

# Existence of partially localized quasiperiodic solutions of homogeneous elliptic equations on $\mathbb{R}^{N+1}$

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## Abstract

We consider the equation

$$\Delta u + u_{yy} + f(u) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}, \quad (1)$$

where  $N \geq 2$  and  $f$  is a smooth function satisfying  $f(0) = 0$  and  $f'(0) < 0$ . We show that for suitable nonlinearities  $f$  of this form equation (1) possesses uncountably many positive solutions which are quasiperiodic in  $y$ , radially symmetric in  $x$ , and decaying as  $|x| \rightarrow \infty$  uniformly in  $y$ . Our method is based on center manifold and KAM-type results and involves analysis of solutions of (1) in a vicinity of a  $y$ -independent solution  $u^*(x)$ —a ground state of the equation  $\Delta u + f(u) = 0$  on  $\mathbb{R}^N$ .

*Key words:* Elliptic equations, entire solutions, quasiperiodic solutions, ground state.

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## 1 Introduction

In this article, we consider the semilinear elliptic equation

$$\Delta u + u_{yy} + f(u) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R}, \quad (1.1)$$

where  $N \geq 2$ ,  $\Delta$  is the Laplace operator in  $x$ , and  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a smooth function satisfying

$$f(0) = 0, \quad f'(0) < 0. \quad (1.2)$$

We are mainly concerned with positive solutions of this equation which decay to 0 in the  $x$ -variables uniformly in  $y$ :

$$\lim_{|x| \rightarrow \infty} \sup_{y \in \mathbb{R}} u(x, y) = 0. \quad (1.3)$$

Equations of the above form arise in a number of problems. For example, one arrives at such equations when looking for solitary waves or stationary states of various nonlinear evolution problems including the Klein-Gordon, Schrödinger, and nonlinear heat equations (see [2] for more details). Depending on the motivation, one may want to study solutions with additional properties, such as nonnegative solutions or finite-energy solutions. Nonnegative solutions of (1.1) are the only relevant steady states for the dynamics of the solutions of the corresponding nonlinear heat equation  $u_t = \Delta u + u_{yy} + f(u)$  with positive initial data. Indeed, by the comparison principle, these solutions stay positive at all times, as long as they exist.

Best understood among positive solutions of (1.1) are the solutions which are (fully) localized in the sense that they decay to 0 in all variables, including  $y$ . A classical result of [15] says that such solutions are radially symmetric about some origin in  $\mathbb{R}^{N+1}$ . When the decay condition is removed, the structure of solutions becomes very complicated and their general classification is probably out of reach. Several authors have exposed possible complexities in the solutions, such as the existence of infinitely many bumps forming along some directions [25], saddle-shaped solutions [4, 10] and more general multiple-end solutions [12, 13, 22], as well as positive solutions having both fronts (transitions) and infinitely many bumps [38]. There is also extensive literature on solutions which are periodic in all but one variable and in the remaining variable they exhibit one or multiple transitions (homoclinic or heteroclinic behavior) between periodic solutions (see, for example, [28, 36] and references therein).

Positive solutions of the kind we study in this paper, namely, solutions that decay in all but one variable and do so uniformly with respect to the remaining variable—occasionally, we refer to such solutions as *partially localized solutions*—form a class somewhere between fully localized and general bounded solutions. The decay in  $x$  rules out most of the complexities mentioned above. Also, it is likely that the decay in  $x$  implies the radial symmetry in  $x$  about some center in  $\mathbb{R}^N$ : although this has not been proved in the full generality, interesting results in this direction can be found in [3, 17]. Thus, the behavior in the  $y$ -variable is the only source of possible complexities in partially localized solutions. To discuss this behavior, and quickly skipping over the simplest case—partially localized solutions constant

in  $y$ , we first recall Dancer’s seminal work [9]. He considered equations of the above form, with special focus on the nonlinearities  $g(u) = u^p - u$  with a Sobolev subcritical  $p > 1$ , and proved the existence of partially localized solutions which are periodic (and nonconstant) in  $y$ . Looking beyond periodic solutions, the existence of quasiperiodic solutions in equations of the form (1.1) is perhaps the next most natural problem to address. We show that for suitable nonlinearities such solutions do indeed exist. Moreover, since our theorem is derived from KAM-type results, it automatically yields an uncountable family of quasiperiodic solutions with mutually distinct frequency vectors (see the next section for a precise statement).

Our method of proving the existence of quasiperiodic solutions, partially localized or other, of elliptic equations on the entire space has its grounding in our earlier work [34]. It builds on spatial dynamics and center manifold techniques for elliptic equations (see [21] for the origins of this method, and, for example, [5, 14, 16, 18, 26, 27, 31, 33, 42] and references therein for further developments) and KAM-type results in a finite-differentiability setting (similar methods had been previously applied to elliptic equations on the strip, see [39, 41]).

In general terms, the method of [34], used also in our subsequent paper [35], consists in the following. We consider equations of the form

$$\Delta u + u_{yy} + a(x)u + f_1(x, u) = 0, \quad (x, y) \in \mathbb{R}^N \times \mathbb{R} = \mathbb{R}^{N+1}, \quad (1.4)$$

where

$$f_1(x, u) = a_2(x)u^2 + a_3(x)u^3 + u^4g(x, u), \quad (1.5)$$

and all the listed functions are sufficiently smooth. First we verify that, under suitable spectral assumptions on the operator  $\Delta + a(x)$ , equation (1.4) admits a class of solutions comprising a finite dimensional manifold, a center manifold of (1.4). Moreover, these solutions are in one-to-one correspondence with solutions of an ordinary differential equation (ODE) on this manifold, the variable  $y$  playing the role of time. The ODE has a Hamiltonian structure [26], and we use a sequence of transformations—a Darboux transformation, a normal form procedure, and action-angle variables—to bring it to a form suitable for an application of the KAM theory: it becomes a Hamiltonian system in a neighborhood of the origin (in a Euclidean space  $\mathbb{R}^{2n}$ ) with the canonical symplectic structure, and in this neighborhood it is a small perturbation of an integrable Hamiltonian system. The main issue in applying a suitable KAM theorem is then the verification of a nondegeneracy condition for the integrable Hamiltonian system. Of course, for this to be applicable to equations of the form (1.4), one needs to do all the aforementioned changes of coordinates with some care, so that the nondegeneracy conditions can be formulated as some verifiable hypotheses on the functions in (1.4), (1.5). This then yields sufficient conditions for the existence of  $y$ -quasiperiodic solutions of (1.4). The papers [34], [35] both give such sufficient conditions, with the following key difference. In [34], the cubic term is given some prominence. When a small parameter is introduced in the coefficients  $a_2, a_3$ , the coefficient  $a_2$  vanishes, as the parameter approaches 0, at least at the same rate as  $a_3$ . In particular, it is essential that  $a_3$  is not identical to 0. In [35], we considered the complementary case: the coefficient  $a_2$  is dominant and there is no condition on  $a_3$ , which may well vanish identically. Technically this case is more involved and, for this reason, our conclusion in [35] is weaker in that quasiperiodic solutions with only two frequencies are considered ([34] contains results on quasiperiodic solutions with any given number  $n \geq 2$  of frequencies).

Our conditions on  $a_2$  in [35], or on  $a_3, a_2$  in [34], require integrals of certain polynomial expressions involving eigenfunctions of  $\Delta + a$  and the functions  $a_2$  or  $a_3$  to be nonzero. The conditions are robust and they are particularly easy to achieve if  $a_2, a_3$  can be perturbed independently of  $a$  and of each other. This may not be possible in some specific classes of equations, such as the homogeneous equations (1.1). Indeed, as we elaborate in the next section, the only way to apply the general scheme from [34], [35] to (1.1) is by taking the Taylor expansion of the nonlinearity  $f$  at some nonconstant solution of (1.1). The resulting coefficients  $a, a_2, a_3$  then all depend on  $f$  (and the nonconstant solution of (1.1)), and they change simultaneously when  $f$  is perturbed. The verification of nondegeneracy conditions for a given nonlinearity is therefore a highly nontrivial task; this will be our main technical hurdle in the paper.

Let us comment on the condition  $N \geq 2$ , which we assume here (we had no such restriction in [34], [35]). A key prerequisite for our method in this paper is the existence of a ground state (a localized positive solution) of the  $N$ -dimensional problem

$$\Delta u + f(u) = 0, \quad x \in \mathbb{R}^N, \quad (1.6)$$

with Morse index greater than 1 (see Section 3.2 for details). It is well known that no such ground state exists if  $N = 1$ , hence this case has to be excluded. By the same token, our method does not apply to equations with some specific nonlinearities where the ground state is known to be unique (up to translations) and to have Morse index 1. This is the case, for example, if  $f(u) = u^p - u$  with  $p$  satisfying  $p > 1$  and, if  $N > 2$ , also  $p < (N + 2)/(N - 2)$  (see [23]). It is an interesting question, which we do not address here, whether partially localized quasiperiodic solutions may exist for such specific equations or in any equation (1.1) with  $N = 1$ .

The rest of the paper is organized as follows. Our main result and an outline of the proof are given in Section 2. Nonlinearities for which quasiperiodic solutions exist are found using Schrödinger operators whose eigenvalues and eigenfunctions satisfy certain conditions, as described in Section 3. In Section 4, we complete the proof of our main result by showing the existence of potentials in the Schrödinger operator such that all the needed conditions are satisfied.

## 2 Statement of the main result

In this section we introduce some terminology and state our main result. Afterward, we give an outline of the proof.

Throughout the paper, for  $k \in \mathbb{N} := \{0, 1, 2, \dots\}$  and  $N \geq 2$ , the space  $L^2_{\text{rad}}(\mathbb{R}^N)$  is the closed subspace of  $L^2(\mathbb{R}^N)$  consisting of radially symmetric functions (that is, the common fixed points of the bounded linear maps  $u \mapsto u \circ R$ ,  $R \in O(N)$ ), and  $H^k(\mathbb{R}^N)$  is the usual Sobolev space on  $\mathbb{R}^N$ . The space  $\mathcal{C}_{\text{rad}}(\mathbb{R}^N)$  is the space of continuous, bounded, radially symmetric real-valued functions on  $\mathbb{R}^N$ , while  $\mathcal{C}^k_{\text{rad}}(\mathbb{R}^N)$  is the space of  $k$  times differentiable, radially symmetric functions on  $\mathbb{R}^N$  with bounded, continuous derivatives up to order  $k$ . When needed, we assume that these spaces are equipped with standard norms. At several places, we abuse the notation slightly by viewing radially symmetric functions as functions of  $x \in \mathbb{R}^N$  or functions of  $r \geq 0$ , depending on the context. This should cause no confusion.

Given integers  $n \geq 2$ ,  $k \geq 1$ , a vector  $\omega = (\omega_1, \dots, \omega_n) \in \mathbb{R}^n$  is said to be *nonresonant up to order  $k$*  if

$$\omega \cdot \alpha \neq 0 \text{ for all } \alpha \in \mathbb{Z}^n \setminus \{0\} \text{ such that } |\alpha| \leq k. \quad (2.1)$$

(Here  $|\alpha| = |\alpha_1| + \dots + |\alpha_n|$ , and  $\omega \cdot \alpha$  is the usual dot product.) If (2.1) holds for all  $k = 1, 2, \dots$ , we say that  $\omega$  is *nonresonant*, or, equivalently, that the numbers  $\omega_1, \dots, \omega_n$  are *rationally independent*.

A function  $u : (x, y) \mapsto u(x, y) : \mathbb{R}^N \times \mathbb{R} \rightarrow \mathbb{R}$  is said to be *quasiperiodic* in  $y$  if there exist an integer  $n \geq 2$ , a nonresonant vector  $\omega^* = (\omega_1^*, \dots, \omega_n^*) \in \mathbb{R}^n$ , and an injective function  $U$  defined on  $\mathbb{T}^n$  (the  $n$ -dimensional torus) with values in the space of real-valued functions on  $\mathbb{R}^N$  such that

$$u(x, y) = U(\omega_1^* y, \dots, \omega_n^* y)(x) \quad (x \in \mathbb{R}^N, y \in \mathbb{R}). \quad (2.2)$$

The vector  $\omega^*$  is called a *frequency vector* of  $u$ .

