corresponding ODEs we conclude that $\phi = \phi^1$ has the form as stated in Lemma 4.6. $\square$

Remark 4.7. Note that (3.6) implies that the term σ^2 in (3.17) and, consequently, the term ω^2 in (4.7) are determined by the quadratic term a_2u^2 of the nonlinearity f only – both are independent of the higher order terms $a_3u^3 + u^4 f_1(x, u; s, b)$. Examining the above proof carefully, one can check that the term ϕ^3 is determined only by ω^2. This shows that ϕ^3 is determined by a_2 and is independent of a_3 and f_1.

We now examine more closely the structure of the Hamiltonian Φ, first in the original coordinates (ξ, η) introduced in Section 4.1, see (4.3), then in the Darboux coordinates from Lemma 4.6. This is the content of the following two results. We write $\Phi(\xi, \eta; s, b)$ for the Hamiltonian, accounting for its dependence of the parameters (s, b). Recall that a_1, a_2, a_3 are the functions in (2.2) and $\varphi = (\varphi_1, \ldots, \varphi_n)$, φ_j being the eigenfunctions of $-\Delta - a_1(x)$ as in Section 2.1.

Lemma 4.8. *There is a neighborhood* $\mathscr{V}$ *of* $(0, 0) \in \mathbb{R}^{2n}$ *such that the Hamiltonian* Φ *defined in* (4.3) *has the following property. For each* $(\xi, \eta) \in \mathscr{V}$ *and* $(s, b) \in \bar{\mathscr{P}}$ *one has*

$$\Phi(\xi, \eta; s, b) = \frac{1}{2}\sum_{j=1}^{n}(-\mu_j \xi_j^2 + \eta_j^2) + \frac{sb}{3}\int_{\mathbb{R}^N} a_2(x)(\xi\cdot\varphi(x))^3\,dx \\ + \frac{b}{4}\int_{\mathbb{R}^N} a_3(x)(\xi\cdot\varphi(x))^4\,dx + s^2b^2\Phi_4'(\xi,\eta) + \Phi''(\xi,\eta;s,b), \quad (4.12)$$

where Φ_4' *is a homogeneous polynomial on* $\mathbb{R}^{2n}$ *of degree 4 and* Φ'' *is a* $\mathscr{C}^K$*-function on* $\mathscr{V} \times \bar{\mathscr{P}}$ *such that* $\Phi''(\xi, \eta; s, b) = \mathcal{O}(|(\xi, \eta)|^5)$ *as* $(\xi, \eta) \to (0, 0)$*, uniformly for* $(s, b) \in \bar{\mathscr{P}}$*.*

The regularity of Φ'' is in fact one degree higher: it is of class $\mathscr{C}^{K+1}$; we take $\mathscr{C}^K$ here for consistency with the statement of Proposition 4.9 below, where a degree of regularity is lost due to the Darboux transformation.

Proof of Lemma 4.8. Recalling (2.2), (4.1), and using an integration by parts, we write the functional $H(u,v)$ as

$$H(u,v) = \frac{1}{2}\int_{\mathbb{R}^N} \big(\Delta u(x) + a_1(x)u(x)\big)u(x)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} v^2(x)\,dx \\ + \frac{sb}{3}\int_{\mathbb{R}^N} u^3(x)\,dx + \frac{b}{3}\int_{\mathbb{R}^N} u^3(x)\,dx + G_1(x,u;s,b), \quad (4.13)$$

where

$$G_1(x,u;s,b) = \int_0^u \vartheta^4 f_1(x,\vartheta;s,b)\,d\vartheta = u^5\int_0^1 \varrho^4 f_1(x,u\varrho;s,b)\,d\varrho.$$

According to (4.3), (3.15), to obtain $\Phi(\xi,\eta)$, we need to substitute

$$u = \xi\cdot\varphi + \sigma_1(\xi,\eta;s,b), \quad v = \eta\cdot\varphi + \sigma_2(\xi,\eta;s,b) \quad (4.14)$$

in (4.13). Clearly, by Lemma 3.3, after substituting for u, the last 3 terms of (4.13) give

$$\frac{sb}{3}\int_{\mathbb{R}^N} a_2(x)(\xi\cdot\varphi(x))^3\,dx + \frac{b}{4}\int_{\mathbb{R}^N} a_3(x)(\xi\cdot\varphi(x))^4\,dx + \Phi''(\xi,\eta;s,b),$$

where Φ'' has the properties as stated in Lemma 4.8 (the function Φ'', and later Φ_4', will be modified in the course of this proof).

Next we substitute for u in the first integral in (4.13). Remembering that σ_1 takes values in the $L^2(\mathbb{R}^N)$-orthogonal complement of $\operatorname{span}\{\varphi_1,\dots,\varphi_n\}$ (cp. (3.13)) and that both $\operatorname{span}\{\varphi_1,\dots,\varphi_n\}$ and its orthogonal complement are invariant under the operator $A_1 = -\Delta - a_1$, we are left with the following integrals (omitting the argument x of the integrands)

$$\frac{1}{2}\int_{\mathbb{R}^N} \big(-A_1(\xi\cdot\varphi)\big)\big(\xi\cdot\varphi\big)\,dx + \frac{1}{2}\int_{\mathbb{R}^N} \big(-A_1\sigma_1(\xi,\eta;s,b)\big)\sigma_1(\xi,\eta;s,b)\,dx. \quad (4.15)$$

The first of these integrals is equal to

$$-\frac{1}{2}\sum_{j=1}^n \mu_j\xi_j^2,$$

due to the relations $A_1\varphi_j = \mu_j\varphi_j$ and the $L^2(\mathbb{R}^N)$-orthonormality of $\{\varphi_1,\dots,\varphi_n\}$. The second integral in (4.15) is equal to $s^2b^2\Phi_4'(\xi,\eta) + \Phi''(\xi,\eta;s,b)$ for some functions Φ_4', Φ'' as in Lemma 4.8(a). This follows from Lemma 3.3, noting also that

σ being a Z-valued $\mathscr{C}^{K+1}$ function implies that $A_1\sigma_1$ is an H^m-valued function of class $\mathscr{C}^{K+1}$.

Finally, substituting $v = \eta \cdot \varphi + \sigma_2(\xi, \eta; s, b)$ in the second integral in (4.13) and using the orthogonality again, we obtain the following integrals:

$$\frac{1}{2}\int_{\mathbb{R}^N} \big(\eta \cdot \varphi\big)^2 \, dx + \frac{1}{2}\int_{\mathbb{R}^N} \big(\sigma_2(\xi, \eta; s, b)\big)^2. \tag{4.16}$$

A similar argument as above shows that the first of these terms is equal to

$$\frac{1}{2}\sum_{j=1}^{n} \eta_j^2$$

and the second one is equal to $s^2 b^2 \Phi_4'(\xi, \eta) + \Phi''(\xi, \eta; s, b)$ for some functions Φ_4', Φ'' as in Lemma 4.8.

Summing up all the terms obtained above and redefining Φ_4', Φ'', we see that the conclusion of Lemma 4.8 holds. □

The next proposition says that the structure of the Hamiltonian as given in Lemma 4.8 remains unchanged after the Darboux change of coordinates given by Lemma 4.6.

Proposition 4.9. *Given $(s, b) \in \bar{\mathscr{P}}$, consider the change of coordinates $(\xi, \eta) = \phi(\xi', \eta'; s, b)$, where ϕ is as in Lemma 4.6, and let $\Phi(\xi', \eta'; s, b)$ stand for the Hamiltonian Φ in the coordinates (ξ', η') (i.e., the function $\Phi(\phi(\xi', \eta'; s, b); s, b)$). Then there is a neighborhood $\mathscr{V}$ of $(0, 0) \in \mathbb{R}^{2n}$ such that the conclusion of Lemma 4.8 remains valid with (ξ, η) replaced by (ξ', η').*

Proof. Substituting $(\xi, \eta) = \phi(\xi', \eta'; s, b)$ in (4.12) and using Lemma 4.6, it is straightforward to verify that the statement of Lemma 4.8 remains valid (with some new functions Φ_4', Φ'') when (ξ, η) is replaced by (ξ', η'). □

Remark 4.10. The proof of Lemma 4.8 (see in particular formulas (4.15), (4.16)) reveals that the function Φ_4' in (4.12) is determined by the quadratic terms of

$$\sigma(\cdot, \cdot; s, b) = (\sigma_1(\cdot, \cdot; s, b), \sigma_2(\cdot, \cdot; s, b)).$$

When applying the transformation $(\xi,\eta)=\phi(\xi',\eta';s,b)$ in (4.12) one gets further contribution to the new function Φ_4' from the cubic terms of $\phi(\cdot,\cdot;s,b)$ only. By Remark 4.7, this means that Φ_4' is determined only by the coefficient a_2 in the nonlinearity f (and is independent of a_3 and f_1).

4.3 The normal form

We now consider the Hamiltonian Φ in the coordinates (ξ',η'), as in Proposition 4.9. According to that proposition,

$$\Phi(\xi',\eta';s,b)=\frac{1}{2}\sum_{j=1}^{n}(-\mu_j(\xi_j')^2+(\eta_j')^2)+\frac{sb}{3}\int_{\mathbb{R}^N}a_2(x)(\xi'\cdot\varphi(x))^3\,dx$$
$$+\frac{b}{4}\int_{\mathbb{R}^N}a_3(x)(\xi'\cdot\varphi(x))^4\,dx+s^2b^2\Phi_4'(\xi',\eta')+\Phi''(\xi',\eta';s,b),\quad(4.17)$$

where Φ_4', Φ'' are as in Lemma 4.8.

The reduced equation (3.16) written in the coordinates (ξ',η') is the Hamiltonian system corresponding to Φ with respect to the standard symplectic form α. In this section, we will use further changes of coordinates, all of which are *canonical* in the sense that they do not alter the symplectic form α.

The main result of this section is the following proposition.

Proposition 4.11. *Let k_B be an integer with $2\le k_B\le K/2-1$, where K is as in (2.5), and let $\Phi=\Phi(\xi',\eta';s,b)$ be as in (4.17) and Proposition 4.9. For each $(s,b)\in\bar{\mathscr{P}}$ there is a smooth map $\bar\phi:\mathscr{V}\to\mathbb{R}^{2n}$ defined on a neighborhood $\mathscr{V}$ of $(0,0)\in\mathbb{R}^{2n}$ such that the following statements are valid:*

(a) $\bar\phi$ is a diffeomorphism onto its image, it is a canonical transformation, and

$$\bar\phi(\bar\xi,\bar\eta)-(\bar\xi,\bar\eta)=\mathcal{O}(|(\bar\xi,\bar\eta)|^3)\ \text{as}\ (\bar\xi,\bar\eta)\to(0,0).$$

(b) Making the (canonical) change of coordinates

$$(\xi',\eta')=\bar\phi(\bar\xi,\bar\eta),\quad(\bar\xi,\bar\eta):=(\bar\xi_1,\dots,\bar\xi_n,\bar\eta_1,\dots,\bar\eta_n),\qquad(4.18)$$

let $\Phi(\bar{\xi},\bar{\eta})$ *stand for the transformed Hamiltonian (that is,* $\Phi(\bar{\xi},\bar{\eta})$ *is actually the function* $\Phi(\bar{\phi}(\bar{\xi},\bar{\eta});s,b)$*). Then, setting* $I_j = (\bar{\xi}_j^2+\bar{\eta}_j^2)/2$ *and* $I=(I_1,\dots,I_n)$*, we have*

$$\Phi(\bar{\xi},\bar{\eta}) = \omega\cdot I + \Phi_0(I) + \Phi_1(\bar{\xi},\bar{\eta}), \tag{4.19}$$

where Φ_0 *is a polynomial in* I *of degree at most* k_B*, and* Φ_1 *is of class* $\mathscr{C}^K$ *and of order* $\mathcal{O}(|(\bar{\xi},\bar{\eta})|^{2k_B+2})$ *as* $(\bar{\xi},\bar{\eta})\to(0,0)$*.*

(c) Φ_0 *is given by*

$$\Phi_0(I) = \frac{b}{2} I\cdot MI + \frac{s^2b^2}{2} I\cdot \tilde{M}I + \hat{P}(I), \tag{4.20}$$

where $\hat{P}(I)$ *is a polynomial in* I *of degree at most* k_B *with no constant, linear, or quadratic terms, and* M*,* $\tilde{M}$ *are* $n\times n$ *matrices with entries independent of* (s,b) *(the coefficients of* $\hat{P}(I)$ *do depend on* (s,b)*). Moreover, the matrix* M *is given explicitly as follows. Setting*

$$\hat{\Theta}(i,j) = \frac{1}{4\omega_i\omega_j}\int_{\mathbb{R}^N} a_3(x)\varphi_i^2(x)\varphi_j^2(x)dx,$$

the matrix M *is given by*

$$M = 3\begin{bmatrix} \hat{\Theta}(1,1) & 2\hat{\Theta}(1,2) & \dots & 2\hat{\Theta}(1,n) \\ 2\hat{\Theta}(2,1) & \hat{\Theta}(2,2) & & \vdots \\ \vdots & & \ddots & 2\hat{\Theta}(n-1,n) \\ 2\hat{\Theta}(n,1) & \dots & 2\hat{\Theta}(n,n-1) & \hat{\Theta}(n,n) \end{bmatrix}. \tag{4.21}$$

Remark 4.12. (i) The only specific information on the dependence of the transformed Hamiltonian on the parameters s, b that will be needed below is obtained from (4.19), (4.20). Just for the sake of completeness, we add at this point that the precise dependence on s, b of the transformation $\bar{\phi}$—and thus of the transformed Hamiltonian—can be established from the normal-form computations. Namely, the map $\phi = \phi(\bar{\xi},\bar{\eta};s,b)$ is of class $\mathscr{C}^{K-2k_B}$ on $\mathscr{V}\times\bar{\mathscr{P}}$, for some neighborhood $\mathscr{V}$ of $(0,0)\in\mathbb{R}^{2n}$. Indeed, the transformation ϕ is the composition of finitely many transformations – Lie transforms of homogeneous polynomial vector fields of degrees $\ell = 3, 4, \ldots$ The vector field of degree ℓ is determined from the so-called homological equation (see equation (4.24) below), which is a linear nonhomogeneous equation in the finite-dimensional space of homogeneous polynomial vector fields of degree ℓ. The matrix of this linear equation, in suitable coordinates (see (4.27) below), is diagonal and its right-hand side is a homogeneous polynomial whose coefficients are at least of class $\mathscr{C}^{K-2k_B}$ in (s, b). This implies that the corresponding transformation can be chosen of class $\mathscr{C}^{K-2k_B}$.

(ii) The matrix $\tilde{M}$ in (4.20) is determined by the function a_2 and is independent of a_3, f_1 (and s, b). We give an argument for this in Remark 4.14.

Proposition 4.11 shows that, after a canonical transformation, the Hamiltonian Φ is the sum of a polynomial Hamiltonian depending only on I, and terms of high order. In our application of a KAM theorem, the terms depending only on I will be taken as an integrable analytic Hamiltonian, while the high order terms will be considered as a small perturbation. Knowing explicitly the matrix M will allow us to verify a nondegeneracy condition for the KAM theorem.

The proof of Proposition 4.11 consists in taking the Birkhoff normal form of the Hamiltonian Φ up to order $|(\bar{\xi}, \bar{\eta})|^{2k_B+1}$ and computing its terms explicitly up to order $|(\bar{\xi}, \bar{\eta})|^4$.

We start by recalling a basic normal form theorem.

Theorem 4.13. *Let $k_0 \geq 4$ and $k \geq k_0 + 1$ be integers, $\Omega \subset \mathbb{R}^{2n}$ be a domain containing the origin, and $H : \Omega \to \mathbb{R}$ be a $\mathscr{C}^k$ map. Assume that $H = H_2 + P$, where*

$$H_2(\xi, \eta) = \sum_{j=1}^{n} \omega_j \frac{\xi_j^2 + \eta_j^2}{2},$$

P is of order $\mathcal{O}(|(\xi, \eta)|^3)$ as $(\xi, \eta) \to (0, 0)$, and $\omega = (\omega_1, \ldots, \omega_n)$ is nonresonant up to order k_0. Then there exist two neighborhoods $\mathscr{U}$ and $\mathscr{V}$ of 0, and a smooth canonical transformation $\nu : \mathscr{U} \to \mathscr{V}$ mapping $(\bar{\xi}, \bar{\eta}) \in \mathscr{U}$ to $(\xi, \eta) \in \mathscr{V}$ such that

$\nu(\bar\xi,\bar\eta) - (\bar\xi,\bar\eta)$ is of order $\mathcal{O}(|(\bar\xi,\bar\eta)|^2)$ as $(\bar\xi,\bar\eta) \to (0,0)$ and one has

$$H \circ \nu = H_2 + Z + R,$$

where

(a) Z depends on $(\bar\xi,\bar\eta)$ only via $I = (I_1,\dots,I_n)$, with $I_j = (\bar\xi_j^2 + \bar\eta_j^2)/2$, and it is a polynomial in I of degree at most $[k_0/2]$ ($[\cdot]$ stands for the integer part).

(b) R is (of class $\mathscr{C}^k$ and) of order $\mathcal{O}(|(\bar\xi,\bar\eta)|^{k_0+1})$ as $(\bar\xi,\bar\eta) \to (0,0)$.

Proofs of this theorem, including algorithms to find the normal form Z, can be found in many texts on Hamiltonian systems (see [7, 32, 37], for example). The theorem tells us that we can write our Hamiltonian as in (4.19), but to explicitly compute the terms of order four (order 2 in I), we need to recall some steps from the proof, as found in the above references.

If h and g are $\mathscr{C}^2$ functions on a domain in $\mathbb{R}^{2n}$, their Poisson bracket $\{h,g\}$ is defined by

$$\{h,g\} := \sum_{j=1}^{n} \left(\frac{\partial h}{\partial \xi_j}\frac{\partial g}{\partial \eta_j} - \frac{\partial h}{\partial \eta_j}\frac{\partial g}{\partial \xi_j} \right). \tag{4.22}$$

In the proof of Theorem 4.13 one successively eliminates the nonresonant terms (as defined below) in the expansion of H. The cubic terms are all nonresonant and they are eliminated by a first transformation. This transformation alters terms of degree 4 and higher, but does not change the quadratic terms. The next transformation eliminates the nonresonant terms from the (altered) fourth-order terms, keeping the quadratic and cubic terms intact and altering the terms of degree 5 and higher; and so on.

