Math 4GR3 (Groups and Rings)
Due: March 22, 2024

Homework Assignment 4

Do all of the questions. Three questions will be graded in detail (five points each), while the remaining questions will be graded for completion (one point each). Assignments will be submitted via Crowdmark (a link will be sent).

When submitting to Crowdmark, *submit a pdf version* of your solution instead of a picture. I suggest using a scanner app on your phone.

Exercise 1. Prove the Rational Root Theorem: Let

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$$

with $a_n \neq 0$. If $\frac{r}{s}$ is a rational number with $\gcd(r,s) = 1$, such that $p(\frac{r}{s}) = 0$, then $r|a_0$ and $s|a_n$.

Exercise 2. If $f(x) = a_n x^n + \cdots + a_0$ is a polynomial in $\mathbb{Z}[x]$ and if p is a prime that does not divide a_n, we can consider the polynomial $\overline{f}(x) = [a_n]x^n + \cdots + [a_0] \in \mathbb{Z}_p[x]$ where $[a_i]$ denotes the equivalence class of a_i in $\mathbb{Z}_p$. It can be shown that if $\overline{f}(x)$ is irreducible in $\mathbb{Z}_p$, the $f(x)$ is irreducible in $\mathbb{Q}[x]$. Use this fact to show the following two polynomials are irreducible in $\mathbb{Q}[x]$:

(1) $7x^3 + 6x^2 + 4x + 6$
(2) $9x^4 + 4x^3 - 3x + 7$

Remark. The proof of this fact can be found in most abstract algebra textbooks. It gives you another tool to check if a polynomial is irreducible.

Exercise 3. Consider the following subring of $\mathbb{Q}$ that is also a domain:

$$R = \left\{ \frac{n}{2^i} \;\middle|\; n \in \mathbb{Z}, i \geq 0 \right\}.$$

Prove that the field of fractions F_R is isomorphic to $\mathbb{Q}$.

Remark. In the above result, we can replace 2 with any prime p and get a similar result. Consequently, there are an infinite number of domains $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$ whose field of fractions is isomorhic to $\mathbb{Q}$.

Exercise 4. Let D be a PID. Prove that every ideal of D is contained in a maximal ideal.

Exercise 5. Let D be a Euclidean Domain with corresponding Euclidean valuation v. Prove that $u \in D$ is a unit if and only $v(u) = v(1)$.

Exercise 6. The ring $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ with $i^2 = -1$ is an Euclidean Domain via the Euclidean valuation $v(a + ib) = a^2 + b^2$.

(1) Find all the units of $\mathbb{Z}[i]$.
(2) Show that if $v(a + bi)$ is a prime number, the element $a + bi$ is an irreducible element of $\mathbb{Z}[i]$.

Hint. The previous question may be helpful.

Exercise 7. The ring $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ with $i^2 = -1$ is an Euclidean Domain via the Euclidean valuation $v(a + ib) = a^2 + b^2$. Find q and r such that $2024 + i = (1 + 2024i)q + r$ with $r = 0$ or $v(r) < v(1 + 2024i)$. (In other words, apply the division algorithm to $z = 2024 + i$ and $w = 1 + 2024i$.)

Exercise 8. An ring has the *descending chain condition* if for every descending chain of ideal

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \cdots$$

there exists an integer N such that $I_N = I_{N+1} = \cdots$.

(1) Show that $\mathbb{Q}[x]$ does not have the descending chain condition.
(2) Prove that an integral domain R is a field if and only if R satisfies the descending chain condition.

Hint. If $0 \neq a \in R$ and if a is not a unit, what can be said about the chain of ideal $(a) \supseteq (a^2) \supseteq (a^3) \supseteq \cdots$?