We emphasize that the nonresonance of the frequency vector is a part of our definition. In particular, a quasiperiodic function is not periodic and, if it has some regularity properties, its image is dense in an  $n$ -dimensional manifold diffeomorphic to  $\mathbb{T}^n$ .

We can now state our main result.

**Theorem 2.1.** *For suitable  $\mathcal{C}^\infty$  functions  $f : \mathbb{R} \rightarrow \mathbb{R}$  with  $f(0) = 0$ ,  $f'(0) < 0$  the following holds. There exists a positive solution  $u(x, y)$  of equation (1.1) such that  $u(x, y)$  is radially symmetric in  $x$ ,  $u(x, y) \rightarrow 0$  as  $|x| \rightarrow \infty$  uniformly in  $y$ , and  $u(x, y)$  is quasiperiodic in  $y$ . In fact, there exist uncountably many such solutions of (3.4) (disregarding translations), their frequency vectors forming an uncountable subset of  $\mathbb{R}^2$ .*

The theorem is proved in the sections below. It can be observed from the details of the proof that the class of nonlinearities for which the conclusion is valid contains nonempty open sets in suitable topologies on spaces of sufficiently smooth functions  $f$  satisfying  $f(0) = 0$ ,  $f'(0) < 0$ .

The outline of our proof is as follows. First, to make use of a result in [35], as recalled in Section 3.1, we write equation (1.1) in the form (1.4). This is achieved by taking the Taylor expansion of  $f$  at a ground state of (1.6). From [35], we obtain a sufficient condition on  $f$  and the ground state for the existence of quasiperiodic solutions, see Section 3.2. In the next step, we invoke a construction from [32]. It shows a relation between eigenfunctions of the Schrödinger operator  $\Delta + a(r)$  with a suitable radial potential and a ground state of a nonlinear equation (1.1) with  $f$  determined by  $a$ . This will allow us to reformulate the sufficient conditions in terms of the potential  $a$  and some eigenfunctions of  $\Delta + a(r)$ , see Section 3.3. The last and most difficult step is the verification of the sufficient conditions for some potentials  $a$ . This will be achieved by taking small perturbations of a specially designed potential  $a(r)$ , see Section 4.

### 3 Sufficient conditions for the existence of quasiperiodic solutions

#### 3.1 A theorem from [35]

We recall a theorem on the existence of quasiperiodic solutions of non-homogeneous elliptic equations on  $\mathbb{R}^{N+1}$  from our previous paper [35]. To that end, consider the equation

$$\Delta u + u_{yy} + a_1(r; s)u + f_1(r, u; s) = 0, \quad x \in \mathbb{R}^N, \quad y \in \mathbb{R}, \quad (3.1)$$

where  $r = |x|$ ,  $s \in (-\delta, \delta)$ , with  $\delta > 0$ , is a parameter, and  $f_1$  is of the form

$$f_1(r, u; s) = a_2(r; s)u^2 + u^3g(r, u; s), \quad (3.2)$$

with  $a_1, a_2, g$  sufficiently smooth, as specified below. Fix constants  $K$  and  $m$  satisfying

$$K \geq 18, \quad m > \frac{N}{2}. \quad (3.3)$$

The smoothness assumptions on  $a_1, a_2$ , and  $g$  are as follows:

- (S1)  $a_1(\cdot; s) \in \mathcal{C}_{\text{rad}}^{m+1}(\mathbb{R}^N)$  for each  $s \in (-\delta, \delta)$ , and the map  $s \in (-\delta, \delta) \mapsto a_1(\cdot; s) \in \mathcal{C}_{\text{rad}}^{m+1}(\mathbb{R}^N)$  is of class  $\mathcal{C}^{K+1}$ .
- (S2)  $a_2(\cdot; s) \in \mathcal{C}_{\text{rad}}^{m+1}(\mathbb{R}^N)$  for each  $s \in (-\delta, \delta)$ , the map  $s \in (-\delta, \delta) \mapsto a_2(\cdot; s) \in \mathcal{C}_{\text{rad}}^{m+1}(\mathbb{R}^N)$  is of class  $\mathcal{C}^{K+1}$ ,  $g \in \mathcal{C}^{K+m+4}(\mathbb{R}^N \times \mathbb{R} \times (-\delta, \delta))$ , it is radially symmetric in  $x \in \mathbb{R}^N$ , and for all  $\vartheta > 0$ , the function  $g$  is bounded on  $\mathbb{R}^N \times [-\vartheta, \vartheta] \times [0, \delta]$  together with all its partial derivatives up to order  $K + m + 4$ .

The next hypotheses concern the Schrödinger operator  $A_1(s) := -\Delta - a_1(r; s)$  acting on  $L_{\text{rad}}^2(\mathbb{R}^N)$  with domain  $H^2(\mathbb{R}^N) \cap L_{\text{rad}}^2(\mathbb{R}^N)$ :

- (A1)(a) There exists  $L < 0$  such that  $\limsup_{r \rightarrow \infty} a_1(r; s) \leq L$  for all  $s \in (-\delta, \delta)$ .
- (A1)(b) For all  $s \in [0, \delta)$ ,  $A_1(s)$  has exactly 2 nonpositive eigenvalues  $\mu_1(s) < \mu_2(s)$ , and one has  $\mu_2(s) < 0$  for all  $s \in (0, \delta)$ , and  $\mu_2(0) = 0$ .
- (NR) Denoting  $\omega_j(s) := \sqrt{|\mu_j(s)|}$ ,  $j = 1, 2$ , the frequency vector  $\omega(s) = (\omega_1(s), \omega_2(s))$  is nonresonant up to order  $K$  for all  $s \in (0, \delta)$ .

Note that, by the radial symmetry, the eigenvalues  $\mu_1(s), \mu_2(s)$  are automatically simple [37]. For  $s \in [0, \delta)$  and  $j \in \{1, 2\}$ , we denote by  $\varphi_j(\cdot; s)$  the eigenfunction of  $A_1(s)$  associated to  $\mu_j(s)$ , normalized in the  $L^2$ -norm. The normalization determines  $\varphi_j$  uniquely up to a sign; we select  $\varphi_j$  such that  $\varphi_j(0; s) > 0$  for each  $s \in [0, \delta)$ .

Our last hypothesis concerns both the coefficient  $a_2$  and the eigenfunction  $\varphi_2$  when  $s = 0$ :

- (A2) One has

$$\int_{\mathbb{R}^N} a_2(x; 0)\varphi_2^3(x; 0)dx \neq 0.$$

The main theorem proved in [35] is the following:

**Theorem 3.1.** *Suppose that the hypotheses (S1), (S2), (A1), (NR) (with  $K, m$  as in (3.3)) and (A2) are satisfied. Then the following statements are valid for each  $s \in (0, \delta)$ , possibly after making  $\delta > 0$  smaller. There exists a solution  $u(x, y)$  of equation (3.1) such that  $u(x, y)$  is radially symmetric in  $x$ ,  $u(x, y) \rightarrow 0$  as  $|x| \rightarrow \infty$ , uniformly in  $y$ , and  $u(x, y)$  is quasiperiodic in  $y$ . In fact, there is an uncountable family of such quasiperiodic solutions (disregarding translations), their frequency vectors forming an uncountable subset of  $\mathbb{R}^2$ .*

We remark that [34, Theorem 2.2] has a similar—and even stronger—conclusion to Theorem 3.1: it yields quasiperiodic solutions with any  $n \geq 2$  number of frequencies if certain conditions on  $a_2, a_3$ , and the eigenfunctions of  $A_1$  are satisfied. However, the fact that both  $a_2$  and  $a_3$  are involved in the hypotheses—unlike in Theorem 3.1 above—makes [34, Theorem 2.2] more difficult to apply in the context of homogeneous problems such as (1.1).

### 3.2 Taylor expansion at a ground state on $\mathbb{R}^N$

In order to apply Theorem 3.1, we consider the Taylor expansion of a (parameter-dependent) nonlinearity  $f$  around a solution of the  $N$ -dimensional problem (1.6). We want the expansion to yield an equation of the form (3.1)–(3.2) such that the hypotheses of Theorem 3.1 are satisfied. Clearly, it would be of no use to take the expansion at the trivial solution  $u \equiv 0$ : since  $-\Delta - f'(0)$  has only continuous spectrum, hypotheses (A1)(a) and (A1)(b) would not be satisfied. Instead, the expansion will be taken at a ground state of (1.6).

By a *ground state* of (1.6) we mean a positive solution  $u^*$  of (1.6) such that  $u^*(x) \rightarrow 0$  as  $|x| \rightarrow \infty$ . Assuming (1.2), a classical result of [15] implies that, after a suitable translation,  $u^*$  is a radially symmetric (around 0) and radially decreasing function. Thus,  $u^*$  depends on  $x \in \mathbb{R}^N$  via  $r = |x|$  only.

To recall a few other relevant concepts, assume  $u^*$  is a ground state of (1.6) and consider the Schrödinger operator  $L(u^*) := -\Delta - f'(u^*(r))$ . Unless stated otherwise, we always consider such operators as unbounded operators on  $L^2_{\text{rad}}(\mathbb{R}^N)$  with domain  $H^2(\mathbb{R}^N) \cap L^2_{\text{rad}}(\mathbb{R}^N)$ . It is well known—see, e.g., [37] for all the basic properties of  $L(u^*)$  listed below—that  $L(u^*)$  is a self-adjoint operator bounded from below whose essential spectrum,  $\sigma_{\text{ess}}(L(u^*))$ , is contained in  $[-f'(0), \infty)$  (the latter uses the fact that  $u^*(\infty) = 0$ ). In particular, the condition  $f'(0) < 0$  implies that  $\sigma(L(u^*)) \cap (-\infty, 0]$  consists of a finite number of isolated eigenvalues, all of which are simple as  $L(u^*)$  is acting on radial functions only. We define the *Morse index* of  $u^*$  as the number of negative eigenvalues of  $L(u^*)$ . Further, we say that  $u^*$  is a *degenerate* ground state if 0 is an eigenvalue of  $L(u^*)$ , otherwise, we say it is *nondegenerate*.

We now introduce a small parameter  $s$  in equations (1.1) and (1.6). Namely, we consider the equations

$$\Delta u + u_{yy} + f(u; s) = 0, \quad (x, y) \in \mathbb{R}^{N+1}, \quad (3.4)$$

and

$$\Delta u + f(u; s) = 0, \quad x \in \mathbb{R}^N, \quad (3.5)$$

where, for some  $\delta > 0$ ,  $f : \mathbb{R} \times (-\delta, \delta) \rightarrow \mathbb{R}$  is a function of class  $\mathcal{C}^2$  (at least) with

$$f(0, s) = 0, \quad f_u(0; s) < 0 \quad (s \in (-\delta, \delta)). \quad (3.6)$$

Even though we will only deal with positive solutions, it will be convenient to assume also that

$$f(u; s) > 0 \quad (u < 0, s \in (-\delta, \delta)). \quad (3.7)$$

Assuming that, for each  $s$ ,  $u^s$  is a ground state of (3.5), set

$$\begin{aligned} a_1(r; s) &:= f_u(u^s(r); s), \\ a_2(r; s) &:= \frac{1}{2} f_{uu}(u^s(r); s), \\ g(r, u; s) &:= \begin{cases} \frac{1}{u^3} (f(u^s(r) + u; s) - f(u^s(r)) - a_1(r; s)u - a_2(r; s)u^2), & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases} \end{aligned} \quad (3.8)$$

Then

$$f(u^s(r) + u; s) = a_1(r; s)u + a_2(r; s)u^2 + u^3 g(r, u; s) \quad (u \in \mathbb{R}, r \geq 0, s \in (-\delta, \delta)), \quad (3.9)$$

and, for any  $s \in (-\delta, \delta)$ ,  $u = u(x, y)$  is a solution of (3.4) if (and only if)  $u = u^s + \tilde{u}$  for some solution  $\tilde{u}$  of (3.1). Moreover, since  $u^s$  is a radial function with  $u^s(\infty) = 0$ , the function  $u(x, y)$  is quasiperiodic in  $y$ , radially symmetric in  $x$ , and decaying to 0 as  $|x| \rightarrow \infty$ , uniformly in  $y$ , if  $\tilde{u}$  has all these properties. These remarks lead to the following sufficient condition for the existence of quasiperiodic solutions of (3.4).

