Math 2R03 (Linear Algebra II)
Due: November 8, 2024

Homework Assignment 7

All of the questions from Part A will be graded. One of the questions from Part B will be graded in detail, while the others will be marked for completion. Assignments will be submitted via *Crowdmark*. You will be graded on your solution *and* how well you write your proof.

Part A. [Short Questions; 4pts]

Exercise 1. Let V be a finite-dimensional real vector space and $T \in \mathcal{L}(V)$. Suppose that the minimal polynomial of T is $p(z) = z^{2024} - 1$. Does T have an upper-triangular matrix with respect to some basis of V? Justify your answer.

Exercise 2. Let $T \in \mathcal{L}(\mathbb{R}^4)$ be the linear operator given by

$$T(x_1, x_2, x_3, x_4) = (6x_1 + x_2, 6x_2 + x_3, 6x_3, 6x_4).$$

For this operator, $\lambda = 6$ is an eigenvalue. Find the generalized eigenspace of of $\lambda = 6$, that is, find $G(6, T)$.

Part B. [Proof Questions; 7pts]

Exercise 3. Consider $T \in \mathcal{L}(V)$ with $\dim V = n$. Suppose there is a subspace $H \subseteq V$ with $\dim H = n - 1$ and a vector v not in H such that

$$T(h) = h \text{ for all } h \in H$$

and

$$T(v) = -v.$$

Show that T is diagonalizable. In particular, show that there is a basis of V such that

$$\mathcal{M}(T) = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

Remark. Note that H is an invariant under T.

Exercise 4. Let V be a complex vector space, and $T \in \mathcal{L}(V)$ such that T is invertible. Prove that if T^2 is diagonalizable, then T is also diagonalizable.

Hint. Use Theorem 5.62. Also, note that if the minimal polynomial of T^2 is $p(z) = (z - \lambda_1) \cdots (z - \lambda_m)$, then the polynomial $q(z) = (z^2 - \lambda_1) \cdots (z^2 - \lambda_m)$ has the property that $q(T) = 0$.

Additional Suggested Problems. [Not graded]

Problems 5.C # 1, 3, 6, 8, 5.D #2, 3, 5, 7, 8.A # 4, 6, 7, 8, 12, 13, 17, 21