NAME: Soln's

STUDENT NUMBER: ______________

# MATH 1XX3 (CALCULUS FOR MATH AND STATS II) FINAL EXAM

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DAY CLASS

DURATION OF EXAM: 2.5 hours

MCMASTER FINAL EXAM

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THIS EXAMINATION PAPER INCLUDES 10 PAGES AND 15 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

SPECIAL INSTRUCTIONS

- Only the McMaster Standard Calculator (Casio fx 991) is allowed.
- All solutions should be written on the exam.
- The total number of points is 75. **Do all of the 15 questions; for Question 9, do either A or B.**
- For multiple choice questions, full marks will be received for questions with the correct circled answer. For incorrect answers, part marks are based upon work provided.
- Use the back of pages if you need more room.

| Page | Possible | Received |
|---|---|---|
| 2 | 10 | |
| 3 | 10 | |
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| 5 | 8 | |
| 6 | 4 | |
| 7 | 10 | |
| 8 | 10 | |
| 9 | 10 | |
| 10 | 5 | |
| Total | 75 | |

**Q. 1.** **[5 pts]** Solve the initial value problem $\frac{dy}{dx} + 2xy - y = 0$ with $y(0) = 10$.

a. $y = 2e^{x-x^2}$ $\boxed{\text{b. } y = 10e^{x-x^2}}$ c. $y = -10e^{x-x^2}$ d. $y = 10e^{x^2-x}$

Separable $\frac{dy}{dx} = y - 2xy \Rightarrow \frac{dy}{y} = (1-2x)dx$ 1

1

$\int \frac{dy}{y} = \int (1-2x)\,dx$

$\ln y = x - x^2 + C$ 1

$\Rightarrow e^{\ln y} = y = e^{x-x^2+C}$

$\Rightarrow y = Ce^{x-x^2}$

Now solve the IVP

$y(0) = 10 = Ce^{0-0^2} = C \cdot 1 \Rightarrow C = 10$ 1

So $\boxed{y = 10e^{x-x^2}}$ 1

**Q. 2.** **[5 pts]** Consider the initial value problem $y' = y^2 + 3x^2 + 3x$ with $y(1) = 0$. You are asked to approximate $y(10)$ using Euler's method with step size of 1. Will your solution underestimate or overestimate the exact solution? Justify your answer.

The solution will $\boxed{\text{underestimate}}$ the exact sol$^{\underline{n}}$.

To see why, note that as $y$ and $x$ increase, so does $y'$

So, if we consider the first ~~approximation~~ step of Euler's method,

we get

$(x,y) = (1,0) \Rightarrow y' = 0 + 3 \cdot 1^2 + 3 \Rightarrow y' = 6$

So at $(1,0)$, we approx $y$ with the line

$y - 0 = 6(x-1) \Rightarrow y = 6x - 6$

At $x = 2$, $6(2) - 6 = 6$ will be less than then the actual value

Since on the interval $1 \le x \le 2$, $y' \geq 6$, i.e. the function is increasing faster than the line of slope 6. This reasoning applies for all future values

1 pt = answer
4 pts = justification (must refer to slope)

**Q. 3. [5 pts]** Find the point(s) on the curve described by

$$x(t) = t^3 - 3t + 2 \quad y(t) = t^3 - 3t^2 + 2$$

where the tangent is horizontal.

a. $(0,0)$. b. $(2,2),(2,-2)$. c. $(-1,1),(2,-2)$. $\boxed{\text{d. } (2,2),(4,-2).}$ e. None of these.

want $0 = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2-6t}{3t^2-3}$ 2

1

$3t^2 - 6t = 3t(t-2) = 0 \iff t=0$ or $t=2$. 1

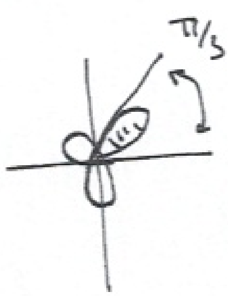
Note that @ $t=0$ or $t=2$, $dx/dt \neq 0$.