**Theorem 3.2.** *Assume that for some  $\delta > 0$ ,  $f : \mathbb{R} \times (-\delta, \delta) \rightarrow \mathbb{R}$  is a  $\mathcal{C}^2$ -function satisfying (3.6), (3.7), and for each  $s \in (-\delta, \delta)$ ,  $u^s$  is a ground state of (3.5). Assume further that the functions  $a_1$ ,  $a_2$ , and  $g$  defined by (3.8) satisfy hypotheses (S1), (S2), (A1), (NR), and (A2) with  $K$ ,  $m$  as in (3.3). Then, possibly after making  $\delta > 0$  smaller, the following statements are valid for each  $s \in (0, \delta)$ . There exists a positive solution  $u(x, y)$  of (3.4) such that  $u(x, y)$  is radially symmetric in  $x$ ,  $u(x, y) \rightarrow 0$  as  $|x| \rightarrow \infty$ , uniformly in  $y$ , and  $u(x, y)$  is quasiperiodic in  $y$ . In fact, there is an uncountable family of such quasiperiodic solutions, their frequency vectors forming an uncountable subset of  $\mathbb{R}^2$ .*

*Proof.* All these statements, except for the positivity of the solution  $u$ , follow directly from Theorem 3.1 and the above remarks. To prove the positivity of  $u$ , we use the maximum principle. It is sufficient to show that  $u \geq 0$ . Indeed,  $u$ , being quasiperiodic in  $y$ , is a nontrivial solution. Therefore, the relations  $u \geq 0$ ,  $f(0; s) = 0$ , and the strong comparison principle give  $u > 0$  in  $\mathbb{R}^{N+1}$ .

Suppose now, for a contradiction, that  $u < 0$  on some nonempty open set  $\Omega \subset \mathbb{R}^{N+1}$ . We take  $\Omega$  maximal, so that also  $u = 0$  on  $\partial\Omega$ . Since  $u(x, y)$  is quasiperiodic in  $y$  and  $u(x, y) \rightarrow 0$  as  $|x| \rightarrow \infty$ , uniformly in  $y$ ,  $u$  has a local minimum at some point in  $\Omega$ . At that point, equation (3.4) cannot be satisfied since  $f(u(x, y); s) > 0$  for  $(x, y) \in \Omega$ , due to (3.7). This contradiction proves the positivity of  $u$ .  $\square$

Let us ponder the sufficient conditions given by this theorem. First of all, for the smoothness hypotheses (S1), (S2) to be satisfied by the functions in (3.8), the functions  $(u, s) \mapsto f(u; s)$  and  $(x, s) \mapsto u^s(x)$  have to be sufficiently smooth.

Next, hypothesis (A1)(b) dictates that  $u^s$  has to be a nondegenerate ground state of (3.5) of Morse index 2 when  $s > 0$ , while for  $s = 0$  it has to be a degenerate ground state



of Morse index 1. We have already mentioned in the introduction that no ground state with Morse index 2 exists if  $N = 1$ , or if  $N > 1$  and the nonlinearity is of some specific form, such as  $f(u) = u^p - u$  with a subcritical  $p > 1$  [23] (references [6, 7, 24, 30, 40] give other structural conditions on the nonlinearity which imply that the ground state is unique up to translations and has Morse index 1). The existence of a degenerate ground state is also a nontrivial issue. Typically, the uniqueness of the ground state comes along with its nondegeneracy (see, for example, [29]).

Examples of nonlinearities  $f$  for which (1.6) possesses a ground state of Morse index 2 are given in [8, 11]; nonlinearities with ground states of an arbitrary Morse index  $k \geq 2$  were found in [32]. Among these examples, the most explicit one is that of [11], where (1.6) is considered on  $\mathbb{R}^3$  and  $f$  is given by

$$f(u) = \lambda u^p + u^q - u, \quad (3.10)$$

$p, q$  being suitable exponents satisfying  $1 < q < p < 5$  and  $\lambda > 0$  is sufficiently large. As shown in [8, 32], once a nonlinearity which gives a ground state of Morse index greater than 1 is found, taking a homotopy to another nonlinearity with a unique ground state of Morse index 1, one obtains a nonlinearity with a degenerate ground state somewhere on the homotopy. Thus, in principle, nonlinearities from any of the papers [8, 11, 32] could be used as a starting point in our method. The results of [32], which we actually use here, give us enough flexibility to verify all the hypotheses of Theorem 3.2. It is not clear to us if our method, or a modification thereof, could be applied with specific nonlinearities, such as the ones in (3.10). Letting the regularity issues aside, hypothesis (A2) is probably very hard to verify for such nonlinearities.

### 3.3 Sufficient conditions in terms of a Schrödinger operator

As mentioned in the previous section, the results of [32] which yield nonlinearities  $f$  such that (1.6) has degenerate and nondegenerate ground states with a prescribed Morse index (see Theorems 1.1 and 1.3' in [32]) are relevant for our method. However, we shall mainly use two results from [32], Lemmas 2.1 and 3.1 in [32], which tell us how such nonlinearities are found using a certain Schrödinger operator. We recall these results in Lemmas 3.3, 3.4, and 3.5 below.

**Lemma 3.3.** *Assume the following hypotheses.*

(a)  $a(r)$  is a continuous function on  $[0, \infty)$  which converges to a negative limit as  $r \rightarrow \infty$ .

(b)  $w \in \mathcal{C}^1([0, \infty))$  is a positive solution of

$$w_{rr} + \frac{N-1}{r}w_r + \left(a(r) - \frac{N-1}{r^2}\right)w = 0, \quad r > 0, \quad (3.11)$$

which satisfies the following conditions:

(i)  $w(0) = 0, w_r(0) > 0,$

(ii)  $e^{\beta r}w(r) \rightarrow 0, e^{\beta r}w_r(r) \rightarrow 0$  as  $r \rightarrow \infty$  for some  $\beta > 0$ .

Then

$$u^*(r) := \int_r^\infty w(t) dt, \quad r = |x| \geq 0, \quad (3.12)$$

defines a ground state of (1.6) for a  $\mathcal{C}^1$  function  $f$  that satisfies (1.2) and for which

$$f'(u^*(r)) = a(r) \quad (r \geq 0). \quad (3.13)$$

On the interval  $[0, u^*(0)]$ ,  $f$  is given explicitly by

$$f(z) = \int_0^z a(\xi(\tau)) d\tau, \quad (3.14)$$

where  $\xi$  is the inverse of  $u^* : [0, \infty) \rightarrow (0, u^*(0)]$ .

For a little bit of intuition about this statement, consider an equation of the form (1.6) with a ground state  $u^*$ . In spherical coordinates, the (radial) function  $u^*$  satisfies the equation

$$u_{rr}^* + \frac{N-1}{r} u_r^* + f(u^*) = 0.$$

Differentiating this equation, we see that  $w(r) = -u_r^*$  is a positive solution of equation (3.11) with  $a(r)$  given by (3.13). The statement in Lemma 3.3 goes in the opposite direction: given  $a(r)$  and a positive solution  $w(r)$  of (3.11), it yields a nonlinearity  $f$  and a ground state  $u^*$ .

Note also that the function  $w$  represents an eigenfunction of the operator  $-\Delta - a(r)$  if it is considered on the full space  $L^2(\mathbb{R}^N)$  with domain  $H^2(\mathbb{R}^N)$  (not restricted to the space of radial functions). In fact,  $w(r)$  being a positive solution of equation (3.11) means that 0 is an eigenvalue of this operator and it is the minimal eigenvalue with a nonradial eigenfunction. This can be seen using separation of variables. The nonradial eigenfunctions corresponding to the eigenvalue zero are the functions  $w(r)x_j/r$ ,  $j = 1, \dots, N$ , and their linear combinations. Another interpretation of  $w$  is that  $w(r)x_1/r$  is the principal eigenfunction for the operator  $-\Delta - a(r)$  on the half-space  $\mathbb{R}_+^N := \{x \in \mathbb{R}^N : x_1 > 0\}$  with Dirichlet boundary condition on  $\partial\mathbb{R}_+^N$ .

Now, if the ground state  $u^*$  given by Lemma 3.3 is to have a given Morse index  $k$ , then the operator  $-\Delta - a(r)$  must have exactly  $k$  negative eigenvalues. In particular, the first  $k$  eigenvalues with radial eigenfunctions must come before the first eigenvalue with a nonradial eigenfunction. Potentials with this property, and some additional useful features, are provided by the following lemma:

**Lemma 3.4.** *For any integer  $n \geq 2$  there exists a  $\mathcal{C}^\infty$  function  $a_0(r)$  on  $[0, \infty)$  such that the following statements are valid:*

- (a) *There exist constants  $k_0 > 0$ ,  $k_\infty > 0$ , and  $\ell > 1$  such that  $a_0 \equiv k_0$  on  $[0, 1/\ell]$  and  $a_0 \equiv -k_\infty$  on  $[\ell, \infty)$ .*
- (b) *Equation (3.11) with  $a = a_0$  has a positive solution  $w$  as in Lemma 3.3(b).*
- (c) *The  $n$ th eigenvalue of  $-\Delta - a_0(r)$  (viewed as an unbounded operator on  $L_{\text{rad}}^2(\mathbb{R}^N)$  with domain  $H^2(\mathbb{R}^N) \cap L_{\text{rad}}^2(\mathbb{R}^N)$ ) is equal to zero.*

This is the first part of [32, Lemma 3.1]; the result was proved there with  $k_0 = k_\infty = 1$ .

**Lemma 3.5.** *Let  $\ell$  and  $a_0$  be as in Lemma 3.4. Then there exist  $\delta > 0$  and a smooth function  $b(r; s)$  on  $[0, \infty) \times (-\delta, \delta)$  satisfying the identities*

$$b(r; 0) = 0 \quad (r \geq 0), \quad b(r; s) = 0 \quad (r \in [0, 1/\ell] \cup [\ell, \infty), s \in (-\delta, \delta)), \quad (3.15)$$

*and the following statement. For each  $s \in (-\delta, \delta)$  statement (b) of Lemma 3.4 holds with  $a_0$  replaced by  $a(\cdot; s) := a_0 + b(\cdot; s)$ , and, denoting by  $\mu_n^s$  the  $n$ th eigenvalue of the operator  $-\Delta - a(\cdot; s)$  (on  $L_{\text{rad}}^2(\mathbb{R}^N)$ ), one has*

$$\left. \frac{d}{ds} \mu_n^s \right|_{s=0} < 0. \quad (3.16)$$

Perhaps a word of explanation for the last statement is due here. Note that the first identity in (3.15) implies that statement (a) of Lemma 3.4 holds with  $a_0$  replaced by  $a(\cdot; s) := a_0 + b(\cdot; s)$ . In particular, the essential spectrum of  $-\Delta - a(\cdot; s)$  is contained in  $[k_\infty, \infty)$  for all  $s \approx 0$ . Therefore,  $\sigma(-\Delta - a(\cdot; s)) \cap (-\infty, k_\infty)$  consists of simple isolated eigenvalues, which we number in an increasing manner. For  $s \approx 0$ ,  $a(\cdot; s)$  is a small perturbation of  $a_0$ . Hence, due to statement (c) and the simplicity of the eigenvalues (in the present radial setting), the  $n$ th eigenvalue  $\mu_n^s$  is well defined and it is a smooth function of  $s$  (see [20] for the underlying perturbation results).

