NAME: ______________________

STUDENT NUMBER: ______________________

MATH 1281 - CHRISTMAS EXAMINATION

Lakehead University

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Instructions: Answer all questions in the space provided. If you need more room, answer on the back of the previous page. Where appropriate, you must provide clear explanations. You have 3 hours.

There are **THREE** parts.

Part A Short Answer Questions – Do all the questions.

Part B Long Answer Questions – Do all the questions.

Part C Longer Answer – Do only *one* of the two questions.

You are *not* allowed to use a calculator.

If doubt exists as to the interpretation of any question, you are urged to submit with the answer paper a clear statement of any assumptions made.

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PART A: SHORT ANSWER QUESTIONS

1. [**2 pts**] Write out the truth table for the compound proposition $(p \wedge q) \rightarrow \neg p$.

2. [**2 pts**] Let A and B be sets. Show that

$$\overline{A} \cap \overline{B} = \overline{A \cup B}.$$

3. [**2 pts**] Give an example of a tautology.

4. [**4 pts**] Let $Q(x, y)$ denote the statement $xy = 1$ where the universe of discourse is $\mathbb{R}$. What is the truth value of

   (*a*) $\exists x \forall y \; Q(x, y)$.

   (*b*) $\forall y \exists x \; Q(x, y)$.

5. [**2 pts**] Let $A = \{\text{dog, cat, mouse}\}$. Write out the elements in $\mathcal{P}(A)$, the power set of A.

6. [**2 pts**] Give an example of a function $f : \mathbb{Z} \to \mathbb{Z}$ that is one-to-one, but not onto.

7. [**2 pts**] Find $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$ where

$$a = 2^2 \cdot 3^3 \cdot 5^5 \cdot 41^{41} \quad \text{and} \quad b = 2^{41} \cdot 7^6 \cdot 13^4 \cdot 23^5.$$

8. [**2 pts**] Use the Euclidean Algorithm to find $\gcd(37, 8)$. Show all your work!

9. [**2 pts**] Find two numbers x such that $2x \equiv 3 \pmod{9}$.

10. [**2 pts**] Provide a simple formula or rule that generates the terms of the integer sequence that begins with:

$$-1, 3, 7, 11, 15, 19, 23, 27, \ldots$$

11. [**2 pts**] Give a recursive definition for the set of all positive integers divisible by 7.

12. [**2 pts**] Count the number of strings of length 10 that can be made from the letters $\{w, x, y, z\}$.

13. [**2 pts**] Evaluate the following sum:

$$\sum_{j\in\{0,2,4\}} \sum_{i=1}^{3} ij.$$

14. [**2 pts**] Find $f(4)$ if $f(0) = 4$ and

$$f(n) = \begin{cases} \frac{f(n-1)}{2} & \text{if } f(n-1) \text{ is even} \\ f(n-1)+1 & \text{if } f(n-1) \text{ is odd.} \end{cases}$$

15. [**2 pts**] Suppose that in a particular class, all the students come from one of the 13 provinces or territories of Canada. How many students must be in the class to guarantee that at least four students come from the same province or territory.

16. [**2 pts**] A mathematics department has 10 people. If the department must form a committee of 4 people, count the number of ways to form the committee.

17. [**2 pts**] What is the coefficient of x^3y^2 in the expansion of $(2x - y)^5$.

18. [**2 pts**] What is the next permutation in lexicographical order after 1247653.

19. [**2 pts**] What is the probability that a five-card poker hand contains a heart.

PART B: LONG ANSWER QUESTIONS

20. [**5 pts**] Show how the statements

1. Catherine, a student in the class, enjoys whale watching.
2. Everyone who enjoys whale watching cares about ocean pollution.

imply the conclusion

There is a person in the class who cares about ocean pollution.

(Use the method we used in class; represent each proposition by a letter or propositional function, and then, using a table format, identify what rule of inferences are used to make the desired conclusion.)

21. [**5 pts**] Use the Principle of Mathematical Induction to prove that $2^n < n!$ for all integers $n \geq 4$.

22. [**5 pts**] Use the Principle of Mathematical Induction to prove that

$$1 + 4 + 7 + 10 + \cdots + (3n - 2) = \frac{n(3n-1)}{2}$$

for all $n \geq 1$.

23. [**5 pts**] Consider the following recursive algorithm:

```
procedure mystery(n)

if n = 1 then
   print ''Finished!''
if (2 divides n) then
   print  ''Not Done'';
   n:=n/2
   mystery(n)
if (2 does not divide n) And (n does not equal 1) then
   print ''Working...''
   n:=3n+1
   mystery(n)
end
```

What is the output of this algorithm with the input `mystery(5)`?

PART C: DO EITHER Question 24 OR Question 25.

24. [**10 pts**] Let $x_1, x_2, \ldots, x_n$ be n variables. A *monomial* is a product of variables and has the form $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ where the a_i's are integers such that $a_i \geq 0$. The *degree* of the monomial $x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ is $a_1 + a_2 + \cdots + a_n$.

For example $x_1^2 x_2^4 x_3$ is a monomial of degree $2 + 4 + 1 = 7$. The following list

$$x_1^2,\ x_1x_2,\ x_1x_3,\ x_2^2,\ x_2x_3,\ x_3^2$$

is the list of all the monomials in $n = 3$ variables of degree 2.

(*a*) Count the number of monomials of degree 3 in $n = 4$ variables. [Hint: $x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}$ is a monomial of degree 3 if and only if $a_1 + a_2 + a_3 + a_4 = 3$.]

(*b*) How many of the monomials in Part (*a*) are divisible by x_1? [Hint: $x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4}$ is a monomial of degree 3 that is divisible by x_1 if and only if $a_1 + a_2 + a_3 + a_4 = 3$ and $a_1 \geq 1$.]

(*c*) How many of the monomials of degree 6 in $n = 4$ variables are not divisible by x_4^2?

(*d*) Give a formula for the number of monomials of degree d in n variables.

(*e*) Give a formula for the number of monomials of degree d in n variables that are divisible by x_1.

Note. You may leave your questions in an unexpanded form, e.g., you can write 5^4 instead of 625, or $C(6, 3)$ instead of 20.

(You may also use the next page)

24. (Extra space for Question 24).

PART C: DO EITHER Question 24 OR Question 25.

25. [**10 pts**] Let S be the set that consists of all the strings that can be made from the letters in the word LAKEHEAD.

A string is picked at random from the set S.

(a) What is the probability that the string is "DEAHEKAL"?

(b) What is the probability that the string begins with the letter A?

(c) What is the probability that the string begins with the letters AA?

(d) What is the probability that the string begins with begins with A and ends with A?

(e) What is the probability that the string ends with A given that the first letter is an A?

Note. You may leave your questions in an unexpanded form, e.g., you can write 5^4 instead of 625, or $C(6,3)$ instead of 20.

(You may also use the next page.)

25. (Extra space for Question 25).