The transformations in this procedure are always found as the Lie transforms corresponding to a polynomial Hamiltonian (which guarantees that they are canonical). The key observation here is as follows. Let χ_ℓ be a homogeneous polynomial on $\mathbb{R}^{2n}$ of degree $\ell \geq 3$ and let ν_ℓ be the time-1 map of the Hamiltonian flow with the Hamiltonian χ_ℓ (ν_ℓ is defined in a neighborhood of the origin and it is a near identity transformation). Let now $H = H_2 + H_3 + \cdots + H_\ell + \text{h.o.t.}$, where H_2 is as in Theorem 4.13, H_j is a homogeneous polynomial of degree j, $j = 2, \ldots, \ell$, and "h.o.t." stands for terms of order greater than ℓ. Then

$$H \circ \nu_\ell = H_2 + H_3 + \cdots + H_\ell + \{H_2, \chi_\ell\} + \text{h.o.t.} \tag{4.23}$$

Thus, if χ_ℓ can be chosen such that

$$\{H_2, \chi_\ell\} = -H_\ell, \tag{4.24}$$

then the terms of degree ℓ can be completely eliminated. This is always possible, with a uniquely determined χ_ℓ, if ℓ is odd. If ℓ is even, only certain terms of degree ℓ, as specified below, can be eliminated by a suitable (nonunique) choice of χ_ℓ.

In the first step of the above procedure, one takes the (unique) solution χ_3 of

$$\{H_2, \chi_3\} = -H_3. \tag{4.25}$$

The corresponding Lie transform ν_3 eliminates the cubic terms and alters the quartic terms as follows (see [7, 32, 37] for details):

$$\begin{aligned} H \circ \nu_3 &= H_2 + H_4 + \frac{1}{2}\{\{H_2, \chi_3\}, \chi_3\} + \{H_3, \chi_3\} + \text{h.o.t.} \\ &= H_2 + H_4 + \frac{1}{2}\{H_3, \chi_3\} + \text{h.o.t.} \end{aligned} \tag{4.26}$$

where "h.o.t." now stands for terms of order 5 or higher and (4.25) was used to get the second equality in (4.26).

Thus, the new degree-four homogeneous polynomial is $H_4 + \frac{1}{2}\{H_3, \chi_3\}$. The second step is to determine which terms in this polynomial can be eliminated by the next transformation ν_4. For this, we use the complex coordinates $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n)$ given by

$$(\alpha_j, \beta_j) = \frac{1}{\sqrt{2}}\big(\xi_j + i\eta_j, i(\xi_j - i\eta_j)\big). \tag{4.27}$$

We remark, without using this fact explicitly below, that when the homological equation (4.24) is rewritten in the coordinates (α, β) and then as a linear system with respect to the basis consisting of the monomials, the coefficient matrix of the system is diagonal. We employ the coordinates (α, β) only to identify the fourth-order terms in (4.26) which are eliminated after the next transformation.

Substituting the inverse relations $\xi_j = \frac{1}{\sqrt{2}}(\alpha_j - i\beta_j)$, $\eta_j = \frac{1}{\sqrt{2}}(\beta_j - i\alpha_j)$, in the Hamiltonian, we obtain a sum of homogeneous polynomials in (α, β) of the same degrees as before the substitution. For the terms of degree 3 we find

$$H_3(\alpha, \beta) = \sum_{\substack{J,L \in \mathbb{N}^2 \\ |J|+|L|=3}} h_3^{JL} \alpha^J \beta^L,$$

for some coefficients h_3^{JL}, which allows us to express χ_3 as

$$\chi_3(\alpha, \beta) = \sum_{\substack{J,L \in \mathbb{N}^2 \\ |J|+|L|=3}} \frac{h_3^{JL}}{i\omega \cdot (L - J)} \alpha^J \beta^L, \tag{4.28}$$

where $J = (j_1, \dots, j_n) \in \mathbb{N}^n$, $L = (\ell_1, \dots, \ell_n) \in \mathbb{N}^n$ are multiindices, $|J| = j_1 + \cdots + j_n$, $\alpha^J = \alpha_1^{j_1} \dots \alpha_n^{j_n}$, and similarly for β^L. For the fourth order term $\tilde{H}_4 := H_4 + \frac{1}{2}\{H_3, \chi_3\}$ in (4.26), we find coefficients h_4^{JL} such that

$$\tilde{H}_4(\alpha, \beta) = \sum_{|J|+|L|=4} h_4^{JL} \alpha^J \beta^L. \tag{4.29}$$

We say a term $h^{JL} \alpha^J \beta^L$ is *resonant* if $\omega \cdot (J - L) = 0$; otherwise, it is nonresonant. Due to the nonresonance assumption on ω, for any $|J| + |L| \leq k_0$, in particular for $|J| + |L| = 4$, the term $h^{JL} \alpha^J \beta^L$ is nonresonant if and only if $J = L$. Now, the crux of the second step consists in choosing a homogeneous polynomial χ_4 (real in the coordinates (ξ, η)) such that the corresponding transformation ν_4 eliminates all nonresonant terms in (4.29) while keeping the resonant terms intact. The final form of the quartic terms in $H \circ \nu_3 \circ \nu_4$ is then obtained by substituting (4.27) in the sum of all the resonant terms,

$$\sum_{|J|=2} h_4^{JJ} \alpha^J \beta^J, \tag{4.30}$$

and noting that for $|J| = 2$ one gets $h_4^{JJ}\alpha^J\beta^J = -h_4^{JJ}I^J$, with $I = (I_1, \dots, I_n)$ as in Theorem 4.13.

To conclude these preparatory remarks, we rewrite (4.30) in a more convenient way. For any $J = (J_1, \dots, J_n)$ with $|J| = 2$, there exist two integers $1 \le j_2 \le j_1 \le n$ such that either $j_2 < j_1$ and

$$J_j = \begin{cases} 1 & \text{if } j = j_1 \text{ or } j = j_2 \\ 0 & \text{otherwise,} \end{cases}$$

or $j_1 = j_2$ and

$$J_j = \begin{cases} 2 & \text{if } j = j_1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, denoting $\bar{h}_4^{j_1,j_2} = h_4^{JJ}$, we have

$$\sum_{|J|=2} h_4^{JJ}\alpha^J\beta^J = \sum_{j_1=1}^{n}\sum_{j_2=1}^{j_1} \bar{h}_4^{j_1,j_2}\alpha_{j_1}\alpha_{j_2}\beta_{j_1}\beta_{j_2}. \tag{4.31}$$

Proof of Proposition 4.11. We apply the above normal form procedure to the Hamiltonian Φ in (4.17). Recalling that $\omega_j = |\mu_j|^{1/2}$, we start with the canonical change of coordinates

$$\xi_j' = \frac{1}{\sqrt{\omega_j}}\xi_j, \quad \eta_j' = \sqrt{\omega_j}\,\eta_j, \tag{4.32}$$

so the quadratic part of Φ becomes

$$\Phi_2(\xi,\eta) := \frac{1}{2}\sum_{j=1}^{n}\omega_j(\xi_j^2 + \eta_j^2),$$

as in Theorem 4.13 (needless to say, (ξ,η) no longer represent the original coordinates on the center manifold). We write the Hamiltonian (4.17) in the coordinates (ξ,η) as follows

$$\Phi(\xi,\eta) = \Phi_2(\xi,\eta) + sb\Phi_3(\xi,\eta) + b\Phi_4(\xi,\eta) + s^2b^2\Phi_4'(\xi,\eta) + \Phi''(\xi,\eta;s,b),$$

where, with $\xi' = (\xi'_1, \dots, \xi'_n)$, $\eta' = (\eta'_1, \dots, \eta'_n)$ and the ξ'_j, η'_j as in (4.32),

$$\begin{aligned} \Phi_3(\xi,\eta) &= \int_{\mathbb{R}^N} \frac{a_2}{3}(\xi'\cdot\varphi)^3\,dx, \\ \Phi_4(\xi,\eta) &= \int_{\mathbb{R}^N} \frac{a_3}{4}(\xi'\cdot\varphi)^4\,dx, \end{aligned} \tag{4.33}$$

and Φ'_4, Φ'' are as in Lemma 4.8 (and (4.17)). Although Φ_3 and Φ_4 are independent of η, for consistency we write them as functions of (ξ,η) anyway.

After the first step of the normal form procedure (cp. (4.25), (4.26)), taking the unique homogeneous cubic polynomial vector field χ_3 satisfying

$$\{\Phi_2, \chi_3\} = -\Phi_3, \tag{4.34}$$

and ν_3 the Lie transform corresponding to $sb\chi_3$, we obtain

$$\Phi\circ\nu_3 = \Phi_2 + \Phi_4 + s^2b^2\Phi'_4(\xi,\eta) + s^2b^2\frac{1}{2}\{\Phi_3,\chi_3\} + \text{h.o.t.}, \tag{4.35}$$

where "h.o.t." stands for terms of order $\mathcal{O}(|(\xi,\eta)|^5)$ (we will not keep track of the parameter dependence in the higher order terms).

After expanding $(\xi'\cdot\varphi)^4$:

$$(\xi'\cdot\varphi)^4 = \sum_{j_1,\dots,j_4=1}^{n} \xi'_{j_1}\xi'_{j_2}\xi'_{j_3}\xi'_{j_4}\varphi_{j_1}\varphi_{j_2}\varphi_{j_3}\varphi_{j_4} = \sum_{j_1,\dots,j_4=1}^{n} \frac{\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4}}{(\omega_{j_1}\omega_{j_2}\omega_{j_3}\omega_{j_4})^{1/2}}\varphi_{j_1}\varphi_{j_2}\varphi_{j_3}\varphi_{j_4},$$

Φ_4 becomes

$$\Phi_4(\xi,\eta) = \frac{1}{4}\sum_{j_1,\dots,j_4=1}^{n} \frac{\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4}}{(\omega_{j_1}\omega_{j_2}\omega_{j_3}\omega_{j_4})^{1/2}} \int_{\mathbb{R}^N} a_3\varphi_{j_1}\dots\varphi_{j_4}dx.$$

Setting

$$\Theta(j_1,j_2,j_3,j_4) = \frac{1}{4(\omega_{j_1}\omega_{j_2}\omega_{j_3}\omega_{j_4})^{1/2}} \int_{\mathbb{R}^N} a_3\varphi_{j_1}\dots\varphi_{j_4}dx,$$

we can write

$$\Phi_4(\xi,\eta) = \sum_{j_1,\dots,j_4=1}^{n} \Theta(j_1,\dots,j_4)\xi_{j_1}\dots\xi_{j_4}. \tag{4.36}$$

As in the above remarks, we use the complex variables (4.27), so

$$\xi_j = \frac{1}{\sqrt{2}}(\alpha_j - i\beta_j), \quad \eta_j = \frac{1}{\sqrt{2}}(\beta_j - i\alpha_j).$$

(This change of variables is used only to identify the resonant terms in (4.36), and we revert to the variables (ξ, η) afterwards.) We must now write the product $\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4}$ in terms of $(\alpha, \beta) = (\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)$. Since resonant terms are given by (4.31), we seek terms of the form $\alpha_j\alpha_\ell\beta_j\beta_\ell$, with $j, \ell \in \{1, \dots, n\}$. One verifies easily that such terms arise from the monomial $\xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4}$ only if $j_{(1)} = j_{(2)}$ and $j_{(3)} = j_{(4)}$, where $((1), (2), (3), (4))$ is a permutation of $(1, 2, 3, 4)$. For any such monomial, we have $j_{(1)} = j_{(2)} = j$ and $j_{(3)} = j_{(4)} = \ell$ for some $j,\ \ell \in \{1, \dots, n\}$ and

$$\begin{aligned} \xi_{j_1}\xi_{j_2}\xi_{j_3}\xi_{j_4} &= \xi_j^2\xi_\ell^2 \\ &= \frac{1}{4}(\alpha_j^2\alpha_\ell^2 - 2i\alpha_j\alpha_\ell^2\beta_j - \alpha_\ell^2\beta_j^2 - 2i\alpha_j^2\alpha_\ell\beta_\ell - 4\alpha_j\alpha_\ell\beta_j\beta_\ell \\ &\qquad + 2i\alpha_\ell\beta_j^2\beta_\ell - \alpha_j^2\beta_\ell^2 + 2i\alpha_j\beta_j\beta_\ell^2 + \beta_j^2\beta_\ell^2). \end{aligned} \tag{4.37}$$

If $j \neq \ell$, the only resonant term in (4.37) is $-\alpha_j\alpha_\ell\beta_j\beta_\ell$. If $j = \ell$, then the resonant terms are $-\alpha_j\alpha_\ell\beta_j\beta_\ell - (1/4)(\alpha_\ell^2\beta_j^2 + \alpha_j^2\beta_\ell^2)$. Thus, for any given j, ℓ, the contribution of $\xi_j^2\xi_\ell^2$ to the resonant terms is given by $-\alpha_j\alpha_\ell\beta_j\beta_\ell$ if $j \neq \ell$ and $-(3/2)\alpha_j^2\beta_j^2$ if $j = \ell$.

Note that if $j = \ell$, then there is only one permutation of (j, j, j, j), whereas if $j \neq \ell$, there are six different permutations of (j, j, ℓ, ℓ); thus, the term ξ_j^4, for j fixed, appears only once in (4.36), while, for $j \neq \ell$ fixed, the term $\xi_j^2\xi_\ell^2 = \xi_\ell^2\xi_j^2$ appears precisely six times.

These remarks imply that, in terms of (α, β),

$$\Phi_4(\xi, \eta) = -\frac{3}{2}\sum_{j=1}^{n}\Theta(j, j, j, j)\alpha_j^2\beta_j^2 - 6\sum_{j=1}^{n}\sum_{\ell=1}^{j-1}(-\Theta(j, j, \ell, \ell))\alpha_j\beta_j\alpha_\ell\beta_\ell + \\ + \text{nonresonant terms.}$$

Since $\alpha_j\beta_j = i(\xi_j^2 + \eta_j^2)/2$,

$$\Phi_4(\xi,\eta) = \frac{3}{2}\sum_{j=1}^{n}\Theta(j,j,j,j)\left(\frac{\xi_j^2+\eta_j^2}{2}\right)^2 + 6\sum_{j=1}^{n}\sum_{\ell=1}^{j-1}\Theta(j,j,\ell,\ell)\left(\frac{\xi_j^2+\eta_j^2}{2}\right)\left(\frac{\xi_\ell^2+\eta_\ell^2}{2}\right) + \text{nonresonant terms.}$$

This can be written, with $I_j = (\xi_j^2+\eta_j^2)/2$, $I = (I_1,\dots,I_n)$, $\hat\Theta(j,\ell) = \Theta(j,j,\ell,\ell)$, as

$$\begin{aligned}\Phi_4(\xi,\eta) &= \frac{3}{2}\sum_{j=1}^{n}\hat\Theta(j,j)I_j^2 + 3\sum_{\substack{j,\ell=1\\ j\neq\ell}}^{n}\hat\Theta(j,\ell)I_jI_\ell + \text{nonresonant terms}\\ &= \frac{1}{2}I\cdot MI + \text{nonresonant terms,}\end{aligned}$$

where M is as in (4.21). Here, under “nonresonant terms” we group the terms which get eliminated after the second transformation in the normal form procedure.

Similarly,

$$s^2b^2\left(\Phi_4'(\xi,\eta) + \frac{1}{2}\{\Phi_3,\chi_3\}\right) = s^2b^2\frac{1}{2}I\cdot\tilde M I + \text{nonresonant terms,}$$

for some $n\times n$ matrix $\tilde M$ determined only by $\Phi_4'(\xi,\eta) + \{\Phi_3,\chi_3\}/2$. Thus, the second transformation results in the quartic terms $(I\cdot MI + s^2b^2 I\tilde M I)/2$, as stated in Proposition 4.11.

The subsequent steps in the normal form procedure do not alter the terms up to order 4. Carrying out the procedure up to order $2k_B+1$, we obtain, as a consequence of Theorem 4.13, all the statements of Proposition 4.11. $\square$

Remark 4.14. It will be useful to note that since the matrix $\tilde M$ is determined only by $\Phi_4'(\xi,\eta) + \{\Phi_3,\chi_3\}/2$, it is independent of a_3. Indeed, Φ_3 and χ_3 are determined by a_2 (see (4.33), (4.34)) and, by Remark 4.10, the same is true of Φ_4'.

Chapter 5

An application of a KAM-type result: proofs of Theorems 2.4, 2.6

In this chapter, we find quasiperiodic solutions of the reduced equation via an application of a KAM-type theorem. The application is rather standard: after the results from the previous chapters, we are dealing with a finite-dimensional Hamiltonian system with an elliptic equilibrium at $(0,0)$ whose frequencies are rationally independent to a high order. The main issue is the verification of a nondegeneracy condition. And, of course, we need a finite-differentiability version of the KAM theorem. We use a theorem by Pöschel [57] for this purpose.

To recall the theorem, consider a Hamiltonian $H : \mathbb{T}^n \times \Omega \to \mathbb{R}$ given by

$$H(\theta, I) = H^0(I) + H^1(\theta, I), \tag{5.1}$$

where $\Omega \subset \mathbb{R}^n$ is a domain, and $\mathbb{T}^n$ is the n-dimensional torus (in other words, $H^1(\theta, I)$ is periodic in $\theta_1, \dots, \theta_n$ with a common period, 2π, say). The Hamiltonian system corresponding to H is

$$\begin{aligned} \dot{\theta} &= \nabla_I H(\theta, I), \\ \dot{I} &= -\nabla_\theta H(\theta, I). \end{aligned} \tag{5.2}$$

We make the assumption that H^0 is analytic on Ω and its *frequency map* $\omega^*(I) := \nabla H^0(I) : \Omega \to \mathbb{R}^n$ is a diffeomorphism onto its image

$$V := \{\omega^*(I) : I \in \Omega\};$$

in particular, the Hessian matrix

$$\frac{\partial^2 H^0}{\partial I^2}(I)$$

is nonsingular on Ω. Moreover, we assume that there is a complex neighborhood Ω^ρ of Ω,

$$\Omega^\rho = \bigcup_{I\in\Omega} \{\zeta \in \mathbb{C}^n : |\zeta - I| < \rho\} \tag{5.3}$$

with $\rho > 0$, such that H^0 has an analytic extension to Ω^ρ whose Hessian is nonsingular on Ω^ρ and $\omega^*(I)$ is a one-to-one map of Ω^ρ onto its image in $\mathbb{C}^n$.