Lemma 3.5 is essentially the second part of [32, Lemma 3.1]. Although it was not emphasized there that the function  $b$  with the indicated properties exists for *any* smooth function  $a_0$  satisfying (a)–(c), this is how the lemma is proved in [32]. The only other difference of the present statement from [32, Lemma 3.1] is that in [32] the constants  $k_0, k_\infty$  were specifically taken to be equal to 1. This makes almost no difference in the proof; the only minor modification one needs to make is a rescaling of the Bessel functions as in the following remark.

**Remark 3.6.** For  $s \in (-\delta, \delta)$  and  $a = a(\cdot; s)$  as in Lemma 3.5, the solution  $w = w(\cdot; s)$  of (3.11) as in Lemma 3.3(b) is not unique (it is unique, up to a scalar multiple), but it can be chosen in such a way that  $w(r; s)$  is a  $\mathcal{C}^\infty$  function on  $[0, \infty) \times (-\delta, \delta)$  satisfying

$$\begin{aligned} w(r; s) &= w_1(r) & (r \in [0, 1/\ell], s \in (-\delta, \delta)), \\ w(r; s) &= \gamma(s)w_2(r) & (r \in [\ell, \infty), s \in (-\delta, \delta)), \end{aligned} \quad (3.17)$$

where  $\gamma : (-\delta, \delta) \rightarrow \mathbb{R}$  is smooth (see [32, Remark 3.2] for details), and  $w_1(r), w_2(r)$  are independent of  $s$ : since the functions  $a = a(\cdot; s)$ , for  $s \in (-\delta, \delta)$ , satisfy the identities

$$a(r; s) \equiv k_0 \quad (r \in [0, 1/\ell]), \quad a(r; s) \equiv -k_\infty \quad (r \geq \ell), \quad (3.18)$$

$w_1(r), w_2(r)$  are explicitly given by

$$w_1(r) = c_1 r^{1-N/2} J_{N/2}(r\sqrt{k_0}), \quad w_2(r) = c_2 r^{1-N/2} K_{N/2}(r\sqrt{k_\infty}), \quad (3.19)$$

where  $c_1, c_2$  are nonzero constants and  $J_{N/2}, K_{N/2}$  are, respectively, the Bessel function (of the first kind) and the modified Bessel function (of the second kind) of index  $N/2$ .

Let now  $a_0$ ,  $a = a(r; s)$  be as in Lemmas 3.4, 3.5 with  $n = 2$ , and take the solution  $w(r; s)$  of (3.11) as in Remark 3.6. Lemma 3.3 yields a family of nonlinearities  $f(u; s)$ ,  $s \in (-\delta, \delta)$ , along with corresponding ground states  $u^s$  of (3.5), to which we want to apply Theorem 3.2.

In accord with (3.12), we first set

$$u(r; s) := \int_r^\infty w(t; s) dt \quad (r \geq 0, s \in (-\delta, \delta)). \quad (3.20)$$

By Remark 3.6,  $u$  is a smooth function on  $[0, \infty) \times (-\delta, \delta)$  and one has

$$\begin{aligned} u(r; s) &= \gamma(s)u_2(r) & (r \in [\ell, \infty), s \in (-\delta, \delta)), \\ u(r; s) &= u_1(r) + \beta(s) & (r \in [0, 1/\ell], s \in (-\delta, \delta)), \end{aligned} \quad (3.21)$$

where

$$\begin{aligned} u_2(r) &= \int_r^\infty w_2(t) dt & (r \geq \ell), \\ u_1(r) &= \int_r^{1/\ell} w_1(t) dt & (r \in [0, 1/\ell]), \\ \beta(s) &= \int_{1/\ell}^\ell w(t; s) dt + \gamma(s) \int_\ell^\infty w_2(t) dt. \end{aligned} \quad (3.22)$$

From this it also follows that  $u$  is smooth when considered as a function of  $x \in \mathbb{R}^N$  and  $s \in (-\delta, \delta)$ , radially symmetric in  $x$  (obviously, just the smoothness near  $x = 0$  is an issue here). Indeed, using (3.22), (3.19), and the Frobenius expansion for the Bessel function, one can see that  $u_1(r)$  is analytic near  $r = 0$  and its Taylor series involves only even powers of  $r$ .

Next, for each  $s \in (-\delta, \delta)$ , we use (3.14) with  $a = a(r; s)$  and  $u^* = u(r; s)$ . This defines a function  $f(z, s)$  on

$$U := \{(z, s) \in \mathbb{R}^2 : z \in [0, u(0; s)], s \in (-\delta, \delta)\}.$$

Clearly,  $f$  is smooth in the interior of  $U$ . Moreover, relations (3.14), (3.13), and (3.18) imply that, for any  $s \in (-\delta, \delta)$ ,  $f(u; s) = -k_\infty u$  for  $u$  near 0, and  $f_u(u; s) = k_0$  for  $u$  near  $u(0; s)$ . It is therefore easy to extend  $f$  to  $\mathbb{R} \times (-\delta, \delta)$  in such a way that the extension (which we still denote by  $f$ ) is of class  $\mathcal{C}^\infty$ ,  $f(u; s) > 0$  if  $u < 0$ , and, possibly after making  $\delta > 0$  smaller,  $f$  and all its derivatives are bounded.

Consider the functions  $a_1, a_2, g$  as in (3.8):

$$a_1(r; s) := f_u(u(r; s); s), \quad (3.23)$$

$$a_2(r; s) := \frac{1}{2} f_{uu}(u(r; s); s), \quad (3.24)$$

$$g(r, u; s) := \begin{cases} \frac{1}{u^3} (f(u(r; s) + u; s) - f(u(r; s)) - a_1(r; s)u - a_2(r; s)u^2), & \text{if } u \neq 0, \\ 0, & \text{if } u = 0. \end{cases} \quad (3.25)$$

According to our convention, when needed, the functions  $a_1(\cdot; s)$ ,  $a_2(\cdot; s)$ ,  $g(\cdot, u; s)$  are viewed as functions of  $x \in \mathbb{R}^N$  (depending on  $x$  via  $r = |x|$ ). As we now demonstrate, the hypotheses of Theorem 3.2, with the exception of (A2), are satisfied for these functions.

**Lemma 3.7.** *Making  $\delta > 0$  smaller, if necessary, one achieves that the functions  $a_1$ ,  $a_2$ , and  $g$  defined above satisfy hypotheses (S1), (S2), (A1), (NR) with  $K$ ,  $m$  as in (3.3).*

*Proof.* Using (3.23), (3.24), (3.13), and the definition of  $f$  (cp. (3.14)), we find the following identities for  $a_1$ ,  $a_2$ :

$$\begin{aligned} a_1(r; s) &= a(r; s), \\ 2a_2(r; s) &= \frac{a'(r; s)}{u'(r; s)} = -\frac{a'(r; s)}{w(r; s)}, \end{aligned} \tag{3.26}$$

where the prime denotes the derivative with respect to  $r$ . In particular,  $a_1(r; s) = k_0$ ,  $a_2(r; s) = 0$  for  $r$  near 0, and  $a_1(r; s) = -k_\infty$ ,  $a_2(r; s) = 0$  for all large enough  $r$ . It follows that the regularity hypotheses (S1), (S2) with  $K$ ,  $m$  as in (3.3)—in fact, with arbitrary finite  $K$  and  $m$ —are satisfied by  $a_1$ ,  $a_2$ , and also (A1)(a) is satisfied with  $L = -k_\infty$ . To verify the regularity hypothesis (S2) for the function  $g$ , we use the Taylor expansion for  $f(\cdot; s)$  to write  $g$  in the integral form:

$$g(r, z; s) = \int_0^1 (1-t)(f_{uu}(u(r; s) + tz; s) - f_{uu}(u(r; s))) dt.$$

As noted above,  $u$  is smooth when considered as a function of  $x \in \mathbb{R}^N$  and  $s \in (-\delta, \delta)$ . Moreover, in view of (3.21), (3.19), all derivatives of  $u$  are bounded (we may need to make  $\delta$  smaller here so that the derivatives of  $\gamma(s)$  are all bounded on  $(-\delta, \delta)$ ). Using this and the above definition of the (extended) function  $f$ , one shows easily that (S2) is satisfied.

Next, (3.16) and Lemmas 3.4 and 3.5 imply that  $\mu_2^0 = 0$  and  $\mu_2^s < 0$  for  $s > 0$ ,  $s \approx 0$  (remember that we have taken  $n = 2$ , and the eigenvalues below the essential spectrum of  $-\Delta - a(\cdot; s)$  are numbered in an increasing manner). Making  $\delta > 0$  smaller, we achieve that for  $s \in [0, \delta)$  the operator  $-\Delta - a(\cdot; s)$  has exactly two nonpositive eigenvalues,  $\mu_1^s < \mu_2^s$ , which are strictly negative for  $s \in (0, \delta)$ . Hypothesis (A1)(b) is thus satisfied.

Finally, consider the frequency vector  $\omega(s) = (\omega_1(s), \omega_2(s))$ ,  $\omega_j(s) := \sqrt{|\mu_j^s|}$ ,  $j = 1, 2$ . Since  $\mu_1^0 < \mu_2^0 = 0$ , appealing to the continuity of the eigenvalues in  $s$ , we infer that

$$0 < \omega_2(s) < \frac{\omega_1(s)}{2K}$$

for all  $s \in (0, \delta)$ , possibly after  $\delta > 0$  is made smaller. This implies that hypothesis (NR) is satisfied.  $\square$

Before proceeding further, we summarize where we stand in terms of the applicability of Theorem 3.2:

**Corollary 3.8.** *Let  $a_0(r)$ ,  $a(r; s)$  be as in Lemmas 3.4, 3.5 with  $n = 2$ , and  $w(r; s)$  as in Remark 3.6. Let  $u^s := u(\cdot; s)$  and  $f(u; s)$  be defined (and extended) as above. Assume that*

$$\int_0^\infty \frac{a_0'(r)}{w(r; 0)} \varphi_2^3(r) r^{N-1} dr \neq 0, \tag{3.27}$$

where  $\varphi_2$  is an eigenfunction of  $-\Delta - a_0(r)$  corresponding to the eigenvalue  $\mu_2^0 = 0$  (cp. statement (c) of Lemma 3.4). Then all hypotheses of Theorem 3.2 are satisfied.

*Proof.* Lemma 3.7 verifies all the hypotheses of Theorem 3.2 except for (A2). In view of (3.26) and the relation  $a(\cdot; 0) = a_0$  (see Lemma 3.5), hypothesis (A2) is the same as (3.27).  $\square$

*Remark.* The integral in (3.27) is well defined, since  $w$  is positive in  $(0, \infty)$  and  $a'_0$  has compact support (cp. Lemma 3.4(a)(b)).

## 4 Completion of the proof of Theorem 2.1

For the proof of our main result, we need to find a function satisfying the conditions of Corollary 3.8. Specifically, we seek a smooth function  $a(r)$  on  $[0, \infty)$  with the following properties:

- (a1) There exist constants  $k_0 > 0$ ,  $k_\infty > 0$ , and  $\ell > 1$  such that  $a \equiv k_0$  on  $[0, 1/\ell]$  and  $a \equiv -k_\infty$  on  $[\ell, \infty)$ .
- (a2) Equation (3.11) has a positive solution  $w$  as in Lemma 3.3(b).
- (a3) The operator  $-\Delta - a(r)$  (viewed as an unbounded operator on  $L^2_{\text{rad}}(\mathbb{R}^N)$  with domain  $H^2(\mathbb{R}^N) \cap L^2_{\text{rad}}(\mathbb{R}^N)$ ) has exactly two nonpositive eigenvalues  $\mu_1 < \mu_2$  with  $\mu_2 = 0$ .
- (a4) One has

$$\int_0^\infty \frac{a'(r)}{w(r)} \varphi_2^3(r) r^{N-1} dr \neq 0,$$

where  $w$  is as in (a2) and  $\varphi_2$  is an eigenfunction of  $-\Delta - a(r)$  corresponding to the eigenvalue  $\mu_2^0 = 0$ .

