NAME: ____________________

STUDENT NUMBER: ____________________

MATH 2255 - Final Exam

Lakehead University

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Instructions: Answer all questions in the space provided. If you need more room, answer on the back of the previous page. Where appropriate, you must provide clear explanations.

You are *not* allowed to use a calculator.

If doubt exists as to the interpretation of any question, you are urged to submit with the answer paper a clear statement of any assumptions made.

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1. **[4 pts]** Solve the system of linear equations

$$\begin{aligned} 2x_2 + 3x_3 &= 8 \\ 2x_1 + 3x_2 + x_3 &= 5 \\ x_1 - x_2 - 2x_3 &= -5 \end{aligned}$$

You may use any method introduced in the course.

2. **[6 pts]** Give examples of the following:

(a) a system of linear equations that is inconsistent.

(b) a homogeneous system of linear equations homogeneous.

(c) a system of linear equations with only one solution.

3. **[5 pts]** Find the inverse of the following matrix

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$$

You may use any method introduced in the course.

4. **[5 pts]** Compute the determinant of the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 5 & 2 & 6 \\ 0 & 1 & 0 & 0 \\ 1 & 4 & 2 & 1 \end{bmatrix}$$

You may use any method we learned in class.

5. Consider the following matrix

$$A = \begin{bmatrix} 2 & -4 & 0 \\ 3 & -1 & 4 \\ -1 & 2 & 2 \end{bmatrix}$$

(a) **[4 pts]** Find the LU-factorization of A.

(b) **[2 pts]** Use part (a) to solve $A\mathbf{x} = \mathbf{b}$ when $\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$.

6. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation, and suppose that we know that

$$T(\mathbf{e}_1) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad T(\mathbf{e}_2) = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \quad T(\mathbf{e}_3) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the standard basis elements of $\mathbb{R}^3$.

(*a*) **[2 pts]** What is the standard matrix A for the linear transformation T? That is, what is the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$?

(*b*) **[3 pts]** Is T a one-to-one mapping? Justify your answer.

(*c*) **[3 pts]** Is T a onto mapping? Justify your answer.

7. [**6 pts**] Consider the following subsets of $\mathbb{R}^3$:

$$W_1 = \left\{ \begin{bmatrix} a \\ -a \\ 2a \end{bmatrix} \mid a \in \mathbb{R} \right\} \quad \text{and} \quad W_2 = \left\{ \begin{bmatrix} a \\ b \\ a+b+1 \end{bmatrix} \mid a, b \in \mathbb{R} \right\}$$

For each subset, determine whether they are subspaces of $\mathbb{R}^3$.

8. **[6 pts]** Consider the following three polynomials of $\mathbb{P}_2$:

$$\mathbf{p}_1(t) = 1 + t - 2t^2, \quad \mathbf{p}_2(t) = 2 + 5t - t^2, \quad \mathbf{p}_3(t) = t + t^2.$$

Determine if $\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t)$ are linearly independent.

9. Consider the sets

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix} \right\} \quad \text{and} \quad \mathcal{C} = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}.$$

(a) **[2 pts]** Explain why $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{R}^2$.

(b) **[2 pts]** Find the vector $\mathbf{x}$ whose $\mathcal{B}$-coordinate vector $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$.

(c) **[2 pts]** Find the $\mathcal{C}$-coordinate of the vector $\mathbf{x} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$.

(d) **[2 pts]** Find the change-of-coordinates matrix $\mathcal{C} \overset{P}{\leftarrow} \mathcal{B}$ from $\mathcal{B}$ to $\mathcal{C}$.

(e) **[2 pts]** Use Part (d) to find $\mathcal{B} \overset{P}{\leftarrow} \mathcal{C}$.

10. Assume that A is row equivalent to the matrix B:

$$A = \begin{bmatrix} 1 & 1 & 3 & 1 & 6 \\ 2 & -1 & 0 & 1 & -1 \\ -3 & 2 & 1 & -2 & 1 \\ 4 & 1 & 6 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Find a basis for

(a) **[3 pts]** Nul(A).

(b) **[3 pts]** Col(A).

(c) **[3 pts]** Row(A).

11. **[10pts]** The following statements are **FALSE**. Explain why they are false by using a relevant theorem, or by giving a counterexample.

(a) There exists a 3×6 matrix A with $\text{rank}(A) = 4$.

(b) If $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are any n vectors of a vector space V, and $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, then $\dim H = n$.

(c) If A is an $m \times n$, then $\text{Nul}(A)$ is a subspace of $\mathbb{R}^m$.

(d) A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

(e) If the set $\mathcal{B}$ is a basis for $\mathbb{P}_7$, then the set $\mathcal{B}$ has 7 elements.

12. **[4 pts]** Suppose that A is a 5×7 matrix and suppose that $\mathrm{Col}(A) = \mathbb{R}^5$. What is $\dim \mathrm{Nul}(A)$? Justify your answer.

13. **[4 pts]** Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Find the 2×2 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.