So, ~~stangent~~ horizontal when $t=0 \iff (x(0), y(0)) = (2,2)$ 1

and $t=2 \iff (x(2), y(2)) = (4,-2)$ 1

**Q. 4. [5 pts]** Calculate the area enclosed by the polar equation $r = 2\sin 3\theta$. [Hint: The graph consists of three petals. Find the area of one petal, and then multiply by three. Also, $\sin^2\theta = \frac{1-\cos 2\theta}{2}$]

Note that $\sin 3\theta = 0 \iff \theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$




So, one petal will be described as $0 \le \theta \le \pi/3$ 1

$$\text{Area} = 3\int_0^{\pi/3} \frac{1}{2} r^2 \, d\theta = 3\int_0^{\frac{\pi}{3}} \frac{1}{2}(2\sin 3\theta)^2 \, d\theta$$ 1

$$= 3\int_0^{\frac{\pi}{3}} 2\sin^2 3\theta \, d\theta = 3\int_0^{\frac{\pi}{3}} 2\left(\frac{1-\cos 6\theta}{2}\right) d\theta$$ 1

$$= 3\int_0^{\frac{\pi}{3}} 1 - \cos 6\theta \, d\theta = 3\left[\theta - \frac{\sin 6\theta}{6}\right]_0^{\frac{\pi}{3}}$$ 1

$$= 3\left[\left(\frac{\pi}{3} - 0\right) - (0-0)\right] = \boxed{\pi}$$ 1

**Q. 5.** **[4 pts]** Below are four $p$-series. Three of these series converges and the other diverges. Which one diverges?

$\boxed{\text{a. } \sum_{n=1}^{\infty} \frac{1}{n^{0.25}}}$ b. $\sum_{n=1}^{\infty} n^{-\pi}$ c. $\sum_{n=1}^{\infty} \frac{1}{n^4}$ d. $\sum_{n=1}^{\infty} \frac{1}{n^{2\sqrt{2}}}$

A p series converges if and only if $p > 1$. — 2

So, in (a), $p = 0.25 < 1 \Rightarrow$ (a) diverges — 2

**Q. 6.** **[4pts]** Test for convergence or divergence (use any test): $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$.

Use the alternating Series test with $b_n = \frac{1}{n+4}$ — 1

① $b_{n+1} = \frac{1}{n+5} \leq \frac{1}{n+4} = b_n$ for all $n \geq 0$ — 1

② $\lim_{n \to \infty} b_n = 0$ — 1

So, by A.S.T. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$ $\boxed{\text{Converges}}$ — 1

**Q. 7.** **[4 pts]** Does the geometric series

$$\sum_{n=0}^{\infty} 3^n 4^{-n+1}$$

converge or diverge? If it converges, find its sum.

a. Diverges. b. 3. ~~c. 12.~~ $\boxed{\text{d. 16.}}$

$$\sum_{n=0}^{\infty} 3^n 4^{-n+1} = \sum_{n=0}^{\infty} \frac{3^n}{4^{n-1}} = \sum_{n=0}^{\infty} 3\left(\frac{3}{4}\right)^{n-1} \quad \text{— } ①$$

← geometric series with $r = \left(\frac{3}{4}\right)$

$a = 3\left(\frac{3}{4}\right)^{-1} = 3 \cdot \frac{4}{3} = 4$ — ①

Since $|r| < 1$, converges. It converges to

$$\frac{a}{1-r} = \frac{4}{1-3/4} = \frac{4}{1/4} = \boxed{16} \quad \text{— } ①$$

— ①

**Q. 8.** **[4 pts]** Does the infinite series

$$1 - \frac{1}{2^3} + \frac{1}{3^2} - \frac{1}{4^3} + \frac{1}{5^2} - \frac{1}{6^3} + \frac{1}{7^2} - \frac{1}{8^3} + \cdots$$

converge or diverge? Justify your answer?

Let $A = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$

and $B = \frac{1}{2^3} + \frac{1}{4^3} + \frac{1}{6^3} + \cdots$ — ①

Note $A \le \sum_{n=0}^{\infty} \frac{1}{n^2}$ and $B \le \sum_{n=0}^{\infty} \frac{1}{n^3}$. Both p-series converge — ②

By comparison test, both $A$ & $B$ then converge. But then so does $A - B$ by properties of series. — ①

**Q. 9.** [4 pts] **DO <u>ONE</u> OF THE FOLLOWING TWO QUESTIONS.**

A. If the power series representation of $\ln(1+x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

find the power series representation of

$$f(y) = \ln\left(\frac{11+y}{11-y}\right).$$

B. A friendly alien tells you that the answer to life, the universe, and everything is $f(1)$ for some function $f$. The alien cannot tell you what the function $f$ is (it's an alien secret), but it can give you some information about $f$ at other values. The alien tells you that $f(0) = 26$, $f'(0) = 22$, $f''(0) = -16$ and $f'''(0) = 12$. Using this information, approximate the value of $f(1)$.