**Proposition 4.1.** *There exists a smooth function  $a$  on  $[0, \infty)$  such that statements (a1)–(a4) above are all satisfied.*

Before taking on the proof of this proposition, we show how it is used to complete the proof of Theorem 2.1.

*Proof of Theorem 2.1.* For the purpose of this proof, we denote the function provided by Proposition 4.1 by  $a_0$ . Then (a1)–(a3) are the same as statements (a)–(c) in Lemma 3.4 with  $n = 2$ . Let now  $a(r; s)$  be as in Lemma 3.5,  $w(r; s)$  as in Remark 3.6; and let  $u^s := u(\cdot; s)$ ,  $f(u; s)$  be defined as in Lemma 3.3 (cp. Corollary 3.8). Recall from Lemma 3.5 that  $a(\cdot; 0) = a_0$ . Therefore, the function  $w(r)$  in the above statement (a2) and the function  $w(r; 0)$  differ only by a scalar factor (cp. Remark 3.6). Thus, from statement (a4) we infer that (3.27) holds. Corollary 3.8 now tells us that all hypotheses of Theorem 3.2 are satisfied. Using that theorem, we conclude that the statement of Theorem 2.1 holds with  $f = f(\cdot; s)$ , for any  $s \in (0, \delta)$ .  $\square$

The remainder of this section is devoted to proving Proposition 4.1. The outline is as follows. We know that if we take  $a = a_0$  with  $a_0$  as in Lemma 3.4, statements (a1)–(a3) are satisfied. If statement (a4) happens to be satisfied by this function, we are done: Proposition 4.1 is proved. Otherwise, our goal is to find a suitable perturbation  $a$  of  $a_0$  such that all statements (a1)–(a4) are valid. Note that statements (a2) and (a3) are not

robust. Therefore the perturbation has to be made carefully for (a2) and (a3) to remain valid (we will perturb  $a_0$  in a compact subinterval of  $(0, \infty)$  only, so there are no issues with statement (a1)).

For the rest of this section, we fix a constant  $\ell > 0$  and a function  $a_0$  with the following properties:

- (A0)  $a_0(r)$  is a smooth function on  $[0, \infty)$  such that conditions (a1)–(a3) are satisfied with  $a = a_0$  and  $k_0 = k_\infty = 1$ .

The existence of such  $a_0$  is guaranteed by [32, Lemma 3.1] (cp. Lemma 3.4 in this paper). We took  $k_0 = k_\infty = 1$  just for simplicity, it is not essential.

In our perturbation arguments, we use the following notation. For  $a \in \mathcal{C}_{\text{rad}}(\mathbb{R}^N)$ ,  $S(a)$  denotes the Schrödinger operator  $-\Delta - a(r)$  on  $L^2_{\text{rad}}(\mathbb{R}^N)$  with domain  $D(S(a)) = H^2(\mathbb{R}^N) \cap L^2_{\text{rad}}(\mathbb{R}^N)$ . If  $S(a)$  has at least two eigenvalues below its essential spectrum (which is in particular the case if  $a$  is close to  $a_0$  in the supremum norm),  $\mu_2[a]$  stands for the second smallest eigenvalue. By  $\varphi_2[a] \in D(S(a))$  we denote the eigenfunction of  $S(a)$  corresponding to  $\mu_2[a]$  normalized in the  $L^2$ -norm. The normalization determines  $\varphi_2[a]$  uniquely up to a sign; for definiteness, we choose  $\varphi_2[a]$  such that  $\varphi_2[a](0) > 0$ . We remark here that, by a Sturm-Liouville property in the radial setting [37], the function  $r \mapsto \varphi_2[a](r)$  has a unique zero, which is positive (in particular,  $\varphi_2[a](0) \neq 0$ ). As above, without fearing confusion, we abuse the notation slightly and use the same symbol for the function  $\varphi_2[a]$ , and other radial functions below, viewed as a function of  $x \in \mathbb{R}^N$  and as a function of  $r \in [0, \infty)$ . Also, we may omit the argument  $a$  from  $\mu_2$ ,  $\varphi$ , and related functions, for the sake of notational simplicity.

For  $a \in \mathcal{C}_{\text{rad}}(\mathbb{R}^N)$  close to  $a_0$ ,  $\mu_2[a] \in \mathbb{R}$  and  $\varphi_2[a] \in H^2(\mathbb{R}^N) \cap L^2_{\text{rad}}(\mathbb{R}^N)$  are well defined and are smooth functions of  $a$  [20]. The eigenfunction  $\varphi_2[a] \in D(S(a))$  is a solution of the following equation (with  $r = |x|$ ):

$$\Delta\varphi + a(r)\varphi + \mu_2[a]\varphi = 0, \quad x \in \mathbb{R}^N; \quad (4.1)$$

as a function of  $r$ , it is a solution of the following problem:

$$\begin{cases} \varphi_{rr} + \frac{N-1}{r}\varphi_r + a(r)\varphi + \mu_2[a]\varphi = 0, & r > 0, \\ \varphi_r(0) = 0, \quad \varphi_2 \rightarrow 0 \text{ as } r \rightarrow \infty. \end{cases} \quad (4.2)$$

We shall also need to perturb the function  $w$  as in statement (a2). For that we introduce the following eigenvalue problem:

$$\begin{cases} \Delta\psi + a(r)\psi + \nu\psi = 0, & x \in \mathbb{R}_+^N := \{(x_1, x') \in \mathbb{R} \times \mathbb{R}^{N-1} : x_1 > 0\}, \\ \psi(0, x') = 0, & x' \in \mathbb{R}^{N-1}. \end{cases} \quad (4.3)$$

Any eigenfunction  $\psi \in H^2(\mathbb{R}^N)$  of this problem can alternatively be viewed as an eigenfunction of the operator  $-\Delta - a(r)$  considered on the closed subspace  $L^2_o(\mathbb{R}^N)$  of  $L^2(\mathbb{R}^N)$  consisting of all functions odd in  $x_1$  (with domain  $H^2(\mathbb{R}^N) \cap L^2_o(\mathbb{R}^N)$ ). We temporarily denote this operator by  $A_o(a)$ .

As noted in Section 3.3—and as one can see by separation of variables—if  $a$  satisfies statements (a1), (a2), then  $\nu = 0$  is the principal (minimal) eigenvalue of (4.3) and it has an eigenfunction

$$\psi(x) = w(r)x_1/r \quad (r = |x|).$$

Here  $w$  is a positive solution of the equation (with  $\nu = 0$ )

$$w_{rr} + \frac{N-1}{r}w_r + \left(a(r) + \nu - \frac{N-1}{r^2}\right)w = 0, \quad r > 0, \quad (4.4)$$

satisfying  $w(0) = 0$ ,  $w(r) \rightarrow 0$  as  $r \rightarrow \infty$ . Since  $\psi > 0$  and  $a(\infty) = -k_\infty < 0$ , it is a standard consequence of the maximum principle that  $\nu = 0$  is a simple eigenvalue. Also,  $\nu = 0$  is an isolated eigenvalue of the operator  $A_o(a)$ : statement (A0) implies that  $\nu = 0$  is below the essential spectrum of this operator. Applying these remarks to  $a_0$ , we conclude that for any  $a \in \mathcal{C}_{\text{rad}}(\mathbb{R}^N)$  close enough to  $a_0$  the minimal eigenvalue  $\nu[a]$  of (4.3) is defined and it is a smooth function of  $a$  [20]. Moreover, there is a (uniquely determined)  $L^2(\mathbb{R}^N)$ -normalized eigenfunction  $\psi[a]$  of  $A_o(a_0)$  with  $\psi[a] > 0$  in  $\mathbb{R}_+^N$  (the positivity can be proved by standard variational arguments [37]). The function  $a \mapsto \psi[a] \in H^2(\mathbb{R}^N) \cap L_o(\mathbb{R}^N)$  is smooth on a neighborhood of  $a_0$  in  $\mathcal{C}_{\text{rad}}(\mathbb{R}^N)$ . Separation of variables gives

$$\psi[a] = w[a](r)x_1/r, \quad (4.5)$$

where  $w[a]$  is a positive solution of (4.4) with  $\nu = \nu[a]$  (we take this as the definition of  $w[a]$ ).

For brevity, we set

$$\varphi_{2,0} := \varphi_2[a_0], \quad w_0 := w[a_0], \quad \text{and} \quad \psi_0 := \psi[a_0]. \quad (4.6)$$

We take a sufficiently small neighborhood  $\mathcal{U}$  of  $a_0$  in  $\mathcal{C}_{\text{rad}}(\mathbb{R}^N)$  so that

(U)  $\mu_2[a]$ ,  $\varphi_2[a]$ ,  $\nu[a]$ ,  $\psi[a]$  are defined and have the smoothness properties with respect to  $a \in \mathcal{U}$ , as specified above.

If  $a \in \mathcal{U}$  is of class  $\mathcal{C}^1$  as a function of  $r \in [0, \infty)$  and such that the support of  $a'$  is a compact subset of  $(0, \infty)$ , we denote

$$E_a := \int_0^\infty \frac{a'(r)}{w[a](r)} \varphi_2^3[a](r) r^{N-1} dr. \quad (4.7)$$

This is the integral that we want to made different from zero by taking a suitable perturbation of the function  $a_0$  (assuming  $E_{a_0} = 0$ ). Note that  $E_a \in \mathbb{R}$  is well defined, as  $w[a] > 0$  on  $(0, \infty)$ .

We will look for  $a$  within a two-parameter family of potentials

$$\bar{a}(\cdot; t, \tau) = a_0 + \tau b_0 + t b_1 \quad (t \approx 0, \tau \approx 0), \quad (4.8)$$

where  $t$  and  $\tau$  are so small that  $\bar{a}(\cdot; t, \tau) \in \mathcal{U}$  and  $b_0, b_1$  are suitably chosen smooth radial functions. Specifically, we want  $b_0, b_1$  to satisfy the following set of conditions (with  $\ell > 1$  as in (A0)):



(B1) The support of the function  $b_0(r)$  is a compact subset of  $(1/\ell, \ell)$ .

(B2) The support of the function  $b_1(r)$  is a compact subset of  $(\ell, \infty)$ .

(B3) 
$$-\int_{\mathbb{R}^N} b_0 \varphi_{2,0}^2 dx > -\int_{\mathbb{R}^N} b_0 \psi_0^2 dx, \text{ or, equivalently,}$$

$$-\int_0^\infty b_0(r) \varphi_{2,0}^2(r) r^{N-1} dr > -\int_0^\infty b_0(r) w_0^2(r) r^{N-1} dr.$$

(B4) 
$$\int_{\mathbb{R}^N} b_1 \varphi_{2,0}^2 dx = \int_{\mathbb{R}^N} b_1 \psi_0^2 dx.$$

Moreover, for a suitable constant  $C_0$  determined by  $a_0$ , as specified below (see Lemma 4.6), we want the following condition to be satisfied:

(B5) 
$$\int_\ell^\infty b_1(r) \left( C_0 \varphi_{2,0}^2(r) r^{N-1} - \frac{d}{dr} \left( \frac{\varphi_{2,0}^3(r)}{w_0(r)} r^{N-1} \right) \right) dr \neq 0.$$

Note that if (B1) and (B2) hold, then the integrals in (B3), (B4)—written in spherical coordinates—and (B5) are in effect integrals over compact subintervals of  $(0, \infty)$  and are thus well defined.