The perturbation term H^1 is assumed sufficiently small (as specified in the theorem, see equation (5.8)) in a Hölder norm: if $\vartheta > 0$ is a noninteger, $\|H\|_{\mathscr{C}^\vartheta(\mathbb{T}^n\times\Omega)}$ is the infimum of all M satisfying the following inequalities:

$$\|D^J H(\theta, I)\|_{L^\infty(\mathbb{T}^n\times\Omega)} \le M \text{ for all } J \in \mathbb{N}^{2n}, |J| \le [\vartheta],$$

and

$$\sup_{\substack{\theta,\theta'\in\mathbb{T}^n\\ \theta\neq\theta'}} \frac{|D^J H(\theta, I) - D^J H(\theta', I)|}{|\theta-\theta'|^{\vartheta-[\vartheta]}} \le M, \quad \sup_{\substack{I,I'\in\Omega\\ I\neq I'}} \frac{|D^J H(\theta, I) - D^J H(\theta, I')|}{|I-I'|^{\vartheta-[\vartheta]}} \le M$$

for all $J \in \mathbb{N}^{2n}$ such that $|J| = [\vartheta]$. Here $[\vartheta]$ is the integer part of ϑ and, for $J = (j_1, \ldots, j_n, \ell_1, \ldots, \ell_n)$,

$$D^J = \frac{\partial^{|J|}}{\partial\theta_1^{j_1}\ldots\partial\theta_n^{j_n}\partial I_1^{\ell_1}\ldots\partial I_n^{\ell_n}}, \quad |J| = j_1 + \cdots + j_n + \ell_1 + \cdots + \ell_n.$$

A vector $\omega \in \mathbb{R}^n$ is said to be κ, ν-*Diophantine*, for some $\kappa > 0$ and $\nu > n-1$, if

$$|\omega \cdot \alpha| \ge \kappa|\alpha|^{-\nu} \quad (\alpha \in \mathbb{Z}^n \setminus \{0\}). \tag{5.4}$$

For $\kappa > 0$ and $\nu > n-1$, we define

$$V_\kappa := \{\omega \in V : \operatorname{dist}(\omega, \partial V) \geq \kappa \text{ and } \omega \text{ is } \kappa, \nu\text{-Diophantine}\}. \tag{5.5}$$

(We only emphasize the dependence on κ of the set V_κ in our notation, since in our proofs ν will always be fixed.)

The following statement is contained in [57, Theorem A].

Theorem 5.1. *Let Ω, H^0, ρ, and V be as above. Suppose additionally that for some $R > 0$ one has*

$$\left|\frac{\partial^2 H^0}{\partial I^2}(I)\right|, \left|\left(\frac{\partial^2 H^0}{\partial I^2}\right)^{-1}(I)\right| \leq R \quad (I \in \Omega^\rho). \tag{5.6}$$

Fix constants λ, ν and α satisfying

$$\lambda > \nu + 1 > n, \quad \alpha > 1, \quad \alpha \notin \{\ell/\lambda + j : j, \ell \in \mathbb{N}\}. \tag{5.7}$$

Then there exists δ_{KAM}, depending on n, ν, λ, ρ, R (but independent of Ω and κ), such that for any $\kappa \in (0, \rho/R)$ and $H^1 \in \mathscr{C}^{\alpha\lambda+\lambda+\nu}(\mathbb{T}^n \times \Omega)$ satisfying

$$\|H^1\|_{\mathscr{C}^{\alpha\lambda+\lambda+\nu}(\mathbb{T}^n \times \Omega)} \leq \kappa^2 \delta_{\mathrm{KAM}} \tag{5.8}$$

the Hamiltonian $H = H^0 + H^1$ has the following property. There exists a diffeomorphism $T : \mathbb{T}^n \times V \to \mathbb{T}^n \times \Omega$ of class $\mathscr{C}^\alpha$ such that for each $I \in \Omega$ with $\omega^*(I) \in V_\kappa$ the manifold $\tilde{\mathbb{T}}_I := T(\mathbb{T}^n \times \omega^*(I))$ is invariant under the flow of (5.2) *and the solution of* (5.2) *with the initial condition $T(\theta_0, \omega^*(I))$, $\theta_0 \in \mathbb{T}^n$, is given by $T(\theta_0 + \omega^*(I)t, \omega^*(I))$, $t \in \mathbb{R}$.*

We remark that [57, Theorem A], besides having additional statements, is more precise in using a weaker norm in (5.8) and giving a stronger (anisotropic) regularity of the transformation T.

The stated property of the diffeomorphism T can be phrased, as it usually is, in the following way: T *conjugates* the flow of (5.2) to a flow for which each torus $\mathbb{T}^N \times \{\bar{\omega}\}$, $\bar{\omega} \in V_\kappa$ is invariant and whose restriction to this torus is a linear flow with frequencies $\bar{\omega}$. The transformation T is not necessarily canonical, but this is of no concern to us.

The theorem provides a class of quasiperiodic solutions of (5.2) whose frequencies cover V_κ. Of course, to use this conclusion, we want $V_\kappa \neq \emptyset$, or, better, $|V_\kappa| > 0$, where $|\cdot|$ stands for the Lebesgue measure.

In an application of the above results, we want to put our Hamiltonian system in the framework of Theorem 5.1. We will be working with the Hamiltonian $\Phi(\bar{\xi}, \bar{\eta})$ as in Proposition 4.11. This is the Hamiltonian of the reduced equation put in the normal form up to a high order (the order is to be specified). We introduce the *action-angle* variables $I = (I_1, \dots, I_n) \in \mathbb{R}^n$, $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{T}^n$ by

$$(\bar{\xi}_j, \bar{\eta}_j) = \sqrt{2I_j}(\cos\theta_j, \sin\theta_j).$$

The change of coordinates from $(\bar{\xi}_j, \bar{\eta}_j)$ to (θ, I) is defined in regions where $I_j = \bar{\xi}_j^2 + \bar{\eta}_j^2 > 0$ for all $j \in \{1, \dots, n\}$, and it is well known that this transformation is canonical. Thus, the relation between the Hamiltonian and the corresponding Hamiltonian system, after both have been written in the (θ, I)-coordinates, is as in (5.2).

The domain Ω we will be working with is

$$\Omega = \Omega_q := \{I \in \mathbb{R}^n : \ q \leq I_j \leq 2q \ \ (j = 1, \dots, n)\} \tag{5.9}$$

where $q > 0$ is sufficiently small, as specified below (we write Ω_q when we want to stress the dependence on q).

In the next lemma, we fix constants α, λ, and ν such that

$$3n > \alpha\lambda + \lambda + \nu \text{ and relations (5.7) hold.} \tag{5.10}$$

One shows easily that such a choice is possible (for example, take α, λ, ν as in (5.7) with $\lambda \approx \nu + 1 \approx n$, $\alpha \approx 1$).

Lemma 5.2. *Suppose the hypotheses* (A1), (A2), (S1), (S2), *and* (NR) *are satisfied. Set* $k_B = [K/2] - 1$, *and let* $\Phi(\bar{\xi}, \bar{\eta})$ *be as in Proposition 4.11 and* M, $\tilde{M}$

as in (4.20). *Assume further that* $(s,b)\in\bar{\mathscr{P}}$ *is such that the following condition is satisfied:*

$$\text{the } n\times n\text{-matrix } M+s^2b\tilde{M} \text{ is nonsingular.} \tag{5.11}$$

With Ω *as in* (5.9), *let* $\Phi(\theta,I)$, $(\theta,I)\in\mathbb{T}^n\times\Omega$, *stand for the Hamiltonian* Φ *in the coordinates* (θ,I). *Fix constants* α, λ, ν *satisfying* (5.10). *Then there exists* $q^*>0$ *such that for each* $q\in(0,q^*)$ *the following statements are valid. One can write* $\Phi(\theta,I)=H^0(I)+H^1(\theta,I)$, *where:*

(a) H^0 *is a polynomial in* I, *and there are* R, $\rho>0$ *such that* (5.6) *holds (with* Ω^ρ *as in* (5.3)*) and the map* $I\mapsto\omega^*(I)=\nabla H^0(I)$ *is one-to-one on* Ω^ρ. *We denote by* V *the image of* Ω *under this map* ω^*.

(b) $H^1\in\mathscr{C}^{\alpha\lambda+\lambda+\nu}(\mathbb{T}^n\times\Omega)$ *and, with* R, ρ *as in statement (a) and* $\delta_{\mathrm{KAM}}=\delta_{\mathrm{KAM}}(n,\nu,\alpha,\rho,R)$ *as in Theorem 5.1, there is* $\kappa\in(0,\rho/R)$ *such that* (5.8) *holds and* $|V_\kappa|>0$ *(*V_κ *is defined in* (5.5)*).*

The choice of functions H^0, H^1 in this statement is naturally given by Proposition 4.11:

$$\begin{aligned} H^0(I)&=\omega\cdot I+b\frac{1}{2}I\cdot MI+\frac{s^2b^2}{2}I\cdot\tilde{M}I+\hat{P}(I),\\ H^1(\theta,I)&=\Phi(\theta,I)-H^0(I). \end{aligned} \tag{5.12}$$

In particular, the frequency map is given by

$$\omega^*(I)=\omega+b(MI+s^2b\tilde{M}I)+\nabla\hat{P}(I), \tag{5.13}$$

where the vector polynomial $\nabla\hat{P}(I)$ has no constant or linear terms.

In preparation for the proof of Lemma 5.2, we estimate the size of the set V_κ, when κ is proportional to q.

Lemma 5.3. *Assume that* (5.11) *holds. Consider the frequency map* (5.13) *on* $\Omega=\Omega_q$ *and let* V *be its range. There exist constants* q^*, $m_1>0$, *and* $C_1>0$ *(independent of* q*) such that*

$$|\{\bar{\omega}\in V\colon \mathrm{dist}(\bar{\omega},\partial V)\geq C_1q\}|\geq m_1q^n \quad (q\in(0,q^*)). \tag{5.14}$$

Proof. For $q > 0$ small, the map ω^* is a bijection from Ω onto V, such that both ω^* and its inverse have a Lipschitz constant independent of q. This is a consequence of (5.13), (5.11). The result follows from this and the following obvious estimate, where $C > 0$ and $\varepsilon > 0$ are independent of q and ε is sufficiently small:

$$|\{I \in \Omega_q : \operatorname{dist}(I, \partial\Omega_q) > \varepsilon q\}| > Cq^n. \qquad \square$$

Lemma 5.4. *Let ω^*, V be as in Lemma 5.3. Then for all sufficiently small $q > 0$, $\kappa > 0$ one has*

$$|V \setminus D(\kappa, \nu)| \leq c\kappa q^{n-1},$$

where $c > 0$ is a constant (independent of q and κ).

Proof. Note that for small $q > 0$ the set V is contained in a ball of radius $2b\|M + s^2 b\tilde{M}\|q\sqrt{n}$, hence also in an n-dimensional cube Q with the edge of length $4b\|M + s^2 b\tilde{M}\|q\sqrt{n}$. This implies (see, for example, [66, Theorem 9.3]) that

$$|Q \setminus D(\kappa, \nu)| \leq c\kappa q^{n-1},$$

where c depends only on n, ν and $b\|M + s^2 b\tilde{M}\|$. Since $V \subset Q$, our assertion follows. $\square$

Proof of Lemma 5.2. Let α, λ, and ν be as in (5.10). Since $K > 6(n+1)$, $k_B := [K/2] - 1$ satisfies

$$K \geq 2k_B + 2 > 6(n+1) > 2([\alpha\lambda + \lambda + \nu] + 1) + 3. \qquad (5.15)$$

Define H^0, H^1 as in (5.12), where the notation comes from Proposition 4.11.

Note, first of all, that H^0 is a polynomial in I. Due to the assumption (5.11), if $q^* > 0$, $\rho > 0$ are sufficiently small and $q \in (0, q^*)$, then H^0 extends to (the same polynomial on) Ω_q^ρ (Ω_q is as in (5.9)), its frequency map ω^* is one-to-one on Ω_q^ρ, and for some R independent of $q \in (0, q^*)$ relations (5.6) hold. This verifies the properties of H^0 stated in (a) with some constants ρ, R, which will no longer be changed.

We denote by V the image of Ω_q under the map $\omega^* = \nabla H^0$.

Remember that in Proposition 4.11, the Birkhoff normal form is taken up to the order $2k_B+1$, so, in the variables $(\bar{\xi}, \bar{\eta})$, H^1 is a $\mathscr{C}^K$ map of order $\mathcal{O}(|(\bar{\xi},\bar{\eta})|^{2k_B+2})$ as $(\bar{\xi},\bar{\eta}) \to (0,0)$. In the variables (θ, I), H^1 is a $\mathscr{C}^K$ map on $\mathbb{T}^n \times \Omega_q$ and, since for $I \in \Omega_q$ one has $I_j > q$ for all j (this controls the singularities introduced by differentiating $\sqrt{I_j}$) and $|I| \leq q\sqrt{n}$, using (5.15) and making q^* smaller, if necessary, we find a positive constant C_2 such that

$$\|H^1\|_{\mathscr{C}^{\alpha\lambda+\lambda+\nu}(\mathbb{T}^n\times\Omega_q)} \leq C_2 q^3 \quad (q \in (0, q^*)). \tag{5.16}$$

Recall further that the frequency map $\omega^* : \Omega_q \to V$ is as in (5.13). Making q^* smaller if necessary, we find constants C_1 and m_1, as in Lemma 5.3, such that (5.14) holds. With c as in Lemma 5.4, we set $C_3 := \min\{C_1, m_1/(2c)\}$ and take $\kappa := C_3 q$. Making q^* smaller again, we achieve that for each $q \in (0, q^*)$ one has $0 < \kappa < \rho/R$ and the estimates in Lemmas 5.3 and 5.4 both apply. This yields $|V \setminus D(\kappa, \nu)| \leq m_1 q^n/2$ and $|\{\bar{\omega} \in V : \operatorname{dist}(\bar{\omega}, \partial V) \geq \kappa\}| \geq m_1 q^n$, thus

$$|V_\kappa| \geq \frac{m_1}{2} q^n.$$

Finally, let $\delta_{\mathrm{KAM}} > 0$ be the constant in (5.8) (independent of κ and Ω). Making q^* smaller one more time, we make the following hold:

$$q^* \leq \frac{C_3^2}{C_2}\delta_{\mathrm{KAM}}.$$

Then, for $q \in (0, q^*)$, $\kappa = C_3 q$, relation (5.16) gives

$$\|H^1\|_{\mathscr{C}^{\alpha\lambda+\lambda+\nu}(\mathbb{T}^n\times\Omega_q)} \leq C_2 q^3 = \frac{C_2}{C_3^2}\kappa^2 q \leq \frac{C_2}{C_3^2}\kappa^2 \frac{C_3^2}{C_2}\delta_{\mathrm{KAM}} = \kappa^2 \delta_{\mathrm{KAM}},$$

so (5.8) is satisfied. Thus all statements in (b) have been verified and the proof is complete. □

We can now give the proofs of Theorems 2.4 and 2.6.

Proof of Theorem 2.4. Under the assumptions of Theorem 2.4, the matrix M in (5.11) is nonsingular, hence (5.11) holds if either $b \neq 0$ is fixed and s is sufficiently small (possibly $s = 0$), or s is fixed and $b \neq 0$ is sufficiently small. Lemma 5.2 tells us that Theorem 5.1, with a suitable choice of the constants, applies to our Hamiltonian Φ in the action-angle variables (θ, I) and, moreover, $|V_\kappa| > 0$. This yields, as noted above, quasiperiodic solutions of the corresponding Hamiltonian system with trajectories contained in $\mathbb{T}^n \times \Omega_q$: there are such quasiperiodic solutions with frequency vector ω^*, for all $\omega^* \in V_\kappa$. Adjusting $q > 0$, we can make these solutions as close to $\mathbb{T}^n \times \{0\}$ as we like.

We now reverse all the changes of variables (action-angle variables, normal form transformation, the Darboux transformation) to get back to the reduced equation (4.8), and find its quasiperiodic solutions $(\xi(y), \eta(y))$ with frequencies covering V_κ. The trajectories of these solutions are contained in a small neighborhood of $0 \in \mathbb{R}^{2n}$. For any such solution, we have

$$\Lambda(\xi(y), \eta(y)) \in \mathscr{N} \quad (y \in \mathbb{R}),$$

where Λ is as in (3.15) and $\mathscr{N}$ is the neighborhood of $0 \in Z = H^{m+2}(\mathbb{R}^N) \times H^{m+1}(\mathbb{R}^N)$ from Theorem 3.1. By Theorem 3.1(b),

$$U(y) = (U_1(y), U_2(y))^T = \xi(y) \cdot \psi + \eta(y) \cdot \zeta + \sigma(\xi(y), \eta(y))$$

is a solution of system (3.7). Letting

$$u(x, y) = U_1(y)(x) = \xi(y) \cdot \varphi(x) + \sigma_1(\xi(y), \eta(y))(x), \tag{5.17}$$

where $\varphi(x) = (\varphi_1(x), \dots, \varphi_n(x))$ and $\sigma = (\sigma_1, \sigma_2)$, we obtain a solution of (2.1). This solution is quasiperiodic in y, in the sense of the definition given in Section 2.2. The frequencies of the quasiperiodic solutions obtained this way cover the set V_κ of positive measure.

It remains to show that the solution $u(x, y)$ in (5.17) decays to 0 as $|x| \to \infty$, uniformly in y. This follows immediately from the fact that the set $\{u(\cdot, y) : y \in \mathbb{R}\}$ is contained in a compact set—continuous image of a torus—in $H^{m+2}(\mathbb{R}^N)$, with $m > N/2$. $\square$

Remark 5.5. (a) The above proof shows that if the standing hypotheses (A1), (S1), (NR), (S2) are satisfied, and (5.11) holds, with matrices M, $\tilde{M}$ as in Proposition 4.11, then the conclusion of Theorem 2.4 holds. The analytic dependence of the matrix in (5.11) on s and b implies that if (5.11) holds for some s with $b \neq 0$ fixed, then it holds for all s, save for isolated values, and, likewise, if it holds for some $b \neq 0$ (with s fixed), then it holds for all $b \neq 0$, save for isolated values.

