MATH 2R03 (THEORY OF LINEAR ALGEBRA)
DUE: SEPTEMBER 27, 2024

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HOMEWORK ASSIGNMENT 3

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the other will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

**Part A.** [Short Questions; 4pts]

**Exercise 1**. Let $U$ be a subspace of the vector space $F^{2024}$ with $\dim U = 1015$. Suppose $W$ is another subspace of $F^{2024}$ such that $U + W = F^{2024}$ and $\dim U \cap W \geq 1000$. Find upper and lower bounds on $\dim W$.

**Exercise 2**. Let $p_1(x) = x + x^3$ and $p_2(x) = 2024$ be elements in the vector space $V = \mathcal{P}_3(\mathbb{R})$. Extend $\{p_1(x), p_2(x)\}$ to a basis of $V$.

*Remark.* You need to find a basis of $\mathcal{P}_3(\mathbb{R})$ where the first two elements in your basis are $p_1(x)$ and $p_2(x)$.

**Part B.** [Proof Questions; 6pts]

**Exercise 3**. Let $V = \mathcal{P}_3(\mathbb{R})$ be the vector space of polynomials of degree at most three.

(1) Suppose $q_0, q_1, q_2, q_3 \in V$ are four polynomials such that $\deg q_i = i$ for $i = 0, \ldots, 3$. Prove that $\{q_0, q_1, q_2, q_3\}$ is a basis of $V$.
(2) Find four polynomials $p_0, p_1, p_2, p_3 \in V$ such that $\deg p_i = 3$ for all $i = 0, \ldots, 3$ and $\{p_0, \ldots, p_3\}$ is a basis for $V$.

*Remark.* There is nothing special about $m = 3$ in this case. Given polynomials $q_0, q_1, \ldots, q_m \in V = \mathcal{P}_m(\mathbb{R})$ with $\deg q_i = i$ for $i = 0, \ldots, m$, then you can prove that $\{q_0, \ldots, q_m\}$ is a basis for $V$.

**Exercise 4**. A matrix $A$ is *skew-symmetric* if $A^T = -A$, where $A^T$ denotes the transpose. For example, the matrix

$$A = \begin{bmatrix} 0 & 1 & -2024 \\ -1 & 0 & 3 \\ 2024 & -3 & 0 \end{bmatrix}$$

is skew-symmetric. Let $V = F^{3,3}$ be the vector space of all $3 \times 3$ matrices, and let

$$W = \{A \in V \mid A \text{ is skew-symmetric}\}.$$

The set $W$ is a subspace of $V$ (you don't need to prove this).

(1) What is $\dim W$? Justify your answer.
(2) Let $U$ be the subspace of all *symmetric matrices*, i.e., $U = \{A \in V \mid A^T = A\}$. Prove $\dim U \leq 6$. (It turns out that $\dim U = 6$, but you don't need to justify this fact.)

**Additional Suggested Problems.** [Not graded]

Problems 2.A # 3, 13 2.B #3, 7, 10, 2.C #11, 14, 17