The existence of functions  $b_1, b_2$  with the above properties, as guaranteed by the next lemma, is key to our method.

**Lemma 4.2.** *For any given constant  $C_0$  (and function  $a_0$  as in (A0)), there exist smooth, radially symmetric functions  $b_0, b_1$ , such that (B1)–(B5) are satisfied.*

We give a proof of this result, based on properties of modified Bessel functions, at the end of this section.

Without specifying the constant  $C_0$  yet, assume that smooth radial functions  $b_0, b_1$  satisfying (B1)–(B5) have been chosen. We take  $\varepsilon_0 > 0$  such that for  $|t|, |\tau| < \varepsilon_0$  one has  $\bar{a}(\cdot; t, \tau) \in \mathcal{U}$ , so  $\mu_2[\bar{a}(\cdot; t, \tau)], \varphi_2[\bar{a}(\cdot; t, \tau)], \nu[\bar{a}(\cdot; t, \tau)],$  and  $\psi[\bar{a}(\cdot; t, \tau)]$  are all well defined and depend smoothly on  $(t, \tau)$ . A priori, the eigenfunctions  $\varphi_2[\bar{a}(\cdot; t, \tau)]$  and  $\psi[\bar{a}(\cdot; t, \tau)]$  depend smoothly on  $(t, \tau)$  as  $H^2(\mathbb{R}^N)$ -valued functions, but combining this with elliptic regularity results (and the smoothness of  $a_0, b_0, b_1$ ) we also have the smoothness in many other spaces, for example  $\mathcal{C}^m(\mathbb{R}^N)$  for any  $m > 0$ . This is useful for justifying some computations below. Note also that since we are dealing with eigenvalues below the essential spectrum, the corresponding eigenfunctions always decay exponentially as  $|x| \rightarrow \infty$  (see [1, 19], for example).

**Lemma 4.3.** *Let  $\varepsilon_0$  be as above. There exists  $\varepsilon \in (0, \varepsilon_0)$  such that the following statements hold:*

- (a)  $\frac{\partial}{\partial \tau} (\mu_2[\bar{a}(\cdot; t, \tau)] - \nu[\bar{a}(\cdot; t, \tau)]) > 0$  for all  $(t, \tau) \in [-\varepsilon, \varepsilon]^2$ .
- (b) There exists  $\delta_1 \in (0, \varepsilon)$  such that  $\mu_2[\bar{a}(\cdot; t, -\varepsilon)] - \nu[\bar{a}(\cdot; t, -\varepsilon)] < 0 < \mu_2[\bar{a}(\cdot; t, \varepsilon)] - \nu[\bar{a}(\cdot; t, \varepsilon)]$  for all  $t \in (-\delta_1, \delta_1)$ .

(c) For each  $t \in (-\delta_1, \delta_1)$ , with  $\delta_1$  as in (b), there exists a unique solution  $\tau = \tau(t)$  of

$$\mu_2[\bar{a}(\cdot; t, \tau)] - \nu[\bar{a}(\cdot; t, \tau)] = 0. \quad (4.9)$$

Moreover,  $t \mapsto \tau(t)$  is a smooth function on  $(-\delta_1, \delta_1)$  satisfying  $\tau(0) = \tau'(0) = 0$ .

**Remark 4.4.** A consequence of statement (c) of the lemma is that the one-parameter family of potentials

$$\bar{a}(\cdot; t, \tau(t)) - \mu_2[\bar{a}(\cdot; t, \tau(t))], \quad t \approx 0, \quad (4.10)$$

satisfies that the second eigenvalue of (4.1) and the principal eigenvalue of (4.3) (both equations considered with  $a$  given by (4.10)) do not change with  $t$  and remain equal to zero. (This is also true for (4.4).) Hence, for all potentials in this family statements (a2), (a3) are satisfied. By (B1), (B2), statement (a1) is also satisfied, after adjusting  $\ell$ , with the constants  $k_0 = 1 - \mu_2[\bar{a}(\cdot; t, \tau(t))]$ ,  $k_\infty = 1 + \mu_2[\bar{a}(\cdot; t, \tau(t))]$ . These constants are close to 1 if  $t \approx 0$ , due to  $\mu_2[\bar{a}(\cdot; 0, \tau(0))] = \mu_2[a_0] = 0$ . In a subsequent step, we will address the validity of statement (a4) for some potentials in this family. Note that the eigenfunctions  $\varphi_2[\bar{a}(\cdot; t, \tau)]$ ,  $w[\bar{a}(\cdot; t, \tau)]$ , and hence the integral in (a4), are unaffected when the potential is shifted by  $\mu_2$ .

The following result will be used in the proof of Lemma 4.3.

**Lemma 4.5.** Denoting by “ $\dot{\cdot}$ ” the derivative of a given function with respect to either  $t$  or  $\tau$ , one has:

(i)  $\dot{\mu} = - \int_{\mathbb{R}^N} \dot{\bar{a}} \varphi_2^2 dx$ , and  $\dot{\varphi}_2$  is given by the unique solution (in the radial space) of

$$\begin{cases} \Delta \dot{\varphi}_2 + \bar{a} \dot{\varphi}_2 + \mu_2 \dot{\varphi}_2 = -\dot{\bar{a}} \varphi_2 + \varphi_2 \int_{\mathbb{R}^N} \dot{\bar{a}} \varphi_2^2 dx, \\ \int_{\mathbb{R}^N} \dot{\varphi}_2 \varphi_2 dx = 0. \end{cases} \quad (4.11)$$

(ii)  $\dot{\nu} = - \int_{\mathbb{R}^N} \dot{\bar{a}} \psi^2 dx$ , and  $\dot{\psi}$  is given by the unique solution (in the space of functions odd in  $x_1$ ) of

$$\begin{cases} \Delta \dot{\psi} + \bar{a} \dot{\psi} + \nu \dot{\psi} = -\dot{\bar{a}} \psi + \psi \int_{\mathbb{R}^N} \dot{\bar{a}} \psi^2 dx, \\ \int_{\mathbb{R}^N} \dot{\psi} \psi dx = 0. \end{cases}$$

*Proof.* For statement (i), recall (cp. (4.1)) that  $\varphi_2$  satisfies

$$\begin{aligned} \Delta \varphi_2 + \bar{a} \varphi_2 + \mu_2 \varphi_2 &= 0, \\ \int_{\mathbb{R}^N} \varphi_2^2 dx &= 1. \end{aligned}$$

Differentiating these equations with respect to either  $t$  or  $\tau$ , we find

$$\begin{aligned}\Delta\dot{\varphi}_2 + \bar{a}\dot{\varphi}_2 + \mu_2\dot{\varphi}_2 + \dot{\bar{a}}\varphi_2 + \dot{\mu}_2\varphi_2 &= 0, \\ \int_{\mathbb{R}^N} \varphi_2\dot{\varphi}_2 \, dx &= 0.\end{aligned}$$

Multiplying the first equation by  $\varphi_2$  and integrating by parts, we obtain

$$\dot{\mu}_2 = - \int_{\mathbb{R}^N} \dot{\bar{a}}\varphi_2^2 \, dx,$$

and the rest of statement (i) follows easily. Statement (ii) is proved in a similar way, using (4.3) instead of (4.1).  $\square$

*Proof of Lemma 4.3.* By definition,  $\mu_2[\bar{a}(\cdot; 0, 0)] = \mu_2[a_0] = 0$  and  $\nu[\bar{a}(\cdot; 0, 0)] = \nu[a_0] = 0$  (cp. (A0)). Similarly,  $\varphi_2[\bar{a}(\cdot; 0, 0)] = \varphi_{2,0}$  and  $\psi[\bar{a}(\cdot; 0, 0)] = \psi_0$  (cp. (4.6)). Also,  $\frac{\partial}{\partial \tau} \bar{a}(\cdot; t, \tau) = b_0$ . At  $(t, \tau) = (0, 0)$  we have

$$\frac{\partial}{\partial \tau} (\mu_2[\bar{a}(\cdot; t, \tau)] - \nu[\bar{a}(\cdot; t, \tau)]) \Big|_{(t, \tau) = (0, 0)} = - \int_{\mathbb{R}^N} b_0 \varphi_{2,0}^2 \, dx + \int_{\mathbb{R}^N} b_0 \psi_0^2 \, dx > 0, \quad (4.12)$$

by Lemma 4.5 and (B3). Since  $\mu_2$  and  $\nu$  depend smoothly on  $(t, \tau)$ , the  $\tau$ -derivative is positive for all  $(t, \tau) \in [-\varepsilon, \varepsilon]^2$  if  $\varepsilon > 0$  is sufficiently small. This proves statement (a). Applying statement (a) with  $t = 0$ , and replacing  $\varepsilon$  by a smaller positive number,  $\varepsilon/2$  say, we obtain in particular that

$$\begin{aligned}\mu_2[\bar{a}(\cdot; 0, -\varepsilon)] - \nu[\bar{a}(\cdot; 0, -\varepsilon)] &< 0, \\ \mu_2[\bar{a}(\cdot; 0, \varepsilon)] - \nu[\bar{a}(\cdot; 0, \varepsilon)] &> 0.\end{aligned}$$

Consequently, by continuity, there is  $\delta_1 \in (0, \varepsilon)$  such that

$$\begin{aligned}\mu_2[\bar{a}(\cdot; t, -\varepsilon)] - \nu[\bar{a}(\cdot; t, -\varepsilon)] &< 0, \quad (|t| \leq \delta_1), \\ \mu_2[\bar{a}(\cdot; t, \varepsilon)] - \nu[\bar{a}(\cdot; t, \varepsilon)] &> 0, \quad (|t| \leq \delta_1).\end{aligned}$$

This proves statement (b). The above relations and the positivity of the  $\tau$ -derivative imply that for each  $t \in (-\delta_1, \delta_1)$  there is a unique  $\tau = \tau(t)$  satisfying (4.9). The implicit function theorem gives the smoothness of the map  $t \mapsto \tau(t)$ . By uniqueness,  $\tau(0) = 0$ . Expanding the equality

$$\frac{d}{dt} (\mu_2[\bar{a}(\cdot; t, \tau(t))] - \nu[\bar{a}(\cdot; t, \tau(t))]) \Big|_{t=0} = 0$$

and rearranging, we obtain

$$\begin{aligned}- \left( \frac{\partial}{\partial \tau} (\mu_2[\bar{a}(\cdot; 0, \tau)] - \nu[\bar{a}(\cdot; 0, \tau)]) \Big|_{\tau=0} \right) \tau'(0) &= \frac{\partial}{\partial t} (\mu_2[\bar{a}(\cdot; t, 0)] - \nu[\bar{a}(\cdot; t, 0)]) \Big|_{t=0} \\ &= - \int_{\mathbb{R}^N} b_1 \varphi_{2,0}^2 \, dx + \int_{\mathbb{R}^N} b_1 \psi_0^2 \, dx \\ &= 0,\end{aligned}$$

where we have used the formulas from Lemma 4.5, the relation  $\frac{\partial}{\partial t} \bar{a}(\cdot; t, 0) \Big|_{t=0} = b_1$ , and the relation in (B4). Since the  $\tau$ -derivative in the left hand side is positive by statement (a), necessarily  $\tau'(0) = 0$ .  $\square$

With  $\delta_1$  and  $\tau(t)$  as in the previous lemma, consider the family

$$\alpha(\cdot; t) := \bar{a}(\cdot; t, \tau(t)) = a_0 + \tau(t)b_0 + tb_1, \quad t \in (-\delta_1, \delta_1). \quad (4.13)$$

Note that, since  $\tau(0) = \tau'(0) = 0$ , we have

$$\begin{aligned} \alpha(\cdot; 0) &= a_0, \\ \frac{d}{dt}\alpha(\cdot; t) \Big|_{t=0} &= \tau'(0)b_0 + b_1 = b_1 = \frac{d}{dt}a(\cdot; t, 0) \Big|_{t=0}. \end{aligned} \quad (4.14)$$

We examine the integral  $E_{\alpha(\cdot; t)}$  given by (4.7), with  $\alpha(\cdot; t)$  in place of  $a$ . Observe that by conditions (A0), (B1), and (B2), the functions  $\alpha'(\cdot; t)$ ,  $|t| < \varepsilon$ , (the derivative with respect to  $r$ ) have support contained in a fixed compact subinterval of  $(0, \infty)$ . Thus the integral in (4.7) is in effect an integral over this compact interval, which implies that  $E_{\alpha(\cdot; t)}$  is (well defined and) a smooth function of  $t$ . Our goal is to show that  $E_{\alpha(\cdot; t)} \neq 0$  for all sufficiently small  $t > 0$ . This is obvious by continuity if  $E_{a_0} \neq 0$ . If  $E_{a_0} = 0$ , that is,

$$\int_0^\infty \frac{a'_0(r)}{w_0(r)} \varphi_{2,0}^3(r) r^{N-1} dr = 0, \quad (4.15)$$

we want to show that the derivative of  $E_{\alpha(\cdot; t)}$  at  $t = 0$  is different from zero. We compute the derivative in the following result.