(b) If the parameters (s, b) are fixed, (5.11) can be viewed as a sufficient condition (assuming also the standing hypotheses (A1), (S1), (NR), (S2)) for the conclusion of Theorem 2.4 to be valid. In fact, (5.11) is a condition on the functions a_3 (which appears in the definition of the matrix M) and a_2, which determines the matrix $\tilde{M}$, see Remark 4.12(ii). If $a_2 = 0$, which is equivalent to taking $s = 0$, then this condition just requires that the matrix M be nonsingular. For $a_2 \neq 0$ the condition is rather implicit and hard to verify without parameters.

(c) The nondegeneracy of the Hessian $D^2H^0(I)$ is called Kolmogorov's nondegeneracy condition. Other nondegeneracy conditions (Arnold's isoenergetic condition, Rüssman's condition) are also known to imply the existence of quasiperiodic solutions for Hamiltonian systems in $\mathbb{R}^{2n}$. Theorems based on such conditions could be used here as well, leading to different sufficient conditions in place of (5.11). However, we stress again that because of the center manifold reduction, only $\mathscr{C}^k$ versions of KAM theorems are applicable in our setting, even when the nonlinearity in the original problem (2.1) is analytic.

Proof of Theorem 2.6. Assume that a_2, f_1 are as in (S2), a_1 is as in (S1), hypotheses (A1), (NR), (A3) hold, and s, b are fixed with $b \neq 0$. As noted in Remark 5.5(a), the conclusion of Theorem 2.4 holds provided $a_3 \in \mathscr{C}_{\mathrm{b}}^{m+1}(\mathbb{R}^N)$ is such that (5.11) holds. Thus, in order to prove Theorem 2.6, all we have to do is show that the set $\mathcal{B}$ of all $a_3 \in \mathscr{C}_{\mathrm{b}}^{m+1}(\mathbb{R}^N)$ for which (5.11) holds is open and dense in $\mathscr{C}_{\mathrm{b}}^{m+1}(\mathbb{R}^N)$.

To stress the dependence on a_3, we write the matrix in (5.11) as $M(a_3) + s^2 b\tilde{M}$ ($\tilde{M}$ is independent of a_3, see Remark 4.12(ii)). Obviously, $M(a_3)$ depends continuously on a_3 which gives the openness of $\mathcal{B}$.

To prove the density, we first find $\tilde{b}_3 \in L^2(\mathbb{R}^N)$ such that

$$\int_{\mathbb{R}^N} \tilde{b}_3(x)\varphi_i^2(x)\varphi_j^2(x)\, dx = \delta_{ij}, \quad i, j \in \{1, \ldots, n\}, \tag{5.18}$$

where δ_{ij} is the Kronecker delta. Such $\tilde{b}_3$ exists, due to the linear independence of the functions $\varphi_i^2\varphi_j^2$, $1 \le i \le j \le n$, since the linear operator

$$b_3 \mapsto \left(\int_{\mathbb{R}^N} b_3\varphi_i^2\varphi_j^2\, dx\right)_{1\le i\le j\le n}$$

is easily verified to be surjective onto $\mathbb{R}^{n(n+1)/2}$.

Next, we approximate $\tilde{b}_3$ by $b_3 \in \mathscr{C}_0^\infty(\mathbb{R}^N)$ so that

$$\left|\int_{\mathbb{R}^N} \tilde{b}_3(x)\varphi_i^2(x)\varphi_j^2(x)\, dx - \int_{\mathbb{R}^N} b_3(x)\varphi_i^2(x)\varphi_j^2(x)\, dx\right| < \varepsilon$$

for all $1 \le i, j \le n$. If ε is sufficiently small, then the matrix $M(b_3)$ is nonsingular: $\det M(b_3) \neq 0$, and we fix such b_3.

Now, for any $a_3 \in \mathscr{C}_{\mathrm{b}}^{m+1}(\mathbb{R}^N)$ and $t > 0$, we have

$$\det(M(a_3 + tb_3)) = \det(M(a_3) + tM(b_3)) = t^n \det\left(\frac{1}{t}M(a_3) + M(b_3)\right) \neq 0$$

if t is sufficiently large. Thus $t \mapsto \det(M(a_3 + tb_3))$ is a nonconstant analytic function, so we can find arbitrarily small $t > 0$ such that $\det(M(a_3 + tb_3)) \neq 0$. This proves the density of $\mathcal{B}$. $\square$

Remark 5.6. Clearly, the above proof works in the radial setting—with the space $\mathscr{C}_{\mathrm{b}}^{m+1}(\mathbb{R}^N)$ replaced by $\mathscr{C}_{\mathrm{rad}}^{m+1}(\mathbb{R}^N)$—if a_1 and the eigenfunctions $\varphi_1, \ldots, \varphi_n$ are radial.

Chapter 6

An application of a KAM-type result using Arnold's condition

This chapter is devoted to obtaining the existence of quasiperiodic solutions of the Hamiltonian system corresponding to the Hamiltonian Φ (derived in Proposition 4.11), using Arnold's isoenergetic nondegeneracy condition, which will be stated below (see (AN)). This will be done by considering a suitable modification of the Hamiltonian Φ for which Kolmogorov's condition (cp. Remark 5.5(c)) holds. Throughout this chapter we do not assume smallness of either s or b in (2.2).

The following proposition and further considerations in this chapter are based on [14].

Proposition 6.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Let $G : \mathbb{T}^n \times \Omega \to \mathbb{R}$ be a Hamiltonian of class $\mathscr{C}^K$ of the form*

$$G(\theta, I) = G^0(I) + G^1(\theta, I),$$

which satisfies $|G(\theta, I)| < 1/4$ for all $(\theta, I) \in \mathbb{T}^n \times \Omega$. Let $\mathscr{G}(\theta, I) := G(\theta, I) + (G(\theta, I))^2$. Write

$$\mathscr{G}(\theta, I) = \mathscr{G}^0(I) + \mathscr{G}^1(\theta, I),$$

where

$$\mathscr{G}^0(I) = G^0(I) + (G^0(I))^2,$$
$$\mathscr{G}^1(\theta, I) = \mathscr{G}(\theta, I) - \mathscr{G}^0(I).$$

Assume $\mathscr{G}$ satisfies the hypotheses of Theorem 5.1, with $\mathscr{G}^0$ and $\mathscr{G}^1$ in lieu of H^0 and H^1, respectively, and $|\mathscr{G}(\theta, I)| < 1/8$ for all $(\theta, I) \in \mathbb{T}^n \times \Omega$. Let $\omega^* = \nabla \mathscr{G}^0$ be the frequency map corresponding to $\mathscr{G}^0$, T be the diffeomorphism from Theorem 5.1 (applied to $\mathscr{G}$), and $I^* \in \Omega$ be such that $\omega^*(I^*) \in V_\kappa$, with V_κ the set defined in (5.5)*. Then there exists a constant c such that the manifold $T\left(\mathbb{T}^n \times \left\{\frac{1}{1+2c}\omega^*(I^*)\right\}\right)$ is invariant under the flow of*

$$\begin{aligned} \dot{\theta} &= \nabla_I G(\theta, I), \\ \dot{I} &= -\nabla_\theta G(\theta, I), \end{aligned} \tag{6.1}$$

and the solution of (6.1) *with the initial condition $T\left(\theta_0, \frac{1}{1+2c}\omega^*(I^*)\right)$, $\theta_0 \in \mathbb{T}^n$, is given by*

$$T\left(\theta_0 + \frac{1}{1+2c}\omega^*(I^*)t, \frac{1}{1+2c}\omega^*(I^*)\right), \quad (t \in \mathbb{R}). \tag{6.2}$$

Note that since $\omega^*(I^*) \in V_\kappa$, the solution in (6.2) is quasiperiodic, with frequency vector $\frac{1}{1+2c}\omega^*(I^*)$. Thus, the proposition implies that the problem of finding quasiperiodic solutions of the Hamiltonian system corresponding to G can be reduced to finding such solutions for the system corresponding to $\mathscr{G}$, as long as $\mathscr{G}$ satisfies the hypotheses of Theorem 5.1. Of course, this result is of interest when the estimate (5.6) does not hold for the Hamiltonian G, in which case we will show that, under some assumptions on G (see (AN) below), the Hamiltonian $\mathscr{G}$ satisfies (5.6).

Remark 6.2. Suppose I^*, $J^* \in \Omega$ are such that $\omega^*(I^*)$, $\omega^*(J^*) \in V_\kappa$, and denote by $c(I^*)$ and $c(J^*)$ the constant from Proposition 6.1 applied to I^* and J^*, respectively. Since the map T from Theorem 5.1 is a diffeomorphism, it is easy to see that the corresponding solutions of (6.1) given by (6.2) are distinct provided

$$\frac{1}{1+2c(I^*)}\omega^*(I^*) \neq \frac{1}{1+2c(J^*)}\omega^*(J^*),$$

in particular, this condition holds if $\omega^*(I^*)$ and $\omega^*(J^*)$ are not multiples of each other.

Proof of Proposition 6.1. Since the manifold $T(\mathbb{T}^n \times \{\omega^*(I^*)\})$ is invariant under the Hamiltonian vector field of $\mathscr{G}$, it is contained in the level set (relative to $\mathscr{G}$) $M_\epsilon := \{(\theta, I) \in \mathbb{T}^n \times \Omega : \mathscr{G}(\theta, I) = \epsilon\}$, for some $\epsilon \in (-1/8, 1/8)$. This set coincides with the ϵ^*-level set of G for $\epsilon^* := (1/2)(-1 + \sqrt{1 + 4\epsilon})$, as found by taking the inverse of the map

$$t \in \left(\frac{\sqrt{2} - 2}{4}, \frac{\sqrt{6} - 2}{4} \right) \mapsto t^2 + t \in (-1/8, 1/8).$$

The gradients of $\mathscr{G}$ and G are related as follows:

$$\nabla \mathscr{G}(\theta, I) = \nabla(G(\theta, I) + (G(\theta, I))^2) = (1 + 2G(\theta, I))\nabla G(\theta, I); \tag{6.3}$$

in particular, when $\nabla \mathscr{G}$ and ∇G are restricted to M_ϵ, one has

$$\nabla \mathscr{G}(\theta, I)\Big|_{M_\epsilon} = (1 + 2\epsilon^*)\nabla G(\theta, I)\Big|_{M_\epsilon}. \tag{6.4}$$

It follows that, up to a multiplicative constant, the Hamiltonian vector fields of G and $\mathscr{G}$ coincide when restricted to M_ϵ.

By Theorem 5.1, the solution of

$$\begin{aligned} \dot{\theta} &= \nabla_I \mathscr{G}(\theta, I), \\ \dot{I} &= -\nabla_\theta \mathscr{G}(\theta, I), \end{aligned} \tag{6.5}$$

with the initial condition $T(\theta_0, \omega^*(I^*))$, is given by $T(\theta_0 + \omega^*(I^*)t, \omega^*(I))$. Using (6.4), it is easy to see that the solution of (6.1) with the initial condition $T(\theta_0, \frac{1}{1+2c}\omega^*(I^*))$ is given by (6.2), with $c = \epsilon^*$. This in turn implies that the manifold $T\left(\mathbb{T}^n \times \left\{\frac{1}{1+2c}\omega^*(I^*)\right\}\right)$ is invariant under the flow of (6.1). □

Remark. The hypotheses $|G(\theta, I)| < 1/4$ for all $(\theta, I) \in \mathbb{T}^n \times \Omega$ is only relevant to ensure $1 + 2G(\theta, I) \neq 0$, and it will clearly hold in our setting by our choice of Ω (cp. (5.9)). Alternatively, one could multiply the term $(G(\theta, I))^2$ (in the definition of $\mathscr{G}$) by $(4 \sup_{(\theta,I)\in\mathbb{T}^n\times\Omega} G(\theta, I))^{-1}$. The sole role of the hypothesis $|\mathscr{G}(\theta, I)| < 1/8$ for all $(\theta, I) \in \mathbb{T}^n \times \Omega$ is to ensure the invertibility of the map $t \mapsto t^2 + t$.

Let Φ, Φ_0 and Φ_1 be as in Proposition 4.11. For $(\theta, I) \in \mathbb{T}^n \times \Omega$, with Ω as in (5.9), let

$$\begin{aligned} G^0(I) &:= \omega \cdot I + \Phi_0(I) \\ G^1(\theta, I) &:= \Phi(\theta, I) - G^0(I) = \Phi_1(\theta, I), \end{aligned} \tag{6.6}$$

and define the matrix

$$\mathcal{M}(I) = \begin{bmatrix} \dfrac{\partial^2 G^0}{\partial I^2}(I) & \dfrac{\partial G^0}{\partial I}(I)^T \\ \\ \dfrac{\partial G^0}{\partial I}(I) & 0 \end{bmatrix} \tag{6.7}$$

(we consider $\dfrac{\partial G^0}{\partial I}$ as a row vector). We make the following assumption on $\mathcal{M}$:

(AN) The $(n+1) \times (n+1)$ matrix $\mathcal{M}(0)$ is nonsingular.

Hypothesis (AN) is called Arnold's nondegeneracy condition, or the isoenergetic nondegeneracy condition.

Define the Hamiltonian

$$\mathscr{G}(\theta, I) = \Phi(\theta, I) + (\Phi(\theta, I))^2. \tag{6.8}$$

This Hamiltonian can be written in the form (5.1) by setting

$$\begin{aligned} H^0(I) &= G^0(I) + (G^0(I))^2 \\ H^1(\theta, I) &= G^1(\theta, I) + 2G^0(I)G^1(\theta, I) + (G^1(\theta, I))^2. \end{aligned} \tag{6.9}$$

We can now state the main result of this chapter:

Theorem 6.3. *Assume hypotheses* (A1), (NR), (S1), (S2), *and* (AN) *are satisfied, and let* $s \in \mathbb{R}$, $b \in \mathbb{R} \setminus \{0\}$ *be arbitrary. Then there exists a solution* $u(x, y)$ *of equation* (2.1) *(with* f *as in* (2.2)*) such that* $u(x, y) \to 0$ *as* $|x| \to \infty$ *uniformly in* y*, and* $u(x, y)$ *is quasiperiodic in* y*. In fact, there is an uncountable family of such quasiperiodic solutions, their frequency vectors forming an uncountable subset of* $\mathbb{R}^n$ *(*n *is as in* (A1)(b)*).*

Remark 6.4. If one is interested in solutions which are radially symmetric in x, one can also consider the version of hypothesis (A1) adapted to this setting, as mentioned in Remark 2.1.

Proof of Theorem 6.3. If $\dfrac{\partial^2 \Phi_0}{\partial I^2}(0)$ is nonsingular, then the result is a direct consequence of Theorem 2.4 and Remark 5.5(b), since (5.11) will hold for all $I \in \Omega$ if $q > 0$ (cp. (5.9)) is sufficiently small.

Now we assume $\dfrac{\partial^2 \Phi_0}{\partial I^2}(0) = \dfrac{\partial^2 G^0}{\partial I^2}(0)$ is singular. For $\mathscr{G}$ as in (6.8) and H^0, H^1 as in (6.9), we will verify the hypotheses of Theorem 5.1. We start by proving the existence of $R > 0$ such that (5.6) holds. Note first that

$$\frac{\partial^2 H^0}{\partial I^2}(I) = (1 + 2G^0(I))\frac{\partial^2 G^0}{\partial I^2}(I) + 2\frac{\partial G^0}{\partial I}(I) \otimes \frac{\partial G^0}{\partial I}(I),$$

where, for vectors $v = (v_1, \dots, v_n)$, $w = (w_1, \dots, w_n) \in \mathbb{R}^n$, $v \otimes w$ is the exterior product of v and w, i.e., $(v \otimes w)_{ij} = v_i w_j$.

Also recall the following determinant identity for block matrices: if A is a $n \times n$ matrix and v, $w \in \mathbb{R}^n$ are (column) vectors, then

$$\begin{vmatrix} A & v \\ w^T & 0 \end{vmatrix} = \det A - \det(A + v \otimes w). \tag{6.10}$$

Applying this identity to the matrix $\mathcal{M}(0)$, and using that $\dfrac{\partial^2 G^0}{\partial I^2}(0)$ is singular,

$$\det \mathcal{M}(0) = -\det\left(\frac{\partial^2 G^0}{\partial I^2}(0) + \frac{\partial G^0}{\partial I}(0) \otimes \frac{\partial G^0}{\partial I}(0)\right).$$

Recalling that $G^0(0) = 0$, we obtain

$$\det \frac{\partial^2 H^0}{\partial I^2}(0) = \det\left(\frac{\partial^2 G^0}{\partial I^2}(0) + 2\frac{\partial G^0}{\partial I}(0) \otimes \frac{\partial G^0}{\partial I}(0)\right) = -2\det \mathcal{M}(0) \neq 0,$$

where the last identity is obtained by applying (6.10) with $2w$ in place of w. Thus, there exists $R > 0$ such that (5.6) holds for H^0 if $q > 0$ is sufficiently small. The verification of (5.8) can be carried out as in Lemma 5.2, using the following remark. The map Φ_1 (as in Proposition 4.11) is a $\mathscr{C}^K$ map of order

$\mathcal{O}(|(\bar{\xi},\bar{\eta})|^{2k_B+2})$ as $(\bar{\xi},\bar{\eta}) \to (0,0)$, so in the variables (θ, I), Φ_1 is a $\mathscr{C}^K$ map on $\mathbb{T}^n \times \Omega_q$ of order $\mathcal{O}(|I|^{k_B+1})$ uniformly in $\theta \in \mathbb{T}^n$. By (6.6) and (6.9), clearly the same applies for $H^1(\theta, I)$.

