Math 2R03 (Theory of Linear Algebra)
Due: November 1, 2024

Homework Assignment 6

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Consider the linear transformation $T \in \mathcal{L}(\mathbb{R}^2)$ give by $T(x, y) = (2024x+2024y, 2024y)$. Prove that $\lambda = 2024$ is the only eigenvalue of T.

Exercise 2. Let $V = \mathbb{R}^4$. Give an example of a linear map $T \in \mathcal{L}(V)$ whose minimal polynomial is $(z - 2024)^3$. Make sure to justify your answer.

Hint. To check if T satisfies $p(T) = a_0 I + a_1 T + \cdots + a_d T^d = 0$ for some polynomial T, you can pick a basis and form $M(T)$, and then check if the matrix $M(T)$ satisfies $a_0 I_n + a_1 M(T) + a_2 M(T)^2 + \cdots + a_d M(T)^d = 0$, where I_n is the correct sized identity matrix, and 0 means the matrix of zeroes.

Part B. [Proof Questions; 6pts]

Exercise 3. Suppose $T \in \mathcal{L}(V)$ and let λ be a positive integer. Prove that λ^2 is an eigenvalue of T^2 if and only if either λ or $-\lambda$ is an eigenvalue of T.

Exercise 4. Let $V = \mathbb{R}^6$ and $T \in \mathcal{L}(V)$. Suppose that $\lambda_1, \lambda_2, \lambda_3$ and λ_4 are the distinct eigenvalues of T. Prove that $(T - \lambda_1 I)^3 (T - \lambda_2 I)^3 (T - \lambda_3 I)^3 (T - \lambda_4 I)^3 = 0$.

Additional Suggested Problems. [Not graded]

Problems 5.A - 2, 5, 21, 23; 5.B - 1, 3, 4, 6, 10, 11

Additional Comment. You will notice that in Chapter 5 of Axler's textbook, there is minimal discussion on how to find or compute eigenvalues (the emphasis is on the existence of these values). In Math 1B03 you learned how to compute eigenvalues using the characteristic polynomial $\det(A - \lambda I_n)$ for an $n \times n$ matrix A. You can use the following procedure to find eigenvalues of $T \in \mathcal{L}(V)$:

- Find $\mathcal{M}(T)$ with respect to a basis for V. This matrix is a $n \times n$ matrix with $n = \dim V$.
- Compute the eigenvalues of the matrix $\mathcal{M}(T)$ as you would in Math 1B03.
- The eigenvalues of $\mathcal{M}(T)$ are the eigenvalues of the operator T.

Note that there are some things that need to proved: (a) this procedure works regardless of what basis you picked, and (b) the eigenvalues of $\mathcal{M}(T)$ and T are the same. These details are covered in Chapter 10, which we will not cover. However, you can assume this result.