A. $f(y) = \ln(11+y) - \ln(11-y) = \ln(11(1+\frac{y}{11})) - \ln(11(1-\frac{y}{11}))$ 2

$= \ln(11) + \ln(1+\frac{y}{11}) - \ln(11) - \ln(1-\frac{y}{11}) = \ln(1+\frac{y}{11}) - \ln(1-\frac{y}{11})$ 1

$= \frac{y}{11} - \frac{y^2}{11^2\cdot 2} + \frac{y^3}{11^3\cdot 3} + \frac{y^4}{11^4\cdot 4} + \cdots - \left[-\frac{y}{11} - \frac{y^2}{11^2\cdot 2} - \frac{y^3}{11^3\cdot 3} - \frac{y^4}{11^4\cdot 4} \cdots\right.$ 1

$= \frac{2y}{11} + \frac{2y^3}{11^3\cdot 3} + \frac{2y^5}{11^5\cdot 5} + \cdots = \boxed{\sum_{n=0}^{\infty} \frac{2y^{2n+1}}{11^{2n+1}(2n+1)}}$ 1

B. Form the degree 3 Taylor poly of f from the given info

$T_3 = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!}$ 1 $= 26 + 22x - \frac{16x^2}{2} + \frac{12x^3}{6}$ 2

$= 26 + 22x - 8x^2 + 2x^3$ 1

So $T_3(1) \approx f(1) \Rightarrow f(1) \approx 26 + 22 - 8 + 2 = \boxed{42}$ 1

**Q. 10. [5 pts]** Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{\sin(9x^2+9y^2)}{2x^2+2y^2}.$$

[Hint: polar coordinates]

$\boxed{\text{a. } \frac{9}{2}}$ b. 0 c. $\frac{2}{9}$ d. 1

Using polar coordinates (i.e. $r = x^2+y^2$) we have — ①

$$\lim_{(x,y)\to(0,0)} \frac{\sin(9x^2+9y^2)}{2x^2+2y^2} = \lim_{r\to 0^+} \frac{\sin 9r}{2r} = \frac{0}{0}$$ ← apply L'hopital — ①

— ①

$$\lim_{r\to 0^+} \frac{\sin 9r}{2r} = \lim_{r\to 0^+} \frac{(\cos 9r)\,9}{2} = \boxed{\frac{9}{2}}$$ — ①

— ①

**Q. 11. [5 pts]** Use the Chain rule to find $\frac{\partial z}{\partial s}$ for

$$z = e^r\cos(\theta) \quad \text{where } r = 6st \text{ and } \theta = \sqrt{s^2+t^2}$$

$\boxed{\text{a. } \frac{\partial z}{\partial s} = e^r\left(6t\cos(\theta) - \frac{s\sin(\theta)}{\sqrt{s^2+t^2}}\right).}$

b. $\frac{\partial z}{\partial s} = e^r\left(6t\cos(\theta) - \frac{se^r\sin(\theta)}{\sqrt{s^2+t^2}}\right).$

c. $\frac{\partial z}{\partial s} = e^r\left(t\cos(\theta) - \frac{s\sin(\theta)}{\sqrt{s^2+t^2}}\right).$

d. $\frac{\partial z}{\partial s} = e^r\left(6t\cos(\theta) - \frac{s\sin(\theta)}{\sqrt{s^2-t^2}}\right).$

Want $$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\cdot\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\cdot\frac{\partial \theta}{\partial s}$$ — ①

$\frac{\partial z}{\partial r} = e^r\cos\theta$ — ①  $\frac{\partial z}{\partial \theta} = -e^r\sin\theta$ — ①  $\frac{\partial r}{\partial s} = 6t$ — ①  $\frac{\partial \theta}{\partial s} = \frac{1}{2\sqrt{s^2+t^2}}\cdot 2s = \frac{s}{\sqrt{s^2+t^2}}$ — ①

So $$\frac{\partial z}{\partial s} = e^r\cos\theta\cdot 6t + -e^r\sin\theta\cdot\frac{s}{\sqrt{s^2+t^2}}$$ ← equivalent to (a)

