Name: ____________________

Student Number: ____________________

# Math 1XX3 (Calculus for Math and Stats II) Final Exam

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Day Class

Duration of Exam: 2.5 hours

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This examination paper includes 10 pages and 15 questions. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancy to the attention of your invigilator.

Special Instructions

- Only the McMaster Standard Calculator (Casio fx 991) is allowed.
- All solutions should be written on the exam.
- The total number of points is 75. **Do all of the 15 questions; for Question 9, do either A or B**.
- For multiple choice questions, full marks will be received for questions with the correct circled answer. For incorrect answers, part marks are based upon work provided.
- Use the back of pages if you need more room.

| Page | Possible | Received |
|---|---|---|
| 2 | 10 | |
| 3 | 10 | |
| 4 | 8 | |
| 5 | 8 | |
| 6 | 4 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 5 | |
| Total | 75 | |

**Q. 1. [5 pts]** Solve the initial value problem $\frac{dy}{dx} + 2xy - y = 0$ with $y(0) = 10$.

a. $y = 2e^{x-x^2}$ b. $y = 10e^{x-x^2}$ c. $y = -10e^{x-x^2}$ d. $y = 10e^{x^2-x}$

**Q. 2. [5 pts]** Consider the initial value problem $y' = y^2 + 3x^2 + 3x$ with $y(1) = 0$. You are asked to approximate $y(10)$ using Euler's method with step size of 1. Will your solution underestimate or overestimate the exact solution? Justify your answer.

**Q. 3. [5 pts]** Find the point(s) on the curve described by

$$x(t) = t^3 - 3t + 2 \quad y(t) = t^3 - 3t^2 + 2$$

where the tangent is horizontal.

a. $(0, 0)$. b. $(2, 2), (2, -2)$. c. $(-1, 1), (2, -2)$. d. $(2, 2), (4, -2)$. e. None of these.

**Q. 4. [5 pts]** Calculate the area enclosed by the polar equation $r = 2 \sin 3\theta$. [Hint: The graph consists of three petals. Find the area of one petal, and then multiply by three. Also, $\sin^2 \theta = \dfrac{1 - \cos 2\theta}{2}$]

**Q. 5. [4 pts]** Below are four $p$-series. Three of these series converges and the other diverges. Which one diverges?

a. $\sum_{n=1}^{\infty} \frac{1}{n^{0.25}}$ b. $\sum_{n=1}^{\infty} n^{-\pi}$ c. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ d. $\sum_{n=1}^{\infty} \frac{1}{n^{2\sqrt{2}}}$

**Q. 6. [4pts]** Test for convergence or divergence (use any test): $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$.

**Q. 7.** [**4 pts**] Does the geometric series

$$\sum_{n=0}^{\infty} 3^n 4^{-n+1}$$

converge or diverge? If it converges, find its sum.

a. Diverges. b. 3. c. 12. d. 16.

**Q. 8.** [**4 pts**] Does the infinite series

$$1 - \frac{1}{2^3} + \frac{1}{3^2} - \frac{1}{4^3} + \frac{1}{5^2} - \frac{1}{6^3} + \frac{1}{7^2} - \frac{1}{8^3} + \cdots$$

converge or diverge? Justify your answer?

**Q. 9. [4 pts] DO ONE OF THE FOLLOWING TWO QUESTIONS.**

A. If the power series representation of $\ln(1+x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

find the power series representation of

$$f(y) = \ln\left(\frac{11+y}{11-y}\right).$$

B. A friendly alien tells you that the answer to life, the universe, and everything is $f(1)$ for some function $f$. The alien cannot tell you what the function $f$ is (it's an alien secret), but it can give you some information about $f$ at other values. The alien tells you that $f(0) = 26$, $f'(0) = 22$, $f''(0) = -16$ and $f'''(0) = 12$. Using this information, approximate the value of $f(1)$.

**Q. 10. [5 pts]** Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{\sin\left(9x^2+9y^2\right)}{2x^2+2y^2}.$$

[Hint: polar coordinates]

a. $\frac{9}{2}$ b. 0 c. $\frac{2}{9}$ d. 1

**Q. 11. [5 pts]** Use the Chain rule to find $\frac{\partial z}{\partial s}$ for

$$z = e^r\cos(\theta) \quad \text{where } r = 6st \text{ and } \theta = \sqrt{s^2+t^2}$$

a. $\frac{\partial z}{\partial s} = e^r\left(6t\cos(\theta) - \frac{s\sin(\theta)}{\sqrt{s^2+t^2}}\right).$

b. $\frac{\partial z}{\partial s} = e^r\left(6t\cos(\theta) - \frac{se^r\sin(\theta)}{\sqrt{s^2+t^2}}\right).$

c. $\frac{\partial z}{\partial s} = e^r\left(t\cos(\theta) - \frac{s\sin(\theta)}{\sqrt{s^2+t^2}}\right).$

d. $\frac{\partial z}{\partial s} = e^r\left(6t\cos(\theta) - \frac{s\sin(\theta)}{\sqrt{s^2-t^2}}\right).$

**Q. 12. [5 pts]** Find $f_{xxx}$ for the function $f(x,y) = x^2y^4 - 3x^4y$.

a. $f_{xxx} = 12x^2y$ b. $f_{xxx} = -36xy$ c. $f_{xxx} = 3xy$ d. $f_{xxx} = -72xy$ e. $f_{xxx} = -3xy$

**Q. 13. [5 pts]** Find all the critical points of the function $f(x,y) = x^4 + y^4 - 4xy$. For each critical point, determine if it is a local maximum a local minimum, or a saddle point.

**Q. 14. [10 pts]** For this question, consider the function $f(x, y) = x^2 - 4y^2 - 9$ at the point $P = (1, -2)$.

a. What is the equation of the tangent plane at the point $P$?

b. Find the directional derivative $D_{\mathbf{u}} f(x, y)$ where $\mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2})$ is a unit vector.

c. What is $D_{\mathbf{u}}(1, -2)$ with $\mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2})$?

d. What direction at the point $P$ gives the maximum rate of change of $f$?

e. What direction at the point $P$ gives no rate of change of $f$?

**Q. 15. [5 pts]** The function $f(x, y) = x^2 - y^2$ has both a maximum and minimum value when subject to the constraint function $x^2 + y^2 = 1$. Use Lagrange multipliers to find the extreme values. [This is Exercise 3 of Section 14.8]

**Bonus. [1 pts]** What was your favourite topic in the course, and why?

THE END