Taking $q > 0$ smaller if necessary, one has $|\Phi(\theta, I)| < 1/4$ and $|\mathscr{G}(\theta, I)| < 1/8$ for all $(\theta, I) \in \mathbb{T}^n \times \Omega$, so all hypotheses of Proposition 6.1 are satisfied (with $G = \Phi$). In addition, the set V_κ (defined in (5.5)) has positive Lebesgue measure, so there exists an uncountable set $W_\kappa \subset V_\kappa$ such that no two elements of W_κ are multiples of each other. Let Ω^W be the preimage of W_κ via the frequency map $\omega^*(I) = \nabla H^0(I)$. For each $I^* \in \Omega^W$ we apply Proposition 6.1 to find a quasi-periodic solution of the Hamiltonian system corresponding to the Hamiltonian Φ with frequency vector $\frac{1}{1+2c(I^*)}\omega^*(I^*)$, where we denote $c = c(I^*)$ the constant in Proposition 6.1. Since the map $I \in \Omega^W \mapsto \frac{1}{1+2c(I^*)}\omega^*(I^*) \in \mathbb{R}^n$ is injective, the corresponding solutions not only are distinct (cp. Remark 6.2), but their frequency vectors are distinct as well. The rest of the proof is the same as the proof of Theorem 2.4, so we omit the details. $\square$

Chapter 7

Proof of Theorem 2.8

In this chapter we prove the existence of quasiperiodic solutions of (2.3), that is, of the following equation:

$$\Delta u + u_{yy} + a_1(r;s)u + a_2(r;s)u^2 + u^3 g(r,u;s) \quad r \geq 0, \ y \in \mathbb{R}, \tag{2.3}$$

where a_1 and a_2 are sufficiently smooth, radially symmetric functions, $s \in [0, \delta]$ is a parameter, with $\delta > 0$ sufficiently small, and g is a sufficiently smooth function. Our purpose is to find the existence of y-quasiperiodic solutions of (2.3) using the results from Chapter 6. Throughout this chapter we assume hypotheses (A1'), (S1'), (S2'), (NR') (with K, m as in (2.5) and $n = 2$) and (A4) are satisfied.

In order to prove Theorem 2.8, we will show that hypothesis (A4) implies that hypothesis (AN) from Chapter 6 is satisfied by the reduced Hamiltonian corresponding to (2.3) for all $s \in (0, \delta)$ if $\delta > 0$ is sufficiently small. Once this has been established, Theorem 2.8 will be a direct consequence of Theorem 6.3.

We start by noting that the construction from Chapters 3 and 4 applies to equation (2.3) as well, with some minor changes to account for the role of the parameter s, which is different from the role of s in (2.2). We discuss those changes here.

The first difference from the foregoing construction is how the smoothness of the reduction function on the parameter s is obtained. Let $m > N/2$, $X =$

$H^{m+1}_{\mathrm{rad}}(\mathbb{R}^N) \times H^m_{\mathrm{rad}}(\mathbb{R}^N)$, and $Z = H^{m+2}_{\mathrm{rad}}(\mathbb{R}^N) \times H^{m+1}_{\mathrm{rad}}(\mathbb{R}^N)$. Denote $\varphi_j(\cdot; s)$ the eigenfunction of $A_1(s) := -\Delta - a_1(r; s)$ (acting on $L^2_{\mathrm{rad}}(\mathbb{R}^N)$ with domain $H^2_{\mathrm{rad}}(\mathbb{R}^N)$) associated to $\mu_j(s)$, $j \in \{1, 2\}$, normalized in the L^2-norm, and such that $\varphi_j(0; s) > 0$. The center space X_c is now

$$X_c(s) := \{(h, \tilde{h})^T : h, \tilde{h} \in \operatorname{span}\{\varphi_1(\cdot; s), \varphi_2(\cdot; s)\}\}.$$

The abstract form of (2.3) is given by

$$\begin{aligned} \frac{du_1}{dy} &= u_2, \\ \frac{du_2}{dy} &= A_1(s)u_1 - \tilde{f}(u_1; s). \end{aligned} \tag{7.1}$$

We rewrite this further as

$$\frac{du}{dy} = A(s)u + R(u; s), \tag{7.2}$$

where $u = (u_1, u_2)$,

$$\begin{aligned} A(s)(u_1, u_2) &= (u_2, A_1(s)u_1)^T, \\ R(u_1, u_2; s) &= (0, \tilde{f}(u_1; s))^T. \end{aligned} \tag{7.3}$$

Here, for each $s \in (-\delta, \delta)$, $A(s)$ is considered as an operator on X with domain $D(A(s)) = Z$, and R as a $\mathscr{C}^{K+1}$-map from $Z \times (-\delta, \delta)$ to Z.

Because of the s-dependence in the linear operator $A(s)$, we cannot refer to the construction from Chapter 3 for the $\mathscr{C}^{K+1}$-regularity in s—in Theorem 3.1, parameters appear only in the nonlinearity R—and we need to prove the existence of σ differently. We derive it from standard center manifold theorems using the fact that $A(s)$ depends on s in its bounded part only.

Write equation (7.2) in the form

$$\frac{du}{dy} = A_0 u + \bar{R}(u; s), \tag{7.4}$$

where $A_0 := A(0)$ and $\bar{R}(u; s) = (A(s) - A_0)u + R(u; s)$. Due to (S1'), (S2'), $\bar{R} : Z \times (-\delta, \delta) \to Z$ is of class $\mathscr{C}^{K+1}$, just like R. Multiplying $\bar{R}$ by a suitable cutoff function on the Hilbert space $Z\times\mathbb{R}$, one finds a $\mathscr{C}_{\mathrm{b}}^{K+1}$-map $\tilde{R}:Z\times\mathbb{R}\to Z$ having a sufficiently small (global) Lipschitz constant and satisfying $\tilde{R}\equiv\bar{R}$ on a small neighborhood of $(0,0)\in Z\times\mathbb{R}$, say, on $\mathscr{N}\times(-\delta_0,\delta_0)$ ($\mathscr{N}$ is a neighborhood of $0\in Z$ and $\delta_0\in(0,\delta)$). One then applies the global center manifold theorem to equation (7.4) with $\bar{R}$ replaced by $\tilde{R}$, augmented by the "stationary-parameter equation" $ds/dy=0$ (cp. [35, 70]). This yields a $\mathscr{C}_{\mathrm{b}}^{K+1}$-map $\tilde{\sigma}:X_c(0)\times\mathbb{R}\mapsto Z$ taking values in $P_h(0)Z$, such that for each $s\in\mathbb{R}$

$$W^c(s):=\{w+\tilde{\sigma}(w;s):w\in X_c(0)\} \tag{7.5}$$

is the *global center manifold* for (7.4). This means, by definition, that $W^c(s)$ is the set of all points $u_0\in Z$ with the following property: there is a solution $u(y)$ of (7.4) defined for all $y\in\mathbb{R}$ such that $u(0)=u_0$ and

$$\sup_{y\in\mathbb{R}}\|u(y)\|_Z e^{-\epsilon|y|}<\infty \quad (\epsilon>0).$$

In particular, since $u\equiv 0$ is a solution of (7.4) due to the relation $\tilde{R}(0,s)=\bar{R}(0,s)=0$, one has $\tilde{\sigma}(0,s)=0$ for all $s\in(-\delta_0,\delta_0)$. The applicability of [35, 70] to (7.4) is verified in the same way as in Section 3.2.

If $s\geq 0$ and it is small enough, $W^c(s)$ can be written as the graph of a map $\bar{\sigma}(\cdot;s):X_c(s)\mapsto P_h(s)Z$. To find $\bar{\sigma}$, for $w\in X_c(0)$, write $w+\tilde{\sigma}(w;s)$ as

$$w+\tilde{\sigma}(w;s)=P_c(s)(w+\tilde{\sigma}(w;s))+P_h(s)(w+\tilde{\sigma}(w;s)). \tag{7.6}$$

Given any $v\in X_c(s)=P_c(s)Z$, we want to solve the equation

$$P_c(s)w+P_c(s)\tilde{\sigma}(w;s)=v \tag{7.7}$$

for $w\in X_c(0)$. To that goal, define, for any $s\in[0,\delta_0)$,

$$Q(s):=P_c(s)P_c(0)+P_h(s)P_h(0)\in\mathscr{L}(X) \tag{7.8}$$

and note that $Q(0)=I_X$—the identity on X, and $Q(s)w=P_c(s)w$ for $w\in X_c(0)$ (in particular, $Q(s)$ takes $X_c(0)$ to $X_c(s)$). As mentioned above, $P_c(s)\in\mathscr{L}(X)$

is of class $\mathscr{C}^{K+1}$ in $s \in (-\delta_0, \delta_0)$, hence $Q(s) \in \mathscr{L}(X)$ is such as well. It follows that for sufficiently small $s \geq 0$ (say, for $s \in [0, \delta_1)$, with some $\delta_1 \in (0, \delta_0]$), the inverse $Q^{-1}(s) \in \mathscr{L}(X)$ exists and is of class $\mathscr{C}^{K+1}$ in s. For such s and for any $v \in X_c(s)$, equation (3.2) can be equivalently written as

$$w = Q^{-1}(s)P_c(s)v - Q^{-1}(s)P_c(s)P_h(0)\tilde{\sigma}(w; s) \tag{7.9}$$

(we have used the relations $Q(s)w = P_c(s)w$, $P_c(s)v = v$, and $\tilde{\sigma}(w; s) = P_h(0)\tilde{\sigma}(w; s)$). Since $\tilde{\sigma}$ is of class $\mathscr{C}_{\mathrm{b}}^{K+1}$ and $P_c(0)P_h(0) = 0$, we observe that if $\delta_2 \in (0, \delta_1)$ is small enough, then the map on the right-hand side of (7.9) is a 1/2-contraction in $w \in X_c(0)$—assuming the norm from X on $X_c(0)$—for all $s \in [0, \delta_2)$ and $v \in X$ (not just $v \in X_c(s)$). The uniform contraction principle implies that equation (7.9) has unique solution $w \in X_c(0)$ given by

$$w = \Upsilon(v, s), \tag{7.10}$$

where $\Upsilon : X \times (-\delta_2, \delta_2) \to X_c(0)$ is a $\mathscr{C}^{K+1}$ map. We now define $\bar{\sigma}$ by

$$\bar{\sigma}(v; s) := P_h(s)(\Upsilon(v, s) + \tilde{\sigma}(\Upsilon(v, s); s)). \tag{7.11}$$

Clearly, $\bar{\sigma} : X \times [0, \delta_2) \to Z$ is of class $\mathscr{C}^{K+1}$ and, by (7.6),

$$W^c(s) = \{w + \tilde{\sigma}(w; s) : w \in X_c(0)\} = \{v + \bar{\sigma}(v; s) : v \in X_c(s)\}. \tag{7.12}$$

To conclude, define $\sigma : \mathbb{R}^4 \times [0, \delta_2) \to Z$ by

$$\sigma(\xi, \eta; s) := \bar{\sigma}(\xi \cdot \psi(s) + \eta \cdot \zeta(s); s) \quad ((\xi, \eta) \in \mathbb{R}^4,\ s \in [0, \delta_2)), \tag{7.13}$$

with $\psi(s)$, $\zeta(s)$ given by $\psi(s) = (\psi_1(s), \psi_2(s))$, $\zeta(s) = (\zeta_1(s), \zeta_2(s))$, for

$$\psi_j(\cdot; s) = (\varphi_j(\cdot; s), 0)^T, \quad \zeta_j(\cdot; s) = (0, \varphi_j(\cdot; s))^T. \tag{7.14}$$

Since the functions $\varphi_j(\cdot; s) \in H^{m+2}(\mathbb{R}^N)$ are of class $\mathscr{C}^{K+1}$ in s, σ is of class $\mathscr{C}^{K+1}$. It is straightforward to verify the statements of Theorem 3.1 as in Section 3.2.

The rest of the construction follows as in Chapter 4, only without distinguishing terms which are small in the parameters s or b from (2.2). (Note that the role of the parameter s in (2.2) is different from the role of s in (2.3).) After the Darboux transformation (cp. Lemma 4.6), the Hamiltonian of the reduced equation takes the form

$$\Phi(\xi',\eta';s) = \frac{1}{2}\sum_{j=1}^{2}(-\mu_j(\xi_j')^2 + (\eta_j')^2) + \frac{1}{3}\int_{\mathbb{R}^N} a_2(\xi'\cdot\varphi)^3 dx + \\ + \Phi_4(\xi',\eta';s) + \Phi''(\xi',\eta';s), \quad (7.15)$$

where $\mu_j = \mu_j(s)$, $\varphi_j = \varphi_j(\cdot;s)$ for $j \in \{1,2\}$, Φ_4 is a homogeneous polynomial in (ξ',η') of degree 4 and Φ'' is of class $\mathscr{C}^K$ on (ξ',η'), of order $\mathcal{O}(|(\xi',\eta')|^5)$ uniformly for $s \in [0,\delta]$.

Lastly, Proposition 4.11 is modified as follows:

Proposition 7.1. *Let k_B be an integer with $2 \le k_B \le K/2 - 1$, where K is as in* (2.5)*, and let $\Phi = \Phi(\xi',\eta';s)$ be the Hamiltonian in* (7.15)*, that is, the Hamiltonian of the reduced equation corresponding to* (2.3)*, written in Darboux coordinates as in Lemma 4.6. Let $\omega = \omega(s)$ be the vector defined in* (NR')*. Then for each $s \in (0,\delta]$ there is a smooth map $\bar\phi : \mathscr{V} \to \mathbb{R}^4$ defined on a neighborhood $\mathscr{V}$ of $(0,0) \in \mathbb{R}^4$ such that the following statements are valid:*

(a) $\bar\phi$ is a diffeomorphism onto its image, it is a canonical transformation, and

$$\bar\phi(\bar\xi,\bar\eta) - (\bar\xi,\bar\eta) = \mathcal{O}(|(\bar\xi,\bar\eta)|^3) \text{ as } (\bar\xi,\bar\eta) \to (0,0).$$

(b) Making the (canonical) change of coordinates

$$(\xi',\eta') = \bar\phi(\bar\xi,\bar\eta), \quad (\bar\xi,\bar\eta) := (\bar\xi_1,\bar\xi_2,\bar\eta_1,\bar\eta_2), \quad (7.16)$$

let $\Phi(\bar\xi,\bar\eta;s)$ stand for the transformed Hamiltonian (that is, $\Phi(\bar\xi,\bar\eta;s)$ is actually the function $\Phi(\bar\phi(\bar\xi,\bar\eta);s)$). Then, setting $I_j = (\bar\xi_j^2 + \bar\eta_j^2)/2$ and $I = (I_1, I_2)$, we have

$$\Phi(\bar\xi,\bar\eta) = \omega\cdot I + \Phi_0(I) + \Phi_1(\bar\xi,\bar\eta), \quad (7.17)$$

where Φ_0 is a polynomial in I of degree at most k_B, and Φ_1 is of class $\mathscr{C}^K$ and of order $\mathcal{O}(|(\bar\xi,\bar\eta)|^{2k_B+2})$ as $(\bar\xi,\bar\eta) \to (0,0)$.

(c) Φ_0 *is given by*

$$\Phi_0(I) = \frac{1}{2} I \cdot MI + \hat{P}(I), \tag{7.18}$$

where $\hat{P}(I)$ *is a polynomial in* I *of degree at most* k_B *with no constant, linear, or quadratic terms, and* M *is a symmetric* 2×2 *matrix with entries depending continuously on* s *for all* $s \in (0, \delta)$.

Our purpose is to understand the asymptotic behavior of the components of the matrix M (in (7.18)) as $s \to 0$ or, equivalently, as $\omega_2 \to 0$, the equivalence being a direct consequence of (A1')(b) and Remark 2.3. In order to do this, we need to study the asymptotic behavior of the terms of degree 4 introduced by the Birkhoff normal form computation, more precisely, the terms introduced in the first step of the algorithm (see (7.23) below).

We begin with some preliminary computations. Let $\Phi(\xi', \eta'; s)$ be the Hamiltonian from (7.15), and denote by Φ_2, Φ_3 and Φ_4 all the terms in Φ of degree 2, 3 and 4, respectively, in (ξ', η').

Remark 7.2. Note that the coefficients of $\Phi_j(\xi', \eta')$, $j \in \{2, 3, 4\}$, are uniformly bounded for $s \in [0, \delta]$.

Consider the change of variables

$$\xi'_j = \frac{1}{\sqrt{\omega_j}} \xi_j, \quad \eta'_j = \sqrt{\omega_j}\, \eta_j,$$

for $j \in \{1, 2\}$. Here ω is the vector defined in (NR'). (Of course, (ξ, η) no longer represent the original coordinates on the center manifold.) The quadratic part of Φ becomes

$$\Phi_2(\xi, \eta; s) := \frac{1}{2} \sum_{j=1}^{2} \omega_j (\xi_j^2 + \eta_j^2).$$

Remark. In an abuse of notation, we denote by $\Phi_2(\xi, \eta; s)$ the function $\Phi_2(\xi(\xi'), \eta(\eta'); s)$, and similarly for other functions.

We now write the cubic terms of Φ, that is,

$$\Phi_3(\xi, \eta; s) = \int_{\mathbb{R}^N} \frac{a_2}{3} (\xi' \cdot \varphi)^3 \, dx, \tag{7.19}$$

explicitly in terms of (ξ,η):

$$(\xi'\cdot\varphi)^3=\sum_{j_1,j_2,j_3=1}^{2}\xi'_{j_1}\xi'_{j_2}\xi'_{j_3}\varphi_{j_1}\varphi_{j_2}\varphi_{j_3}=\sum_{j_1,j_2,j_3=1}^{2}\frac{\xi_{j_1}\xi_{j_2}\xi_{j_3}}{(\omega_{j_1}\omega_{j_2}\omega_{j_3})^{1/2}}\varphi_{j_1}\varphi_{j_2}\varphi_{j_3},$$

so

$$\Phi_3(\xi,\eta;s)=\frac{1}{3}\sum_{j_1,j_2,j_3=1}^{2}\frac{\xi_{j_1}\xi_{j_2}\xi_{j_3}}{(\omega_{j_1}\omega_{j_2}\omega_{j_3})^{1/2}}\int_{\mathbb{R}^N}a_2\varphi_{j_1}\varphi_{j_2}\varphi_{j_3}dx.$$

(Even though Φ_3 is independent of η, for consistency we write it as a function of (ξ,η) anyway.) For $j,k,\ell\in\{1,2\}$, set

$$\Theta_3(j,k,\ell;s)=\frac{1}{3(\omega_j\omega_k\omega_\ell)^{1/2}}\int_{\mathbb{R}^N}a_2\varphi_j\varphi_k\varphi_\ell dx, \tag{7.20}$$

so

$$\Phi_3(\xi,\eta;s)=\sum_{j,k,\ell=1}^{2}\Theta_3(j,k,\ell;s)\xi_j\xi_k\xi_\ell, \tag{7.21}$$

Lemma 7.3. *Let $j,k,\ell\in\{1,2\}$. As $\omega_2\to 0$ (that is, as $s\to 0$),*

$$\Theta_3(j,k,\ell;s)=\mathcal{O}(\omega_2^{-(j+k+\ell-3)/2}).$$

In particular,

$$\Theta_3(2,2,2;s)=\mathcal{O}\big(\omega_2^{-3/2}\big),$$

and if $(j,k,\ell)\neq(2,2,2)$, then

$$\Theta_3(j,k,\ell;s)=\mathcal{O}\big(\omega_2^{-1}\big).$$

Proof. By the continuity of the maps $s\in[0,\delta]\mapsto a_2(\cdot;s)\in\mathscr{C}_{\mathrm{b}}^{m+1}$ and $s\in[0,\delta]\mapsto\varphi_j(\cdot;s)\in L^p_{\mathrm{rad}}(\mathbb{R}^N)$ for $1\le p\le\infty$ and $j=1,2$, it follows that the integral on the right hand side of (7.20) is bounded. Since the negative eigenvalues of $-\Delta-a_1(r;s)$ are isolated, $\mu_1(s)$ stays away from 0 for all $s\in[0,\delta]$, therefore, there exists a positive constant C such that $\omega_1(s)\ge C>0$ for all $s\in[0,\delta]$, and our assertions follow. □

Using a similar reasoning, combined with Remark 7.2, one can prove the following result:

Corollary 7.4. *The coefficients of the polynomial* $\Phi_4(\xi, \eta; s)$ *are of order* $\mathcal{O}(\omega_2^{-2})$ *as* $s \to 0$.