**Lemma 4.6.** *Assume that (4.15) holds. Then, regardless of how the functions  $b_0, b_1$  are defined, as long as they satisfy (B1)–(B4), one has*

$$\dot{E} := \frac{d}{dt}E_{\alpha(\cdot; t)} \Big|_{t=0} = - \int_\ell^\infty b_1(r) \frac{d}{dr} \left( \frac{\varphi_{2,0}^3(r)}{w_0(r)} r^{N-1} \right) dr + C_0 \int_\ell^\infty b_1(r) \varphi_{2,0}^2(r) r^{N-1} dr, \quad (4.16)$$

where  $C_0$  is a constant determined only by  $a_0$  (and independent of  $b_0, b_1$ ).

**Remark 4.7.** Again, by (A0), (B1), and (B2), the integrals in (4.16), as well as similar integrals in the proof of Lemma 4.6 below, are in effect integrals over a compact subinterval of  $(0, \infty)$ .

In the proof of the lemma, the following elementary relations are used. If  $v$  is a radial (integrable) function, then

$$\int_{\mathbb{R}^N} v(x) dx = c_N \int_0^\infty v(r) r^{N-1} dr,$$

and if  $\tilde{v}(x) = \tilde{w}(r)r/x_1$  with  $r = |x|$ , then

$$\int_{\mathbb{R}^N} \tilde{v}(x) dx = \tilde{c}_N \int_0^\infty \tilde{w}(r) r^{N-1} dr.$$

Here  $c_N, \tilde{c}_N$  are positive constants depending only on the dimension  $N$ .

*Proof of Lemma 4.6.* As in (4.16), we use “ $\dot{\cdot}$ ” to indicate the derivative of the functions  $E_{\alpha(\cdot;t)}$ ,  $\varphi_2[\alpha(\cdot;t)]$ ,  $\psi[\alpha(\cdot;t)]$ , and  $w[\alpha(\cdot;t)]$  with respect to  $t$  at  $t = 0$ . Clearly,  $\dot{w}(r)x_1/r = \dot{\psi}(x)$  (cp. (4.3), Lemma 4.5).

We first carry out the proof assuming the integrals in (B4) are different from zero. The simpler special case when they are equal to zero is considered at the end.

Noting that

$$\frac{d}{dt}(\varphi_2^3[\alpha(\cdot;t)])|_{t=0} = 3\varphi_{2,0}^2\dot{\varphi}_2, \quad \frac{d}{dt}((w[\alpha(\cdot;t)])^{-1})|_{t=0} = -w_0^{-2}\dot{w}$$

with  $\varphi_{2,0}$  and  $w_0$  as in (4.6), we find (cp. Remark 4.7, and recall that  $w(r) > 0$  for  $r > 0$ )

$$\begin{aligned} \dot{E} &= \int_0^\infty b_1'(r) \frac{\varphi_{2,0}^3(r)}{w_0(r)} r^{N-1} dr + \\ &\quad + 3 \int_0^\infty a_0'(r) \frac{\varphi_{2,0}^2(r)}{w_0(r)} \dot{\varphi}_2(r) r^{N-1} dr - \int_0^\infty a_0'(r) \frac{\varphi_{2,0}^3(r)}{w_0^2(r)} \dot{w}(r) r^{N-1} dr. \end{aligned} \quad (4.17)$$

We now write  $\dot{\varphi}_2$  and  $\dot{w}$  in terms of  $b_1$ ,  $\varphi_{2,0}$ , and  $w_0$ . Since  $\mu_2[a_0] = 0$ ,  $\nu[a_0] = 0$ , from (4.14), (4.11) we obtain

$$\begin{cases} \Delta \dot{\varphi}_2 + a_0 \dot{\varphi}_2 = -b_1 \varphi_{2,0} + \varphi_{2,0} \int_{\mathbb{R}^N} b_1 \varphi_{2,0}^2 dx \\ \int_{\mathbb{R}^N} \varphi_{2,0} \dot{\varphi}_2 dx = 0. \end{cases} \quad (4.18)$$

Writing the equation in (4.18) in spherical coordinates and using that  $b_1 \equiv 0$  on  $[0, \ell]$ , we obtain the following equation for  $\dot{\varphi}_2$  on  $(0, \ell)$ :

$$(\dot{\varphi}_2)_{rr} + \frac{N-1}{r}(\dot{\varphi}_2)_r + a_0 \dot{\varphi}_2 = \varphi_{2,0} c_N \int_\ell^\infty b_1 \varphi_{2,0}^2 \rho^{N-1} d\rho, \quad r \in (0, \ell), \quad (4.19)$$

with  $\dot{\varphi}_2$  bounded near  $r = 0$ .

Let now  $\zeta$  be any solution of

$$\begin{cases} \zeta_{rr} + \frac{N-1}{r} \zeta_r + a_0 \zeta = c_N \varphi_{2,0} & r \in (0, \ell), \\ \zeta \text{ bounded near } r = 0. \end{cases} \quad (4.20)$$

The existence of  $\zeta$  can be shown by standard ODE techniques, but it also follows from the boundedness of  $\dot{\varphi}_2$  (simply divide (4.19) by the nonzero integral appearing on the right hand side). Whichever way  $\zeta$  is found, it is a function determined only by  $a_0$ , which we fix for the rest of the proof.

Observe that up to a scalar multiple, the function  $\varphi_{2,0}$  is the only bounded solution of the homogeneous equation associated with (4.19), that is, equation (4.19) with the right hand side replaced by 0. This comes from the fact that  $a_0 \equiv 1$  on  $(0, \ell)$  (cp. (A0)), which implies that the bounded solutions are all scalar multiples of  $r^{1-N/2} J_{N/2-1}$ ,  $J_{N/2-1}$  being

the Bessel function (of the first kind) of index  $N/2 - 1$ . This fact and the special form of the right hand side of (4.19) imply that for  $r \in (0, \ell)$  one has

$$\dot{\varphi}_2(r) = \zeta(r) \int_{\ell}^{\infty} b_1 \varphi_{2,0}^2 \rho^{N-1} d\rho + C_{\zeta} \varphi_{2,0}(r), \quad (4.21)$$

where  $C_{\zeta}$  is a constant (depending on  $\zeta$  and  $b_1$ ).

We can write  $\dot{w}$  in a similar form using analogous arguments: the equation for  $\dot{w}$  in spherical coordinates is

$$\dot{w}_{rr} + \frac{N-1}{r} \dot{w}_r + \left( a_0(r) - \frac{N-1}{r^2} \right) \dot{w} = -b_1 w_0 + \varphi_{2,0} \tilde{c}_N \int_{\ell}^{\infty} b_1 w_0^2 \rho^{N-1} d\rho, \quad (4.22)$$

which on  $(0, \ell)$  reduces to

$$\dot{w}_{rr} + \frac{N-1}{r} \dot{w}_r + \left( a_0(r) - \frac{N-1}{r^2} \right) \dot{w} = \varphi_{2,0} \tilde{c}_N \int_{\ell}^{\infty} b_1 w_0^2 \rho^{N-1} d\rho, \quad r \in (0, \ell). \quad (4.23)$$

Since  $w_0$  is, up to a constant multiple, the unique bounded solution of the homogeneous equation associated with (4.23)—this time the bounded solutions are scalar multiples of  $r^{1-N/2} J_{N/2}(r)$ , cp. (3.19)—we have

$$\dot{w}(r) = \gamma(r) \int_{\ell}^{\infty} b_1 w_0^2 \rho^{N-1} d\rho + C_{\gamma} w_0(r) \quad \text{for } r \in (0, \ell), \quad (4.24)$$

where  $\gamma$  (unrelated to the function  $\gamma$  in Remark 3.6) is a particular solution of

$$\gamma_{rr} + \frac{N-1}{r} \gamma_r + \left( a_0(r) - \frac{N-1}{r} \right) \gamma = \tilde{c}_N \varphi_{2,0} \quad (4.25)$$

which is bounded near  $r = 0$  and  $C_{\gamma}$  is a constant (depending on  $b_1$  and  $\gamma$ ). The function  $\gamma$ , which is determined by  $a_0$  alone, is fixed in the rest of the proof.

Substituting (4.21) and (4.24) in (4.17), and using Remark 4.7, we have

$$\begin{aligned} \dot{E} &= \int_{\ell}^{\infty} b_1'(r) \frac{\varphi_{2,0}^3(r)}{w_0(r)} r^{N-1} dr \\ &+ 3 \int_{\ell}^{\infty} b_1(\rho) \varphi_{2,0}^2(\rho) \rho^{N-1} d\rho \int_{1/\ell}^{\ell} a_0'(r) \frac{\varphi_{2,0}^2(r)}{w_0(r)} \zeta(r) r^{N-1} dr \\ &- \int_{\ell}^{\infty} b_1(\rho) w_0^2(\rho) \rho^{N-1} d\rho \int_{1/\ell}^{\ell} a_0'(r) \frac{\varphi_{2,0}^3(r)}{w_0^2(r)} \gamma(r) r^{N-1} dr \\ &+ 3C_{\zeta} \int_{1/\ell}^{\ell} a_0'(r) \frac{\varphi_{2,0}^2(r)}{w_0(r)} \varphi_{2,0}(r) r^{N-1} dr - C_{\gamma} \int_{1/\ell}^{\ell} a_0'(r) \frac{\varphi_{2,0}^3(r)}{w_0^2(r)} w_0(r) r^{N-1} dr. \end{aligned} \quad (4.26)$$

Since  $\text{supp } a_0' \subset (1/\ell, \ell)$ , the last two integrals coincide with the integral in (4.15), so they vanish. Also, using (B2) and (B4),

$$\begin{aligned} \int_{\ell}^{\infty} b_1(\rho) w_0^2(\rho) \rho^{N-1} d\rho &= \int_0^{\infty} b_1(\rho) w_0^2(\rho) \rho^{N-1} d\rho = \int_{\mathbb{R}^N} b_1 \psi_0^2 dx \\ &= \int_{\mathbb{R}^N} b_1 \varphi_{2,0}^2 dx = \int_{\ell}^{\infty} b_1(\rho) \varphi_{2,0}^2(\rho) \rho^{N-1} d\rho. \end{aligned}$$

Using these relations in (4.26) and integrating by parts in the first integral in (4.26), we obtain the desired result, (4.16), with

$$C_0 := 3 \int_{1/\ell}^{\ell} a'_0(r) \frac{\varphi_{2,0}^2(r)}{w_0(r)} \zeta(r) r^{N-1} dr - \int_{1/\ell}^{\ell} a'_0(r) \frac{\varphi_{2,0}^3(r)}{w_0^2(r)} \gamma(r) r^{N-1} dr.$$

This concludes the proof in the case the integrals in (B4) do not vanish.

