NAME: ______________________

STUDENT NUMBER: ______________________

MATH 1281 - CHRISTMAS EXAMINATION

Lakehead University

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Instructions: Answer all questions in the space provided. If you need more room, answer on the back of the previous page. Where appropriate, you must provide clear explanations. You have 3 hours.

There are **THREE** parts.

Part A Short Answer Questions – Do all the questions.

Part B Long Answer Questions – Do all the questions.

Part C Longer Answer – Do only *one* of the three questions.

You are *not* allowed to use a calculator. You may leave your answer in an unexpanded form. For example, you may simply leave your answer as $\binom{4}{2}$.

If doubt exists as to the interpretation of any question, you are urged to submit with the answer paper a clear statement of any assumptions made.

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PART A: SHORT ANSWER QUESTIONS

1. **[10 pts]** In the space provided, match each term with its definition. (Note: There are more definitions than terms.)

Terms.

____ Time complexity

____ Domain of a function f

____ r-permutation

____ Proposition

____ Prime

____ Codomain of function f

____ Tautology

____ r-combination

____ Algorithm

____ Empty set

Definitions.

1. A finite set of precise instructions for performing a computation or solving a problem.
2. The set with no members.
3. An ordered arrangement of r elements of a set.
4. A positive integer greater than 1 with exactly two positive integer divisors.
5. The set A where f is a function from A to B.
6. A compound proposition that is always true.
7. A compound proposition that is always false.
8. The set B where f is a function A to B.
9. A statement that is true or false.
10. An unordered selection of r elements of a set.
11. The set of images of f.
12. The amount of time required for an algorithm to solve a problem.
13. A set that is not finite.

2. **[2 pts]** Find two integers x such that $3x + 7 \equiv 8 \pmod{11}$.

3. **[2 pts]** Find an inverse of 5 modulo 11.

4. **[2 pts]** Use the Euclidean Algorithm to find the $\gcd(36, 49)$. Show all your work!

5. **[2 pts]** For the list of integers below, provide a simple formula or rule that generates the terms of integer sequence that begins:

$$0, 3, 8, 15, 24, 35, 48, 63, \ldots$$

6. **[2 pts]** Evaluate the following sum: $\displaystyle\sum_{i=1}^{3} \sum_{j \in \{1,4,5,6\}} (2j + i)$.

7. **[2 pts]** If $f(0) = 2$ and $f(n)$ is defined recursively by $f(n) = f(n-1) - 3$ for $n \geq 1$, find $f(3)$.

8. **[2 pts]** Give a recursive definition for the set of positive integers divisible by 7.

9. **[2 pts]** Nineteen people have first names Larry, Moe, and Shemp, and last names Chaplin and Keaton. Show that at least four people must have the same first and last names. Justify your answer.

10. **[2 pts]** What is the coefficient of x^9y^3 in the expansion of $(x + 2y)^{12}$.

11. **[2 pts]** How many bit strings of length 8 contain exactly three 1's?

12. [**2 pts**] A donut shop offers 20 kinds of donuts. Assuming that there are at least a dozen of each kind, how many may ways can a dozen donuts be selected?

13. [**2 pts**] How many different strings can be made from the letters in KAKABEKA

14. [**2 pts**] Place the following four permutations in lexicographical order (from smallest to largest):

$$125634, 314256, 123465, 432651$$

.

15. [**2 pts**] What is the probability that a card selected from a deck is a Queen?

16. [**2 pts**] An eight sided die is biased so that 7 appears 3 times as often as each other number, but the other seven outcomes are equally likely. What is the probability of each outcome?

PART B: LONG ANSWER QUESTIONS

17. [**5 pts**] Let m be a positive integer, and let a, b, and c be integers. Prove that if $a \equiv b \pmod{m}$, then $a - c \equiv b - c \pmod{m}$.

18. [**5 pts**] Consider the function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(n) = 2\lfloor \frac{n}{2} \rfloor$. Is this function one-to-one? Is this function onto? Justify your answers.

19. [**5 pts**] Use the Principle of Mathematical Induction to prove that $n^2 + n$ is even for all $n \geq 1$.

20. [**5 pts**] Let $x \neq 1$ be a real number. Use the Principle of Mathematical Induction to prove that

$$1 + x + x^2 + x^3 + \cdots + x^{n-1} + x^n = \frac{x^{n+1} - 1}{x - 1}$$

for all $n \geq 1$.

PART C: Do ONE of Question 21, 22 or 23.

21. [**10 pts**] Show how the statements

1. Catherine, a student in the class, enjoys whale watching and has not seen the Pacific Ocean.
2. Everyone who enjoys whale watching has visited Tofino or visited the Aquarium.
3. Everyone who has visited Tofino has seen the Pacific Ocean.

imply the conclusion

There is a person in the class who has visited the Aquarium.

(Use the method we used in class; represent each proposition by a letter or propositional function, and then, using a table format, identify what rule of inferences are used to make the desired conclusion.)

PART C: Do ONE of Question 21, 22 or 23

22. In chess, the king can move one position in any direction.

(*a*) **[2 pts]** Suppose we decided to limit the king's moves to either moving one position up or one position to the right. Along how many different paths can a king be moved from the lower-left corner position to the upper right-corner position on the standard 8×8 chessboard?

(*b*) **[2 pts]** How many of the above paths in Part (*a*) begin with with 3 moves to the right?

(*c*) **[2 pts]** Repeat Part (*a*) but now assume that we are using a 10×15 chessboard.

(*d*) **[4 pts]** Repeat Part (*a*), but now assume that the king can move one position up, to the right, or diagonally north-east.

(You may also use the next page)

22. (Extra space for Question 22).

PART C: Do ONE of Question 21, 22, or 23

23. [**10 pts**] Let S be the set that consists of all the integer solutions to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 25$$

with $x_i \geq 0$ for all $i = 1, \ldots, 5$.

We can view the elements of S as 5-tuples of the form $(x_1, x_2, x_3, x_4, x_5)$ where $x_1 + \cdots + x_5 = 25$. For example, the 5-tuple $(1, 2, 10, 2, 10)$ is an element of S.

A solution $(x_1, x_2, x_3, x_4, x_5)$ is picked at random from the set S.

(a) [**2 pts**] What is the probability that the solution $(x_1, x_2, x_3, x_4, x_5)$ equals $(1, 3, 5, 7, 9)$?

(b) [**2 pts**] What is the probability that $x_1 \geq 3$?

(c) [**2 pts**] What is the probability that $0 \leq x_1 \leq 2$?

(d) [**4 pts**] What is the probability that $x_2 \geq 10$ given that $x_1 \geq 10$?

Note. You may leave your questions in an unexpanded form, e.g., you can write 5^4 instead of 625, or $C(6, 3)$ instead of 20.

(You may also use the next page.)

23. (Extra space for Question 23).