Recall that the first transformation of the Birkhoff normal form algorithm eliminates all terms of degree 3 in (ξ, η). Let $s > 0$, and let $\chi_3 = \chi_3(\xi, \eta; s)$ be the unique solution of

$$\{\Phi_2, \chi_3\} = -\Phi_3 \tag{7.22}$$

(this is the analogue of (4.25) for the Hamiltonian Φ). If ν_3 is the time-1 map generated by χ_3, (4.26) reads

$$\Phi \circ \nu_3 = \Phi_2 + \Phi_4 + \frac{1}{2}\{\Phi_3, \chi_3\} + \text{h.o.t.}, \tag{7.23}$$

where "h.o.t." stands for terms of order $\mathcal{O}(|(\xi, \eta)|^5)$.

We now consider the change of coordinates (4.27). Denote by $\Phi_2(\alpha, \beta)$ the function $\Phi_2(\xi(\alpha, \beta), \eta(\alpha, \beta))$, and similarly for Φ, Φ_3, and others. These functions, as well as ω_1 and ω_2, depend on s, but for simplicity we suppress the dependence in the notation.

In the following lemma we study the Hamiltonian resulting from the first step of the Birkhoff normal form algorithm. Recall that in the second step of the computation all *nonresonant* terms (see the definition in the paragraph after (4.29)) of degree 4 are eliminated, while resonant terms remain unchanged. We are primarily interested in the asymptotic behavior (as $s \to 0$) of the resonant terms.

Lemma 7.5. *For each* $s \in (0, \delta]$,

$$\Phi \circ \nu_3 = \Phi_2(\alpha, \beta) + \frac{5}{12\omega_2^4}\left(\int_{\mathbb{R}^N} a_2(x; s)\varphi_2^3(x; s)dx\right)^2 \alpha_2^2\beta_2^2 + \tilde{\Phi}(\alpha, \beta) + \\ + \textit{nonresonant terms} + \textit{h.o.t.}, \tag{7.24}$$

where $\tilde{\Phi}$ *is a homogeneous polynomial in* (α, β) *of degree 4, with coefficients of order* $\mathcal{O}(\omega_2^{-7/2})$ *as* $\omega_2 \to 0$ *(that is, as* $s \to 0$*), and "h.o.t." stands for terms of order* $\mathcal{O}(|(\alpha, \beta)|^5)$.

Proof. Let χ_3 be the unique solution of (7.22). If ν_3 is the time-1 map generated by χ_3, then (7.23) holds, where the coefficients of Φ_4 are of order $\mathcal{O}(\omega_2^{-2})$ as $\omega_2 \to 0$ by Corollary 7.4, and "h.o.t." stands for terms of order $\mathcal{O}(|(\bar{\xi}, \bar{\eta})|^5)$ as $(\xi, \eta) \to (0, 0)$.

Our interest will be focused on the term $(1/2)\{\Phi_3, \chi_3\}$ in (7.23). Using Lemma 7.3 and (7.21), we can write

$$\Phi_3(\xi, \eta) = \Theta_3(2, 2, 2)\xi_2^3 + \Phi_3' =: \bar{\Phi}_3 + \Phi_3', \tag{7.25}$$

where $\bar{\Phi}_3(\xi, \eta) = \Theta_3(2, 2, 2)\xi_2^3$, and Φ_3' is a homogeneous polynomial in (ξ, η) of degree 3, whose coefficients are of order $\mathcal{O}(\omega_2^{-1})$ as $\omega_2 \to 0$ by Lemma 7.3. Equation (7.22) can be rewritten as

$$\{\Phi_2, \chi_3\} = -\bar{\Phi}_3 - \Phi_3'.$$

We can find unique $\bar{\chi}_3$ and χ_3', both homogeneous polynomials in (ξ, η) of degree 3, such that

$$\{\Phi_2, \bar{\chi}_3\} = -\bar{\Phi}_3, \tag{7.26}$$

$$\{\Phi_2, \chi_3'\} = -\Phi_3', \tag{7.27}$$

so

$$\chi_3(\xi, \eta) = \bar{\chi}_3(\xi, \eta) + \chi_3'(\xi, \eta) \tag{7.28}$$

is the unique solution of (7.22).

Recall that if $J = (j_1, j_2) \in \mathbb{N}^2$ is a multiindex, we write $\alpha^J = \alpha_1^{j_1}\alpha_2^{j_2}$. When the homological equation $\{\Phi_2, \chi_3\} = \psi$ is expressed in terms of the monomials $\alpha^J\beta^L$, with

$$J,\ L \in \mathbb{N}^2, \quad |J| + |L| = 3, \tag{7.29}$$

the operator $\{\Phi_2, \cdot\}$ has a diagonalizable matrix, which is similar to $\mathrm{diag}(i\omega \cdot (L - J))_{|J|+|L|=3}$. The transition matrix from the basis $\{\xi^J\eta^L : |J| + |L| = 3\}$ to the basis $\{\alpha^J\beta^L : |J| + |L| = 3\}$ is independent of ω_1, ω_2, thus, inverting the matrix of the linear operator $\{\Phi_2; \cdot\}$ introduces a singularity of order at most $\mathcal{O}(\omega_2^{-1})$.

These remarks, together with (7.27), imply that the coefficients of the polynomial χ_3' are of order $\mathcal{O}(\omega_2^{-2})$ as $s \to 0$.

In the coordinates (α, β) defined in (4.27),

$$\begin{aligned}\bar{\Phi}_3(\alpha,\beta) &= \frac{\Theta_3(2,2,2)}{\sqrt{2^3}}(\alpha_2 - i\beta_2)^3 \\ &= \frac{\Theta_3(2,2,2)}{2\sqrt{2}}(\alpha_2^3 - 3i\alpha_2^2\beta_2 - 3\alpha_2\beta_2^2 + i\beta_2^3). \end{aligned} \tag{7.30}$$

We use (4.28) to find $\bar{\chi}_3$ —by dividing each term $\alpha^J\beta^L$ in $\bar{\Phi}_3$ by $i\omega \cdot (L - J)$, with $J = (j_1, j_2)$, $L = (\ell_1, \ell_2)$ as in (7.29). Note that none of the terms on the right hand side of (7.30) involves α_1 or β_1, so $j_1 = \ell_1 = 0$, and the condition $|J| + |L| = 3$ becomes $j_2 + \ell_2 = 3$, hence, $i\omega \cdot (L - J) = i\omega_2(\ell_2 - j_2)$. Thus,

$$\bar{\chi}_3(\alpha,\beta) = \frac{\Theta_3(2,2,2)}{2\sqrt{2}i\omega_2}\left(\frac{-1}{3}\alpha_2^3 + 3i\alpha_2^2\beta_2 - 3\alpha_2\beta_2^2 + \frac{i}{3}\beta_2^3\right).$$

In particular, by Lemma 7.5, $\bar{\chi}_3$ is a homogeneous polynomial in (α, β) of degree 3 whose coefficients are of order $\mathcal{O}(\omega_2^{-5/2})$ as $\omega_2 \to 0$.

Using the formulas for $\bar{\Phi}_3$ and $\bar{\chi}_3$, it follows that

$$\begin{aligned}\{\bar{\Phi}_3, \bar{\chi}_3\} &= \frac{\partial\bar{\Phi}_3}{\partial\alpha_2}\frac{\partial\bar{\chi}_3}{\partial\beta_2} - \frac{\partial\bar{\chi}_3}{\partial\alpha_2}\frac{\partial\bar{\Phi}_3}{\partial\beta_2} \\ &= \frac{\Theta_3(2,2,2)^2}{8i\omega_2}\big[(3\alpha_2^2 - 6i\alpha_2\beta_2 - 3\beta_2^2)(3i\alpha_2^2 - 6\alpha_2\beta_2 + i\beta_2^2) - \\ &\qquad (-\alpha_2^2 + 6i\alpha_2\beta_2 - 3\beta_2^2)(-3i\alpha_2^2 - 6\alpha_2\beta_2 + 3i\beta_2^2)\big]\end{aligned}$$

$$\begin{aligned}&= \frac{\Theta_3(2,2,2)^2}{8i\omega_2}[3i + 36i - 9i - (-3i - 36i + 9i)]\alpha_2^2\beta_2^2 + \\ &\qquad + \text{nonresonant terms} \\ &= \frac{15\Theta_3(2,2,2)^2}{2\omega_2}\alpha_2^2\beta_2^2 + \text{nonresonant terms} \\ &= \frac{15}{18\omega_2^4}\left(\int_{\mathbb{R}^N} a_2(x)\varphi_2^3(x)dx\right)^2 \alpha_2^2\beta_2^2 + \text{nonresonant terms,}\end{aligned}$$

where we have used (7.20) and the fact that the resonant terms are of the form (4.31). Using (7.25) and (7.28),

$$\frac{1}{2}\{\Phi_3,\chi_3\} = \frac{1}{2}\left(\{\bar{\Phi}_3,\bar{\chi}_3\} + \{\bar{\Phi}_3,\chi_3'\} + \{\Phi_3',\bar{\chi}_3\} + \{\Phi_3',\chi_3'\}\right).$$

From our previous observations, we can easily find the asymptotic behavior of the last three terms on the right hand side: the coefficients of the polynomials $\{\bar{\Phi}_3,\chi_3'\}$ and $\{\Phi_3',\bar{\chi}_3\}$ are of order $\mathcal{O}(\omega_2^{-7/2})$ as $\omega_2 \to 0$, while the coefficients of $\{\Phi_3',\chi_3'\}$ are of order $\mathcal{O}(\omega_2^{-3})$. Setting

$$\tilde{\Phi} = \frac{1}{2}\left(\{\bar{\Phi}_3,\chi_3'\} + \{\Phi_3',\bar{\chi}_3\} + \{\Phi_3',\chi_3'\}\right) + \Phi_4,$$

all statements of the lemma are satisfied and the proof is complete. □

As in (6.6), define

$$\begin{aligned} G^0(I;s) &:= \omega(s)\cdot I + \Phi_0(I;s) \\ G^1(\theta,I;s) &:= \Phi(\theta,I;s) - G^0(I;s) = \Phi_1(\theta,I;s), \end{aligned} \tag{7.31}$$

where Φ, Φ_0 and Φ_1 are as in Proposition 7.1. Using the conclusion of Lemma 7.5, in the next lemma we study the determinant of the matrix

$$\mathcal{M}(s)(I) = \begin{bmatrix} \dfrac{\partial^2 G^0}{\partial I^2}(I;s) & \dfrac{\partial G^0}{\partial I}(I;s)^T \\ \\ \dfrac{\partial G^0}{\partial I}(I;s) & 0 \end{bmatrix} \tag{7.32}$$

Lemma 7.6. *Let $M = (m_{ij})$, $i,j = 1,2$, be the matrix from* (7.18)*, and let $\mathcal{M}(s)(I)$ be the matrix in* (7.32)*, defined for $I \in \Omega$, where $\Omega = \Omega_q$ is as in* (5.9)*, with $q > 0$ sufficiently small. If $\delta > 0$ is sufficiently small, then*

$$\det \mathcal{M}(s)(0) = \begin{vmatrix} m_{11} & m_{12} & \omega_1 \\ m_{21} & m_{22} & \omega_2 \\ \omega_1 & \omega_2 & 0 \end{vmatrix} \neq 0$$

for all $s \in (0,\delta)$.

Proof. Note that the matrix M in (7.18) is determined by the first two steps of the Birkhoff normal form algorithm, since the third and subsequent steps do not alter terms of degree less or equal than 4 (in (ξ', η')). This is straightforward from remarks after (4.22). If G^0 is as in (7.31), it is easy to see that

$$\left.\frac{\partial G^0}{\partial I}\right|_{I=0} = (\omega_1, \omega_2), \qquad \left.\frac{\partial^2 G^0}{\partial I^2}\right|_{I=0} = M,$$

where $I = (I_1, I_2)$ is as in statement (b) of Proposition 7.1. Note that

$$\frac{1}{2} I \cdot MI = \frac{1}{2}\left(m_{11} I_1^2 + (m_{12} + m_{21}) I_1 I_2 + m_{22} I_2^2\right).$$

Also, $m_{12} = m_{21}$, and if (α_j, β_j), $j \in \{1, 2\}$, are as in (4.27), then

$$\alpha_j \beta_j = i(\xi_j^2 + \eta_j^2)/2 = i I_j.$$

The asymptotic behavior of the coefficient m_{22} is obtained from Lemma 7.5. More precisely, a term of the form $\alpha_2^2 \beta_2^2$ is present in (7.24) either in the second term of the right hand side (and the coefficient is explicitly known) or in $\tilde{\Phi}(\alpha, \beta)$, in which case its coefficient is of order $\mathcal{O}(\omega_2^{-7/2})$ by Lemma 7.5. Thus, m_{22} can be written as

$$m_{22} = -\frac{5}{12\omega_2^4}\left(\int_{\mathbb{R}^N} a_2(x; s) \varphi_2^3(x; s) dx\right)^2 + \tilde{m}_{22}, \tag{7.33}$$

where $\tilde{m}_{22}$ is of order $\mathcal{O}(\omega_2^{-7/2})$ as $\omega_2 \to 0$. The integral in (7.33) depends continuously on s, thus, by hypothesis (A4), it is nonzero for all $s \in [0, \delta]$ if $\delta > 0$ is sufficiently small. Since the terms $m_{11} I_1^2$ and $2 m_{12} I_1 I_2$ are contained in $\tilde{\Phi}$ in (7.24), Lemma 7.5 implies that m_{11} and m_{12} are of order $\mathcal{O}(\omega_2^{-7/2})$ as $\omega_2 \to 0$.

Expanding the determinant $\det \mathcal{M}(s)(0)$ along the last row,

$$\begin{aligned}\det \mathcal{M}(s)(0) &= \omega_1(m_{12}\omega_2 - m_{22}\omega_1) - \omega_2(m_{11}\omega_2 - m_{21}\omega_1) \\ &= -m_{22}\omega_1^2 + \omega_1\omega_2(m_{12} + m_{21}) - m_{11}\omega_2^2.\end{aligned}$$

Since $m_{12} = m_{21}$, m_{11} and m_{12} are $\mathcal{O}(\omega_2^{-7/2})$ as $s \to 0$, and $\omega_1(s) \geq C > 0$ for all $s \in (0, \delta)$, using (7.33) we find

$$\det \mathcal{M}(s)(0) = -\omega_1^2 m_{22} + \mathcal{O}(\omega_2^{-7/2}) = \psi(s)\omega_2^{-4} + \mathcal{O}(\omega_2^{-7/2}),$$

where $\psi(s)$ is a positive function such that, for some positive constant $C > 0$, $\psi(s) \geq C > 0$ for all $s > 0$ sufficiently small. It follows that $\det \mathcal{M}(s)(0) \to \infty$ as $s \to 0$, so $\det \mathcal{M}(\cdot)(0) \not\equiv 0$. Note that $\mathcal{M}(s)$ depends continuously on s by Proposition 7.1(c) and Remark 2.3, thus, if $\delta > 0$ is sufficiently small, then $\det \mathcal{M}(s)(0) \neq 0$ for all $s \in (0, \delta)$. □

Proof of Theorem 2.8. By Lemma 7.6, if $\delta > 0$ is sufficiently small, then for all $s^* \in (0, \delta)$ the matrix $\mathcal{M}(s^*)(I)$ is nonsingular for all $I \in \Omega_q$ as long as $q > 0$ is sufficiently small. It follows that the Hamiltonian $G(\theta, I) = G^0(I; s^*) + G^1(\theta, I; s^*)$ satisfies all the assumptions of Theorem 6.3 in the radial setting (in the sense of Remark 6.4), which gives the desired conclusion. □

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Appendix A

Hypotheses for the center manifold theorem

In this appendix we complete the verification of hypotheses (H1) and (H3) from Section 3.1 for the center manifold theorem.

Throughout this appendix we denote by $\|\cdot\|_\ell$ the usual norm of $H^\ell(\mathbb{R}^N)$, where $\ell \geq -1$ is an integer, and $\|\cdot\|_{k,p}$ the norm in $W^{k,p}(\mathbb{R}^N)$; in particular, $\|\cdot\|_{0,p}$ is the norm in $L^p(\mathbb{R}^N)$. For the sake of brevity, we will omit the domain $\mathbb{R}^N$ from the spaces H^ℓ, $W^{k,p}$, and L^p.

A.1 Smoothness of the Nemytskii operator

In this section, we consider a function $f : \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, where $\mathbb{R}^d$ with $d \in \mathbb{N}$ is a parameter space ($d = 0$ is the case with no parameters). Under suitable assumptions, we prove the $\mathscr{C}^k$-smoothness of the corresponding Nemytskii operator $\tilde{f}$ acting on the space $H^\ell \times \mathbb{R}^d$. In Section 3.2, the results proved here are applied with $d = 2$, $\ell = m+1$, $k = K+1$, and f as in (2.2), (S2). Actually, for that application it would be sufficient to consider $\tilde{f}$ as a map defined on $H^{m+2}(\mathbb{R}^N)$ (with values in $H^{m+1}(\mathbb{R}^N)$) and $m > N/2$, so our result here is slightly more general than needed above.