If the integrals in (B4) are equal to 0, one can take  $\zeta \equiv 0 \equiv \gamma$ . The relations (4.21) and (4.24) are then valid and the above computations still apply. They lead to (4.16) with  $C_0 = 0$ .  $\square$

We can now complete the proof of Proposition 4.1.

*Proof of Proposition 4.1.* Assuming (A0), Lemma 4.2 guarantees the existence of smooth radial functions  $b_1, b_0$  satisfying conditions (B0)–(B4), as well as condition (B5) with  $C_0$  as in Lemma 4.6. For such functions we have, according to Lemma 4.6,  $E_{\alpha(\cdot;0)} \neq 0$  or  $\dot{E} \neq 0$ . In either case,  $E_{\alpha(\cdot;t)} \neq 0$  for all sufficiently small  $t > 0$ . Therefore, using Lemma 4.3 and Remark 4.4, we conclude that statements (a1)–(a4) are satisfied by  $a = \alpha(\cdot;t) = a_0 + \tau(t)b_0 + tb_1$  if  $t > 0$  is sufficiently small.  $\square$

It remains to prove Lemma 4.2.

*Proof of Lemma 4.2.* To simplify the notation, we set

$$\varphi_2 = \varphi_{2,0} = \varphi_2[a_0], \quad w = w_0 = w[a_0].$$

Let  $C_0$  be an arbitrary constant.

We start by noting that the functions  $\varphi_2$  and  $w$  are linearly independent on any interval in  $(0, \infty)$ . This is obvious from equations (4.2) and (4.4) (with  $a = a_0$ ) satisfied by  $\varphi_2$  and  $w$ , respectively. Therefore also the functions  $\varphi_2^2$  and  $w^2$  are linearly independent on any interval in  $(0, \infty)$ . Using this observation with the interval  $(1/\ell, \ell)$ , we infer that the linear operator

$$b_0 \in L^2(1/\ell, \ell) \mapsto \left( \int_{1/\ell}^{\ell} b_0(r) \varphi_2^2(r) r^{N-1} dr, \int_{1/\ell}^{\ell} b_0(r) w^2(r) r^{N-1} dr \right) \in \mathbb{R}^2$$

is surjective onto  $\mathbb{R}^2$ . The surjectivity and the density of  $\mathcal{D}(1/\ell, \ell)$ —the space of smooth, compactly supported functions—in  $L^2(1/\ell, \ell)$  clearly imply the existence of a smooth radial function  $b_0$  satisfying (B1), (B3).

By a similar surjectivity argument, if the functions

$$(\varphi_2^2(r) - w^2(r)) r^{N-1}, \quad C_0 \varphi_2^2(r) r^{N-1} - \left( \frac{\varphi_2^3(r)}{w(r)} r^{N-1} \right)' \quad (4.27)$$

are linearly independent on  $(\ell, \infty)$ , we can find a smooth radial function  $b_1$  such that (B2), (B5) hold simultaneously with

$$\int_{\ell}^{\infty} b_1(r) (\varphi_2^2(r) - w^2(r)) r^{N-1} dr = 0. \quad (4.28)$$

Since (4.28) and (B2) imply (B4), the proof will be completed once we show that the functions (4.27) are linearly independent on  $(\ell, \infty)$ .

We prove this by contradiction. Assume that, to the contrary, there is a constant  $C_1$  such that

$$C_0\varphi_2^2(r)r^{N-1} - \left(\frac{\varphi_2^3(r)}{w(r)}r^{N-1}\right)' = C_1(\varphi_2^2(r) - w^2(r))r^{N-1} \quad (r \in (\ell, \infty)). \quad (4.29)$$

Dividing the equation in (4.29) by  $r^{N-1}$ , we get

$$C_0\varphi_2^2 - \left(\frac{\varphi_2^3(r)}{w(r)}\right)' - \frac{N-1}{r}\frac{\varphi_2^3(r)}{w(r)} = C_1(\varphi_2^2(r) - w^2(r)) \quad (r \in (\ell, \infty)). \quad (4.30)$$

Since  $a_0(r) \equiv -1$  for  $r > \ell$  and  $\mu_2[a_0] = \nu[a_0] = 0$ , we can explicitly solve equations (4.2) and (4.4) for  $r > \ell$ . In view of the boundedness of the functions  $\varphi_2, w$ , we obtain that for  $r \in (\ell, \infty)$

$$\begin{aligned} \varphi_2(r) &= \tilde{\varphi}_2(r) := \bar{c}_2 r^{1-N/2} K_{N/2-1}(r), \\ w(r) &= \tilde{w}(r) := \bar{c} r^{1-N/2} K_{N/2}(r), \end{aligned}$$

where  $\bar{c}_2, \bar{c}$  are constants, and  $K_j$  stands the modified Bessel function of the second kind of index  $j, j \in \{N/2-1, N/2\}$ . The constants  $\bar{c}_2, \bar{c}$  are both nonzero as none of the functions  $\varphi_2, w$  can vanish identically on  $(\ell, \infty)$ : since each function is a solution of a second order ODE, if  $\bar{c}_2 = 0$  or  $\bar{c} = 0$ , then  $\varphi_2$  or  $w$  would vanish identically in  $(0, \infty)$ , in contradiction to the definition of the eigenfunctions  $\varphi_2(r)$  and  $\psi_0(x) = w(r)x_1/r$ .

The above relations show that the identity (4.30) is valid with  $\varphi_2$  and  $w$  replaced by  $\tilde{\varphi}_2$  and  $\tilde{w}$ , respectively. In addition to this identity holding on  $(\ell, \infty)$ , we have, for some  $C_3 \neq 0$ ,

$$\tilde{w}(r) = C_3 \tilde{\varphi}_2'(r) \quad (r > \ell). \quad (4.31)$$

This (well-known identity between the modified Bessel functions) is obtained by differentiating both sides of equation (4.2) (cp. (4.4)).

From (4.31) and (4.2)—the equation satisfied by  $\tilde{\varphi}_2$  on  $(\ell, \infty)$ —we find the following relation (which, again, is just one of well-known identities in the theory of Bessel functions):

$$\tilde{w}' = C_3 \tilde{\varphi}_2'' = C_3 \left( -\frac{N-1}{r} \tilde{\varphi}_2' + \tilde{\varphi}_2 \right) = -\frac{N-1}{r} \tilde{w} + C_3 \tilde{\varphi}_2. \quad (4.32)$$

Expanding the derivative in (4.30),

$$C_0\tilde{\varphi}_2^2 - \frac{3\tilde{w}\tilde{\varphi}_2^2\tilde{\varphi}_2' - \tilde{\varphi}_2^3\tilde{w}'}{\tilde{w}^2} - \frac{N-1}{r}\frac{\tilde{\varphi}_2^3}{\tilde{w}} = C_1(\tilde{\varphi}_2^2 - \tilde{w}^2),$$

and substituting from (4.31) and (4.32), we find

$$C_0\tilde{\varphi}_2^2 - \frac{3}{C_3}\tilde{\varphi}_2^2 + \frac{\tilde{\varphi}_2^3}{\tilde{w}^2} \left( -\frac{N-1}{r}\tilde{w} + C_3\tilde{\varphi}_2 \right) - \frac{N-1}{r}\frac{\tilde{\varphi}_2^3}{\tilde{w}} = C_1(\tilde{\varphi}_2^2 - \tilde{w}^2),$$



or, rearranging,

$$\left(C_0 - \frac{3}{C_3} - C_1\right) \tilde{\varphi}_2^2 - 2 \frac{N-1}{r} \frac{\tilde{\varphi}_2^3}{\tilde{w}} + C_3 \frac{\tilde{\varphi}_2^4}{\tilde{w}^2} = -C_1 \tilde{w}^2.$$

Dividing by  $\tilde{w}^2$  and letting

$$h(r) := \frac{\tilde{\varphi}_2(r)}{\tilde{w}(r)} = \frac{\bar{c}_2 K_{N/2-1}(r)}{\bar{c} K_{N/2}(r)},$$

we obtain

$$\left(C_0 - \frac{3}{C_3} - C_1\right) h^2 - 2 \frac{N-1}{r} h^3 + C_3 h^4 = -C_1. \quad (4.33)$$

A priori, this identity holds on  $(\ell, \infty)$ . However, recalling that  $\tilde{w}$  and  $\tilde{\varphi}_2$  extend to analytic functions on  $\mathbb{C} \setminus (-\infty, 0]$  (see [43], for example) which are also continuous from above on  $(-\infty, 0)$  (that is, from the upper portion of the complex plane), the identity holds in  $\mathbb{C} \setminus \{0\}$ , save for the (isolated) points where  $K_{N/2} = 0$ .

We will use a few additional properties of the modified Bessel functions, all of which can be found in [43]. If  $0 < 2n \in \mathbb{N}$ , then, as  $r \rightarrow 0+$  (on the real axis), one has

$$\begin{aligned} K_n(r) &= Cr^{-n} + \mathcal{O}(r^{-n+1}), \\ K_0(r) &= -C \log r + \mathcal{O}(r). \end{aligned}$$

This implies

$$h(r) \approx \begin{cases} r & \text{if } N > 2 \\ r \log r & \text{if } N = 2; \end{cases}$$

in either case,  $h \rightarrow 0$ ,  $h^3/r \rightarrow 0$  as  $r \rightarrow 0+$ . Using this fact and (4.33), we deduce that  $C_1 = 0$ .

Dividing (4.33) by  $h^2$ , we get the following identity for  $h$ :

$$\left(C_0 - \frac{3}{C_3}\right) - 2 \frac{N-1}{r} h + C_3 h^2 = 0. \quad (4.34)$$

If  $N = 2$ , taking  $r \rightarrow 0+$ , the second term in (4.34) diverges, while the others remain bounded, so (4.34) cannot hold. This contradiction completes the proof in the case  $N = 2$ .

Now assume  $N \geq 3$ . Dividing (4.33) by  $h^4$ , we obtain the following identity

$$\left(C_0 - \frac{3}{C_3}\right) \bar{h}^2 - 2 \frac{N-1}{r} \bar{h} + C_3 = 0 \quad (4.35)$$

for the function

$$\bar{h}(r) := \frac{1}{h} = \frac{\bar{c}}{\bar{c}_2} \frac{K_{N/2}(r)}{K_{N/2-1}(r)}.$$

Similarly to (4.33), this identity may be assumed to hold on  $\mathbb{C} \setminus \{0\}$ , save for the isolated points where  $K_{N/2-1}$  is equal to 0.

Since  $N/2 \geq 3/2$ , the function  $K_{N/2}$  has at least one zero  $r^* \in \mathbb{C} \setminus \{0\}$  (see [43, Section 15.7] for results concerning the zeros of the functions  $K_n$ ). At the same time,  $r^*$  is not a zero of  $K_{N/2-1}$ . This follows from the following recurrence relation

$$K_{n-1}(r) - K_{n+1}(r) = \frac{-2n}{r} K_n(r) \quad (n > 0, r \in \mathbb{C} \setminus (-\infty, 0]) \quad (4.36)$$

and the fact that  $K_\sigma$  has no zeros for  $0 \leq \sigma < 3/2$ . (If  $K_{N/2-1}(r^*) = 0$ , a successive application of (4.36) leads to either  $K_1(r^*) = 0$  or  $K_{1/2}(r^*) = 0$ .) Evaluating (4.35) at  $r = r^*$  (if  $r^* \in (-\infty, 0)$ , which is necessarily the case for  $N = 3$ , the evaluation goes by taking the limit of the values at  $r^* + it$  as  $t \rightarrow 0+$ ), we obtain  $C_3 = 0$ . This and (4.31) give  $w \equiv 0$ , which is a contradiction. With this contradiction, we have completed the proof in the case  $N \geq 3$ .  $\square$

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