Math 2R03 (Theory of Linear Algebra)
Due: October 25, 2024

Homework Assignment 5

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let $T \in \mathcal{L}(V, V)$. Suppose that $\lambda = 0$ is an eigenvalue of T. Prove that $v \in V$ is an eigenvector of T corresponding to $\lambda = 0$ if and only if $v \in \text{null}(T) \setminus \{0\}$.

Remark. Recall that eigenvectors are not allowed to be the zero vector. That is why we are excluding the zero vector from $\text{null}(T)$.

Exercise 2. Let $V = F^\infty$. Consider the two linear maps $T, S \in \mathcal{L}(F^\infty, F^\infty)$ where

$$T(a_1, a_2, a_3, \ldots) = (a_3, a_4, \ldots)$$

and

$$S(a_1, a_2, a_3, \ldots) = (0, 0, a_1, a_2, \ldots).$$

Prove that $TS = I$, but $ST \neq I$.

Part B. [Proof Questions; 6pts]

Exercise 3. Let V be a vector space with $\dim V = 5$ and let $T \in \mathcal{L}(V)$. Suppose that for every subspace $W \subseteq V$ with $\dim W = 4$, the linear operator T is invariant on W. Prove that *every* non-zero vector $v \in V$ is an eigenvector of T.

Hint. For any vector $v \in V$, we can find 4 more vectors w_1, w_2, w_3, w_4 of V such that $\{v, w_1, w_2, w_3, w_4\}$ is a basis of V. So, $W_1 = \text{span}(v, w_2, w_3, w_4)$, $W_2 = \text{span}(v, w_1, w_3, w_4)$, $W_3 = \text{span}(v, w_1, w_2, w_4)$, and $W_4 = \text{span}(v, w_2, w_3, w_4)$ are subspaces of V of dimension 4.

Exercise 4. Let V be a finite dimensional vector space, and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

Remark. Note that Exercise 2 shows that this exercise is false if V is not finite dimensional.

Additional Suggested Problems. [Not graded]

Problems 3.D - 11, 14, 15; Chap 4 - 4, 5, 6; 5.A - 7, 8, 9