While there are many texts on continuity and smoothness of Nemytskii and substitution operators in Sobolev spaces (see, for example, the monographs [5, 62, 68]), we were not able to locate the results in the form we need. For bounded domains, the smoothness of Nemytskii operators in Sobolev spaces is treated in detail in [68]. It is not difficult, although not completely trivial, to modify the proofs in [68] so that they also apply to the Sobolev spaces on $\mathbb{R}^N$ if suitable assumptions on f are made. We give here a different proof based on the boundedness of Nemytskii operators and the converse to Taylor's theorem. Although we only consider the spaces H^ℓ, $\ell > N/2$, we make no use of the Hilbert space structure here. The same proof works for Nemytskii operators on $W^{\ell,p}(\mathbb{R}^N)$, $p \in (1,\infty)$, if $\ell > N/p$.

We state the result in the following theorem, first, for operators without parameters, then with parameters. Given a function $f \in \mathscr{C}^{k+1+\ell}(\mathbb{R}^N \times \mathbb{R})$, the Nemytskii operator $\tilde{f}$ of f takes a function u on $\mathbb{R}$ to a function $\tilde{f}(u)$ defined by

$$\tilde{f}(u)(x) = f(x, u(x)) \quad (x \in \mathbb{R}^N). \tag{A.1}$$

We will only be dealing with functions $u \in H^\ell$, with $\ell > N/2$. In view of the Sobolev imbedding theorem, we may assume that u is continuous on $\mathbb{R}^N$ (more precisely, it has a continuous representative, but we will not be making this distinction). Thus, $\tilde{f}(u)(x)$ is defined for all $x \in \mathbb{R}^N$.

When f depends on a parameter $\tau \in \mathbb{R}^d$, $f = f(x, u; \tau)$, we define its Nemytskii operator $\tilde{f}$ by

$$\tilde{f}(u;\tau)(x) = f(x, u(x); \tau). \tag{A.2}$$

For $j = 1, \dots, k$, we denote by $\mathscr{L}_s^j(H^\ell, H^\ell)$ the space of all bounded symmetric j-linear maps from H^ℓ to itself; it is equipped with the standard operator norm.

Theorem A.1. *Let $\ell > N/2$ and $k \geq 0$ be integers.*

(a) Assume that $f \in \mathscr{C}^{k+\ell+1}(\mathbb{R}^N \times \mathbb{R})$ and for each $\vartheta > 0$ the function f is bounded on $\mathbb{R}^N \times [-\vartheta, \vartheta]$ together with all its partial derivatives up to order

$k+\ell+1$. *Assume further that* $f(\cdot,0)\in H^\ell$ *and for some constant* $C_1>0$ *one has*

$$\left|\frac{\partial^{\ell+k+1}f(x,y)}{\partial x_i^\ell\,\partial y^{k+1}}-\frac{\partial^{\ell+k+1}f(x,0)}{\partial x_i^\ell\,\partial y^{k+1}}\right|\le C_1|y|$$
$$(x\in\mathbb{R}^N,\ y\in(-1,1),\ i=1,\dots,N). \quad \text{(A.3)}$$

Then the Nemytskii operator $\tilde f$ *takes* H^ℓ *to itself and, considered as an operator on* H^ℓ, $\tilde f$ *is of class* $\mathscr{C}^k$. *Moreover, the* k*-th derivative of* $\tilde f$, *as a map from* H^ℓ *to* $\mathscr{L}_s^k(H^\ell,H^\ell)$, *is Lipschitz on each bounded subset of* H^ℓ.

(b) Assume that $f\in\mathscr{C}^{k+\ell+2}(\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^d)$, $f(x,0;\tau)=0$ *for all* $x\in\mathbb{R}^N$ *and* $\tau\in\mathbb{R}^d$, *and for each* $\vartheta>0$ *the function* f *is bounded on* $\mathbb{R}^N\times[-\vartheta,\vartheta]\times\{\tau\in\mathbb{R}^d:|\tau|\le\vartheta\}$ *together with all its partial derivatives up to order* $k+\ell+2$. *Then the Nemytskii operator* $\tilde f: H^\ell\times\mathbb{R}^d\to H^\ell$ *is of class* $\mathscr{C}^k$.

Remark A.2. (a) The assumption $f\in\mathscr{C}^{k+\ell+1}(\mathbb{R}^N\times\mathbb{R})$ in statement (a) can be relaxed a little. The continuity of the derivatives of f of order $k+\ell+1$ is not needed; their existence and boundedness on the sets $\mathbb{R}^N\times[-\vartheta,\vartheta]$, $\vartheta>0$, is sufficient.

(b) The mean value theorem implies that (A.3) holds if the regularity of f is "one-degree" higher, that is, $f\in\mathscr{C}^{k+\ell+2}(\mathbb{R}^N\times\mathbb{R})$ and for each $\vartheta>0$ the function f is bounded on $\mathbb{R}^N\times[-\vartheta,\vartheta]$ together with all its partial derivatives up to order $k+\ell+2$. Such a higher regularity is assumed in statement (b) for the sake of simplicity.

(c) In the proof of statement (a), we also show that the derivative $D^j\tilde f(u)$ is given by the pointwise multiplication operator:

$$D^j\tilde f(u)[v,\dots,v](x)=D_y^jf(x,u(x))(v(x))^j\quad(u,v\in H^\ell,\ j=1,\dots,k), \qquad \text{(A.4)}$$

where

$$D_y^jf(x,y)=\frac{\partial^j}{\partial y^j}f(x,y). \qquad \text{(A.5)}$$

In the rest of this section, $\ell > N/2$ is fixed. By the Sobolev imbedding theorem, $H^\ell \hookrightarrow \mathscr{C}_{\rm b}(\mathbb{R}^N)$, thus we view each element of H^ℓ as a continuous function.

We prepare the proof of the theorem by several preliminary results. First of all, we note that in statement (a) we may assume, without loss of generality, that

$$f \in \mathscr{C}_{\rm b}^{k+\ell+1}(\mathbb{R}^N \times \mathbb{R}), \tag{A.6}$$

that is, f and all its partial derivatives up to order $k+\ell+1$ are bounded globally on $\mathbb{R}^N \times \mathbb{R}$ (and not just on sets of the form $\mathbb{R}^N \times [-\vartheta, \vartheta]$). Indeed, smoothness is a local property; thus, to prove statement (a) (including the boundedness of the k-th derivative) we just need to consider the restrictions of $\tilde{f}$ to bounded sets of H^ℓ. Dealing with such restrictions, the values of $f(x,y)$ for large $|y|$ are irrelevant, thanks to the imbedding $H^\ell \hookrightarrow \mathscr{C}_{\rm b}(\mathbb{R}^N)$, thus we can modify $f(x,y)$ for large $|y|$ so as to achieve (A.6).

We recall the following Banach algebra properties of H^ℓ. For the proof see [2, 68], for example.

Lemma A.3. *The space H^ℓ is closed under pointwise multiplication and for any integer $j \geq 1$ one has*

$$\|v_1 \dots v_j\|_\ell \leq C\|v_1\|_\ell \dots \|v_j\|_\ell \quad (v_1, \dots, v_j \in H^\ell), \tag{A.7}$$

where $C = C(j, N, \ell)$ is a constant. Consequently, if $a \in Y$, where $Y = H^\ell$ or $Y = \mathscr{C}_{\rm b}^\ell(\mathbb{R}^N)$, then for any integer $j \geq 1$ the map

$$L_j : (v_1, \dots, v_j) \mapsto a v_1 \dots v_j \tag{A.8}$$

belongs to $\mathscr{L}_s^j(H^\ell, H^\ell)$ and

$$\|L_j\|_{\mathscr{L}_s^j(H^\ell, H^\ell)} \leq C\|a\|_Y, \tag{A.9}$$

for some constant $C = C(j, N, \ell)$ (independent of a).

In the next two lemmas, we show a boundedness and Lipschitz continuity property of Nemytskii operators under lower regularity assumptions.

Lemma A.4. *Assume that* $f \in \mathscr{C}_{\mathrm{b}}^{\ell}(\mathbb{R}^N \times \mathbb{R})$, $f(\cdot,0) \equiv 0$, *and for some constant* $C_1 > 0$ *one has*

$$\left|\frac{\partial^\ell}{\partial x_i^\ell} f(x,y)\right| \le C_1|y| \quad (x \in \mathbb{R}^N,\ y \in (-1,1),\ i = 1,\dots,N). \tag{A.10}$$

Then the Nemytskii operator $\tilde{f}$ *takes* H^ℓ *to itself and it is bounded: for each* $\rho > 0$ *there is a constant* $C(\rho)$ *(depending on* f *and* ρ*) such that for all* $u \in H^\ell$ *with* $\|u\|_\ell \le \rho$ *one has*

$$\|\tilde{f}(u)\|_\ell \le C(\rho).$$

Remark A.5. If the condition $f(\cdot,0) \equiv 0$ is dropped, then the lemma can be applied to the function $f(x,u) - f(x,0)$ if

$$\left|\frac{\partial^\ell f(x,y)}{\partial x_i^\ell} - \frac{\partial^\ell f(x,0)}{\partial x_i^\ell}\right| \le C_1|y| \quad (x \in \mathbb{R}^N,\ y \in (-1,1),\ i = 1,\dots,N).$$

Proof of Lemma A.4. For f independent of x, the result is proved in [62, Section 5.24]. We just need minor modifications of the proof given there to yield the present result.

As in [62], we use the fact that, due to the Fourier-multiplier characterization of H^ℓ, the following expression gives an equivalent norm on H^ℓ:

$$\|v\|'_\ell := \|v\|_{0,2} + \sum_{i=1}^{N} \left\|\frac{\partial^\ell v}{\partial x_i^\ell}\right\|_{0,2}.$$

Thus, to prove the statement, we need to show that for each $u \in H^\ell$ the L^2-norms of the functions

$$f(x,u(x)), \qquad \frac{\partial^\ell}{\partial x_i^\ell}\big(f(x,u(x))\big), \quad i = 1,\dots,N, \tag{A.11}$$

are finite, and are bounded from above by a constant determined by ρ if $\|u\|_\ell \le \rho$.

For $f(x,u(x))$, the estimate is simple. The bound $\|u\|_\ell \le \rho$ yields a bound on $\|u\|_{0,\infty}$. The assumptions $f \in \mathscr{C}_{\mathrm{b}}^{\ell}(\mathbb{R}^N \times \mathbb{R})$, $f(\cdot,0) \equiv 0$, imply that for any $y \in \mathbb{R}$ with $|y| \le \|u\|_{0,\infty}$ one has $|f(x,y)| \le \tilde{C}|y|$, where $\tilde{C} = \tilde{C}(\rho)$ is constant. Therefore

$$|f(x,u(x))| \le \tilde{C}|u(x)| \quad (x \in \mathbb{R}),$$

from which the desired estimate follows immediately.

Next, we estimate the derivatives in (A.11). As in [62], this is done by first taking $u \in \mathscr{C}_0^\infty(\mathbb{R}^N)$ and then using the approximation properties of H^ℓ. Fix any $i \in \{1, \dots, N\}$. Using the chain rule, one shows by induction that

$$\frac{\partial^\ell}{\partial x_i^\ell}\big(f(x, u(x))\big) = \frac{\partial^\ell}{\partial x_i^\ell} f(x, y)\,\Big|_{\,y=u(x)} + Q(x), \tag{A.12}$$

where Q is the sum of finitely many terms of the form $p(x)q(x)$, where

$$p(x) = \frac{\partial^{j+s}}{\partial x_i^j \partial y^s} f(x, y)\,\Big|_{\,y=u(x)} \tag{A.13}$$

for some integers $s \geq 1$, $j \geq 0$ satisfying $j + s \leq \ell$, and

$$q(x) = \frac{\partial^{r_1} u(x)}{\partial x_i^{r_1}} \frac{\partial^{r_2} u(x)}{\partial x_i^{r_2}} \cdots \frac{\partial^{r_s} u(x)}{\partial x_i^{r_s}} \tag{A.14}$$

for some positive integers $r_1, \dots, r_s$ satisfying $j + r_1 + \cdots + r_s = \ell$. In the proof of Theorem 1 in [62, Section 5.2.4], the L^2-norms of the products of the form (A.14) are estimated in terms of a finite number of powers of $\|u\|_\ell$; in particular, the L^2-norms are bounded by a constant determined by ρ if $\|u\|_\ell \leq \rho$. Obviously, the same can then be said of the L^2-norms of the products $p(x)q(x)$, since the function $p(x)$ given by (A.13) is bounded.

It remains to estimate the first term on the right-hand side of (A.12). For that we use (A.10). Each (continuous) function $u \in H^\ell$ with $\|u\|_\ell \leq \rho$ has its range contained in $(-c\rho, c\rho)$ (c is a constant from the Sobolev imbedding). Clearly, (A.10) continues to hold if we take the interval $(-c\rho, c\rho)$ in place of $(-1, 1)$, possibly after replacing the constant C_1 by a larger constant $C_1(\rho)$. Consequently, for $u \in H^\ell$ with $\|u\|_\ell \leq \rho$ the L^2-norm of the function

$$\frac{\partial^\ell}{\partial x_i^\ell} f(x, y)\,\Big|_{\,y=u(x)}$$

is not greater than $C_1(\rho)\|u\|_{0,2} \leq C_1(\rho)\rho$. This, in conjunction with the previous estimates, completes the proof. $\square$

Lemma A.6. *Assume that $f \in \mathscr{C}_{\mathrm{b}}^{\ell+1}(\mathbb{R}^N \times \mathbb{R})$ and for some constant $C_0 > 0$ one has*

$$\left| \frac{\partial^{\ell+1} f(x,y)}{\partial x_i^\ell \partial y} - \frac{\partial^{\ell+1} f(x,0)}{\partial x_i^\ell \partial y} \right| \le C_1 |y| \qquad (x \in \mathbb{R}^N,\ y \in (-1,1),\ i = 1, \dots, N). \tag{A.15}$$

Then for each $\rho > 0$ and any two functions $u, v \in H^\ell$ with $\|u\|_\ell, \|v\|_\ell \le \rho$, one has $\tilde{f}(u) - \tilde{f}(v) \in H^\ell$ and

$$\|\tilde{f}(u) - \tilde{f}(v)\|_\ell \le C_2(\rho) \|u - v\|_\ell, \tag{A.16}$$

where $C_2(\rho)$ is a constant determined by ρ (and independent of u and v).

Proof. Fix $u, v \in H^\ell$ with $\|u\|_\ell, \|v\|_\ell \le \rho$. For each $x \in \mathbb{R}^N$,

$$(\tilde{f}(u) - \tilde{f}(v))(x) = \left(\int_0^1 D_y f\big(x, u(x) + t(u(x) - v(x))\big)\, dt \right) \big(u(x) - v(x)\big). \tag{A.17}$$

Write the integral in (A.17) as follows:

$$\int_0^1 \big(D_y f\big(x, u(x) + t(u(x) - v(x))\big) - D_y f(x,0)\big)\, dt + D_y f(x,0). \tag{A.18}$$

We now apply Lemma A.4 to the function $f_y(x,y) - f_y(x,0)$, which is legitimate by (A.15) (cp. Remark A.5). Thereby we obtain that for each $t \in [0,1]$ the function $f_y(x, u(x) + t(u(x) - v(x)))$ belongs to H^ℓ and its H^ℓ-norm is bounded by a constant $C = C(\rho)$. From this it follows that the integral in (A.18) is also a function in H^ℓ with norm bounded by $C(\rho)$. Since $f_y(x,0)$ is a function in $\mathscr{C}_{\mathrm{b}}^\ell$, we conclude, using (A.17) and the second statement of Lemma A.3 with $j = 1$, that $\tilde{f}(u) - \tilde{f}(v) \in H^\ell$ and its norm is estimated as in (A.16). □

We are in a position to prove Theorem A.1.

Proof of statement (a) of Theorem A.1. As noted above, we may assume without loss of generality that (A.6) holds.

Given $u \in H^\ell$, we have $\tilde{f}(u) = \tilde{f}(u) - \tilde{f}(0) + \tilde{f}(0)$. Since the function $\tilde{f}(0)(x) = f(x,0)$ belongs to H^ℓ by assumption, Lemma A.6 (with $v = 0$) implies that $\tilde{f}(u) \in H^\ell$. Thus $\tilde{f}$ takes H^ℓ into itself.

Next, given any two functions $u, v \in H^\ell$, Taylor's theorem gives, for each $x \in \mathbb{R}^N$, the following expansion

$$(\tilde{f}(u+v) - \tilde{f}(u))(x) = \sum_{j=1}^{k} \frac{1}{j!} D_y^j f(x, u(x))(v(x))^j + R(x, u(x), v(x))(v(x))^k, \tag{A.19}$$

where $D_y^j f(x, y)$ is as in (A.5) and

$$R(x, y, z) = \int_0^1 \frac{(1-t)^{k-1}}{(k-1)!} \big(D_y^k f(x, y+tz) - D_y^k f(x, y)\big)\, dt. \tag{A.20}$$

According to the converse to Taylor's theorem [1, 36], the map $\tilde{f}$ is of class $\mathscr{C}^k$, with the derivatives as in (A.4), provided the following holds. The symmetric multilinear operators $L_j(u)$, $j = 1, \dots, k$, and $L(u, v)$ defined by

$$L_j(u)[v_1, \dots, v_j](x) = D_y^j f(x, u(x)) v_1(x) \dots v_j(x) \quad (v_1, \dots, v_j \in H^\ell), \tag{A.21}$$

$$L(u, v)[v_1, \dots, v_k](x) = R(x, u(x), v(x)) v_1(x) \dots v_k(x) \quad (v_1, \dots, v_k \in H^\ell), \tag{A.22}$$

are bounded, the maps

$$u \mapsto L_j(u) : H^\ell \to \mathscr{L}_s^j(H^\ell, H^\ell), \tag{A.23}$$

$$(u, v) \mapsto L(u, v) : H^\ell \times H^\ell \to \mathscr{L}_s^j(H^\ell, H^\ell) \tag{A.24}$$

are continuous, and $L(u, 0) = 0$. The last property is obvious. Consider now the operator $L_j(u)$, for any $j \in \{1, \dots, k\}$. Observe that Lemma A.6 applies to the function $D_y^j f$. Indeed, condition (A.15) (with f replaced by $D_y^j f$) holds for $j < k$ due to $D_y^j f \in \mathscr{C}_{\mathrm{b}}^{\ell+2}$ and for $j = k$ due to assumption (A.3). Let $\widetilde{D_y^j f}$ be the Nemytskii operator of $D_y^j f$. From Lemma A.6, we obtain, first of all, that for each $u \in H^\ell$,

$$\widetilde{D_y^j f}(u) - \widetilde{D_y^j f}(0) \in H^\ell.$$

Writing

$$\widetilde{D_y^j f}(u) = \widetilde{D_y^j f}(u) - \widetilde{D_y^j f}(0) + \widetilde{D_y^j f}(0)$$

and noting that $\widetilde{D_y^j f}(0)$ is the $\mathscr{C}_{\mathrm{b}}^{\ell}$-function $D_y^j f(x,0)$, we obtain from Lemma A.3 that the j-linear map $L_j(u)$ is bounded. Moreover, using (A.9) and Lemma A.6, we infer that for arbitrary $\rho > 0$ and $u, \bar{u} \in H^\ell$ with $\|u\|_\ell, \|\bar{u}\|_\ell \le \rho$ one has

$$\|L_j(u) - L_j(\bar{u})\|_{\mathscr{L}_s^j(H^\ell, H^\ell)} \le C(\rho)\|u - \bar{u}\|_\ell, \tag{A.25}$$

where $C(\rho)$ is a constant independent of u, $\bar{u}$. This gives the continuity—even Lipschitz continuity on bounded sets—of $u \mapsto L_j(u)$.

The boundedness of $L(u,v)$ and its Lipschitz continuity on bounded subsets of $H^\ell \times H^\ell$ are proved by similar arguments (cp. the proof of Lemma A.6) and we omit the details.

The proof of statement (a) is now complete. □

Proof of statement (b) of Theorem A.1. The hypotheses of statement (b) guarantee (cp. Remark A.2), that statement (a) applies to $f(\cdot,\cdot;\tau)$ for each τ. This implies in particular that $\tilde{f}$ takes $H^\ell \times \mathbb{R}^d$ to H^ℓ.

As in statement (a) (cp. (A.6)), we may assume without loss of generality that

$$f \in \mathscr{C}_{\mathrm{b}}^{k+\ell+2}(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^d). \tag{A.26}$$

To prove that $\tilde{f} : H^\ell \times \mathbb{R}^d \to H^\ell$ is of class $\mathscr{C}^k$, we use the converse to Taylor's theorem again. Given any $u, v \in H^\ell$, $\tau, \varsigma \in \mathbb{R}^d$, we first write down the multivariable Taylor expansion at each $x \in \mathbb{R}^N$. Taking $\tau = (\tau_1, \ldots, \tau_d)$, $\varsigma = (\varsigma_1, \ldots, \varsigma_d)$, and using the standard multiindex notation, we have

$$\begin{aligned}\big(\tilde{f}(u+v;\tau+\varsigma) - \tilde{f}(u;\tau)\big)(x) &= \sum_{j=0}^{k} \sum_{\substack{\beta\in\mathbb{N}^d \\ 1\le j+|\beta|\le k}} \frac{1}{j!\beta!} D_\tau^\beta D_y^j f(x,u(x);\tau)(v(x))^j \varsigma^\beta \\ &\quad + \sum_{j=0}^{k} \sum_{\substack{\beta\in\mathbb{N}^d \\ j+|\beta|=k}} R_{j,\beta}(x,u(x),v(x);\tau,\varsigma)(v(x))^j\varsigma^\beta, \end{aligned} \tag{A.27}$$

where, for $\beta = (\beta_1, \ldots, \beta_d)$,

$$D_\tau^\beta D_y^j f(x,y;\tau) = \frac{\partial^{j+|\beta|}}{\partial\tau_1^{\beta_1} \ldots \partial\tau_d^{\beta_d} \partial y^j} f(x,y;\tau), \tag{A.28}$$

and

$$R_{j,\beta}(x,y,z;\tau,\varsigma)=\frac{1}{j!\beta!}\int_0^1\frac{(1-t)^{k-1}}{(k-1)!}\big(D_\tau^\beta D_y^j f(x,y+tz;\tau+t\varsigma)-D_\tau^\beta D_y^j f(x,y;\tau)\big)\,dt. \tag{A.29}$$

As in the proof of statement (a), the functional coefficients in this expansion define symmetric multilinear maps (by pointwise multiplication). We need to prove that these multilinear maps are bounded on $H^\ell\times\mathbb{R}^d$ and depend continuously in the multilinear-operator norm on $(u,\tau)\in H^\ell\times\mathbb{R}^d$, or, in the case of $R_{j,\beta}$, on $(u,v,\tau,\varsigma)\in H^\ell\times H^\ell\times\mathbb{R}^d\times\mathbb{R}^d$ (the additional needed relations $R_{j,\beta}(x,u(x),0;\tau,0)=0$ are trivial). The boundedness is proved as in (a) (since τ is in a finite dimensional space, we only need to worry about the boundedness in $u\in H^\ell$). Also as in (a), the proof of the continuity amounts to proving the continuous dependence of the Nemytskii operators, viewed as maps from $H^\ell\times\mathbb{R}^d$ to H^ℓ, of the functions

$$D_y^j D_\tau^\beta f(x,y;\tau)-D_y^j D_\tau^\beta f(x,0;\tau)\quad(j=0,\dots,k,\ \beta\in\mathbb{N}^d,\ 1\le j+|\beta|\le k). \tag{A.30}$$

We claim that these Nemytskii operators are Lipschitz on each bounded subset of $H^\ell\times\mathbb{R}^d$. Indeed, each of the functions (A.30) is at least of class $\mathscr{C}_{\mathrm{b}}^{\ell+2}$. Therefore, as in (a), its Nemytskii operator is Lipschitz in u, uniformly for (u,τ) in any given bounded subset of $H^\ell\times\mathbb{R}^d$. The uniform Lipschitz continuity in τ follows from (A.26). Similar considerations show the continuity of the operators defined by $R_{j,\beta}$. □

A.2 Bound on the resolvent

This section is devoted to the proof of the resolvent bound stated in hypothesis (H3) in Section 3.1. We essentially use a proof found in [70], modifying and extending it slightly to account for the differences in our setting.

Recall that in Chapter 3 we defined the operator $A_1=-\Delta-a_1(x)$ on H^m, with domain $D(A_1)=H^{m+2}$. Here we assume, as in (S2), $a_1\in\mathscr{C}_{\mathrm{b}}^{m+1}$, and the integer m satisfies $m > N/2$ (it actually suffices here to assume $a_1 \in \mathscr{C}_{\mathrm{b}}^m$). Recall also that A is the operator on $H^{m+1} \times H^m$, with domain $D(A) = H^{m+2} \times H^{m+1}$, given by

$$A(u_1, u_2) = (u_2, A_1 u_1)^T. \tag{A.31}$$

Below, we suppress the argument x from a_1 for the sake of clarity.

Proposition A.7. *Assume that $a_1 \in \mathscr{C}_{\mathrm{b}}^{m+1}$, where $m > N/2$ is an integer, and A be defined as above. Then there exist $\hat{\omega}_0 > 0$ and a constant C, depending only on m, N, and $\|a_1\|_{m,\infty}$, such that for all $\hat{\omega} \in \mathbb{R}$ satisfying $|\hat{\omega}| > \hat{\omega}_0$ one has*

$$\|(i\hat{\omega} - A)^{-1}\|_{\mathscr{L}(H^{m+1} \times H^m)} \leq \frac{C}{|\hat{\omega}|}. \tag{A.32}$$

The proof of Proposition A.7 goes along similar lines as an example in [70], where a domain with a bounded cross-section is considered. We will use estimates of solutions of the equation

$$-\Delta u - a_1 u + \tau^2 u = v, \tag{A.33}$$

where $\tau \in \mathbb{R}$. By standard results, if $|\tau| > \sqrt{1 + \|a_1\|_{0,\infty}}$, then for each $v \in L^2$ this equation has a unique solution $u \in H^2$. Moreover, if $v \in H^j$, $j \in \{m-1, m\}$, then $v \in H^{j+2}$.

Lemma A.8. *Under the assumptions of Proposition A.7, there exist constants $B_{m,N}$ and $C_{m,N}$, depending only on m and N, such that if $|\tau| > \sqrt{1 + \alpha_1}$, where $\alpha_1 := C_{m,N} \|a_1\|_{m,\infty}$, then the following statements hold:*

(a) If $v \in H^m$ and $u \in H^{m+2}$ is the solution of (A.33), *then*

$$(\tau^2 - \alpha_1)\|u\|_m \leq B_{m,N}\|v\|_m, \tag{A.34}$$

and

$$(\tau^2 - \alpha_1 - 1)^{1/2}\|u\|_{m+1} \leq B_{m,N}\|v\|_m. \tag{A.35}$$

(b) If $v \in H^{m-1}$ and $u \in H^{m+1}$ is the solution of (A.33)*, then*

$$\|u\|_{m+1} \le B_{m,N}\|v\|_{m-1}, \tag{A.36}$$

and

$$(\tau^2 - \alpha_1 - 1)^{1/2}\|u\|_m \le B_{m,N}\|v\|_{m-1}. \tag{A.37}$$

Proof. Recall that for $J = (j_1, \dots, j_n) \in \mathbb{N}^n$

$$D^J = \frac{\partial^{|J|}}{\partial x_1^{j_1} \dots \partial x_n^{j_n}}.$$

For $|J| \le m$, applying D^J to (A.33), we obtain

$$-\Delta(D^J u) - D^J(a_1 u) + \tau^2 D^J u = D^J v. \tag{A.38}$$

Multiplying (A.38) by $D^J u$, integrating by parts, and applying the Hölder inequality, we obtain

$$\int_{\mathbb{R}^N} |\nabla D^J u|^2\, dx + \tau^2\|D^J u\|_{0,2}^2 \le \|D^J v\|_{0,2}\|D^J u\|_{0,2} + \int_{\mathbb{R}^N} D^J(a_1 u) D^J u\, dx. \tag{A.39}$$

Computing the derivative of $a_1 u$ using the Leibniz rule, one finds a constant C', depending only on m and N, such that

$$\int_{\mathbb{R}^N} D^J(a_1 u) D^J u \; dx \le C'\|a_1\|_{m,\infty}\|u\|_m^2.$$

Substituting in (A.39), we obtain

$$\int_{\mathbb{R}^N} |\nabla D^J u|^2 \; dx + \tau^2\|D^J u\|_{0,2}^2 \le \|v\|_m\|u\|_m + C'\|a_1\|_{m,\infty}\|u\|_m^2. \tag{A.40}$$

Dropping the first term of (A.40), and adding over all multiindices J satisfying $|J| \le m$, we obtain

$$\tau^2\|u\|_m^2 \le B_{m,N}(\|v\|_m\|u\|_m + C'\|a_1\|_{m,\infty}\|u\|_m^2).$$

Setting $C_{m,N} = B_{m,N}C'$, $\alpha_1 = B_{m,N}C'\|a_1\|_{m,\infty}$, we have

$$(\tau^2 - \alpha_1)\|u\|_m^2 \le B_{m,N}\|v\|_m\|u\|_m,$$

that is, (A.34) holds.

Equation (A.40) can also be rewritten as

$$\|D^J u\|_1^2 + (\tau^2 - 1)\|D^J u\|_{0,2}^2 \leq \|v\|_m \|u\|_m + C' \|a_1\|_{m,\infty} \|u\|_m^2. \tag{A.41}$$

Since

$$\sum_{\substack{J\in\mathbb{N}^N \\ |J|\leq m}} \|D^J u\|_1^2 \geq \sum_{\substack{J\in\mathbb{N}^N \\ |J|\leq m+1}} \|D^J u\|_{0,2}^2 = \|u\|_{m+1}^2,$$

adding over all multiindices J satisfying $|J| \leq m$ in (A.41), we obtain

$$\|u\|_{m+1}^2 + (\tau^2 - 1 - \alpha_1)\|u\|_m^2 \leq B_{m,N} \|v\|_m \|u\|_m.$$

Since the left hand side dominates $(\tau^2 - 1 - \alpha_1)^{1/2} \|u\|_{m+1} \|u\|_m$, (A.35) follows.

To prove statement (b), we return to (A.38) again. Similar computations as above, but with an extra integration by parts (to move a derivative from v to u), yield

$$\|u\|_{m+1}^2 + (\tau^2 - \alpha_1 - 1)\|u\|_m^2 \leq B_{m,N} \|v\|_{m-1} \|u\|_{m+1}. \tag{A.42}$$

Strictly speaking, in the above computations we assumed that $v \in H^m$ and $u \in H^{m+2}$, but it can be verified easily that the result remains valid if $v \in H^{m-1}$, taking into account that in the worst case one may have $D^J v \in H^{-1}$ (alternatively, one can prove the final estimate by approximating v in H^{m-1} by functions in H^m).

Since $\tau^2 - \alpha_1 - 1 > 0$, we can drop the second term of the left hand side of (A.42) to get (A.36). The left hand side of (A.42) dominates $(\tau^2 - \alpha_1 - 1)^{1/2} \|u\|_{m+1} \|u\|_m$, from which we obtain (A.37). □

Proof of Proposition A.7. In this proof, C_1, C_2, C_3, C_3', and C_4 are constants depending only on m, N, and $\|a_1\|_{m,\infty}$.

Recall that the operator A has $2n$ (purely) imaginary eigenvalues $\pm i\omega_1, \ldots, \pm i\omega_n$, with $\omega_j > 0$. Set $\omega_M = \max_j \omega_j$. Let $\lambda = i\hat{\omega}$, where $\hat{\omega} \in \mathbb{R}$ satisfies $|\hat{\omega}| > \omega_M + 1$ and $\hat{\omega}^2 > \alpha_1 + 1$, with $\alpha_1 = C_{m,N} \|a_1\|_{m,\infty}$, as in Lemma A.8. For

$u = (u_1, u_2)$ and $v = (v_1, v_2)$, consider the equation $Au = \lambda u + v$, or, equivalently,

$$\begin{aligned} u_2 &= \lambda u_1 + v_1 \\ -\Delta u_1 - a_1 u_1 &= \lambda u_2 + v_2. \end{aligned} \tag{A.43}$$

Eliminating u_2 from (A.43), we get

$$-\Delta u_1 - \lambda^2 u_1 - a_1 u_1 = \lambda v_1 + v_2,$$

or,

$$-\Delta u_1 + \hat{\omega}^2 u_1 - a_1 u_1 = \lambda v_1 + v_2. \tag{A.44}$$

If $v_1 \equiv 0$ and $v_2 \in H^m$, applying Lemma A.8(a) to (A.44) gives

$$\begin{aligned} \|u_1\|_m &\leq \frac{B_{m,N}}{\hat{\omega}^2 - \alpha_1} \|v_2\|_m, \\ \|u_1\|_{m+1} &\leq \frac{B_{m,N}}{(\hat{\omega}^2 - \alpha_1 - 1)^{1/2}} \|v_2\|_m \leq \frac{C_1}{|\hat{\omega}|} \|v_2\|_m. \end{aligned} \tag{A.45}$$

Since $u_2 = i\hat{\omega} u_1$,

$$\|u_2\|_m \leq \frac{B_{m,N}|\hat{\omega}|}{\hat{\omega}^2 - \alpha_1} \|v_2\|_m \leq \frac{C_2}{|\hat{\omega}|} \|v_2\|_m. \tag{A.46}$$

Now take $v_2 \equiv 0$, $v_1 \in H^{m+1}$. Eliminating u_1 from (A.43), we get

$$-\Delta u_2 + \hat{\omega}^2 u_2 - a_1 u_2 = -\Delta v_1 - a_1 v_1.$$

From Lemma A.8(b), we deduce

$$\|u_2\|_{m+1} \leq B_{m,N} \|\Delta v_1 + a_1 v_1\|_{m-1} \leq C_3 \|v_1\|_{m+1}, \tag{A.47}$$

and

$$(\hat{\omega}^2 - \alpha_1 - 1)^{1/2} \|u_2\|_m \leq B_{m,N} \|\Delta v_1 + a_1 v_1\|_{m-1} \leq C_3 \|v_1\|_{m+1}. \tag{A.48}$$

Relations (A.48) imply

$$\|u_2\|_m \leq \frac{C_3}{(\hat{\omega}^2 - \alpha_1 - 1)^{1/2}} \|v_1\|_{m+1} \leq \frac{C_3'}{|\hat{\omega}|} \|v_1\|_{m+1}; \tag{A.49}$$

while, using $u_1 = (u_2 - v_1)/(i\omega)$, relations (A.47) yield

$$\|u_1\|_{m+1} \le \frac{C_4}{|\hat{\omega}|}\|v_1\|_{m+1}. \tag{A.50}$$

Combining (A.46), (A.49); (A.45), (A.50); and the fact that $(i\hat{\omega} - A)^{-1}$ is a linear operator, we conclude that (A.32) holds for all $\hat{\omega} \in \mathbb{R}$ satisfying $|\hat{\omega}| \ge \hat{\omega}_0$, where $\hat{\omega}_0 \in \mathbb{R}$ satisfies $\hat{\omega}_0 > \max\{\sqrt{\alpha_1 + 1}, \omega_M + 1\}$. □

Remark A.9. In the parameter-dependent case it is possible to prove the existence of a constant C such that (A.32) holds uniformly for $s \in [0, \delta]$. Indeed, suppose that instead of the operator $A_1 = -\Delta - a_1(x)$ we consider the family of operators $A_1(s) = -\Delta - a_1(x; s)$, where the map $s \in [0, \delta] \mapsto a_1(\cdot; s)$ is continuous in the $\mathscr{C}^m(\mathbb{R}^N)$-norm, and $\delta > 0$. In this setting the operator A, defined in (A.31) depends on s via A_1. If $\hat{\omega}_0$ is sufficiently large, in particular, such that the set $\{i\lambda : |\lambda| > \hat{\omega}_0\}$ does not intersect the spectrum of $A(s)$ for any $s \in [0, \delta]$, then C can be chosen sufficiently large so that (A.32) holds for all $s \in [0, \delta]$. This fact is a direct consequence of C depending only on m, N, and $\|a_1\|_{m,\infty}$.