**Q. 12.** **[5 pts]** Find $f_{xxx}$ for the function $f(x,y) = x^2y^4 - 3x^4y$.

a. $f_{xxx} = 12x^2y$ b. $f_{xxx} = -36xy$ c. $f_{xxx} = 3xy$ d. $\boxed{f_{xxx} = -72xy}$ e. $f_{xxx} = -3xy$

$f_x = 2xy^4 - 12x^3y$ — 1

$f_{xx} = 2y^4 - 36x^2y$ — 2

$f_{xxx} = -72xy$ — 2

↑ up

**Q. 13.** **[5 pts]** Find all the critical points of the function $f(x,y) = x^4 + y^4 - 4xy$. For each critical point, determine if it is a local maximum a local minimum, or a saddle point.

$f_x = 4x^3 - 4y$ Want all $(x,y)$ s.t. $f_x(x,y) = f_y(x,y) = (0,0)$

$f_y = 4y^3 - 4x$ — 1

$4x^3 - 4y = 0 \Leftrightarrow x^3 - y = 0$
$4y^3 - 4x = 0 \Leftrightarrow y^3 - x = 0$ } $x^3 = y$, $y^3 = x$ } $(x^3)^3 = x$

$\Rightarrow x^9 = x$

$\Rightarrow x(x^8 - 1) = 0$

$\Rightarrow x = 0$ or $x = 1$, $x = -1$

$\boxed{\text{So, critical pts } (0,0), (1,1) \text{ \& } (-1,-1)}$ — 1

$f_{xx} = 12x^2$ So $D(x,y) = (12x^2)(12y^2) - (-4)^2$

$f_{yy} = 12y^2$ $= 144x^2y^2 - 16$ — 1

$f_{xy} = -4.$ — 1

At point $(0,0)$, $D(0,0) = -16 < 0$ $\Rightarrow$ saddle point

At point $(1,1)$, $D(1,1) = 144 - 16 > 0$ and $f_{xx}(1,1) = 12 > 0$ $\Rightarrow$ local min

At point $(-1,-1)$ $D(-1,-1) = 144 - 16 > 0$ and $f_{xx}(-1,-1) = 12 > 0$ $\Rightarrow$ local min

— 1

**Q. 14.** [10 pts] For this question, consider the function $f(x, y) = x^2 - 4y^2 - 9$ at the point $P = (1, -2)$.

a. What is the equation of the tangent plane at the point $P$?

b. Find the directional derivative $D_{\mathbf{u}}f(x, y)$ where $\mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2})$ is a unit vector.

c. What is $D_{\mathbf{u}}(1, -2)$ with $\mathbf{u} = (1/\sqrt{2}, 1/\sqrt{2})$?

d. What direction at the point $P$ gives the maximum rate of change of $f$?

e. What direction at the point $P$ gives no rate of change of $f$?

a. $f(1,-2) = 1^2 - 4(-2)^2 - 9 = 1 - 4\cdot 4 - 9 = -24$

$f_x(x,y) = 2x \Rightarrow f_x(1,-2) = 2$

$f_y(x,y) = -8y \Rightarrow f_y(1,-2) = 16$ — 1

Eq'n tangent plane $\boxed{z + 24 = 2(x-1) + 16(y+2)}$ — 1

b. $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u} = \langle f_x, f_y \rangle \cdot \vec{u}$ — 1

$= \langle 2x, -8y \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = \boxed{\frac{2}{\sqrt{2}}x - \frac{8}{\sqrt{2}}y}$ — 1

c. $D_{\vec{u}}(1,-2) = \frac{2}{\sqrt{2}}\cdot 1 - \frac{8(-2)}{\sqrt{2}} = \boxed{\frac{18}{\sqrt{2}}}$ — 1 — 1

d. max rate of change occurs at $\nabla f(1,-2)$ — 1

$\nabla f(1,-2) = \langle f_x(1,-2), f_y(1,-2) \rangle = \boxed{\langle 2, 16 \rangle}$ — 1

e. Note we have no rate of change in direction $\vec{u}$ if

$\nabla f(1,-2) \cdot \vec{u} = 0$ — 1

So, want any vector orthogonal to $\langle 2, 16 \rangle$.

One such vector is $\boxed{\langle -16, 2 \rangle}$. — 1

**Q. 15. [5 pts]** The function $f(x, y) = x^2 - y^2$ has both a maximum and minimum value when subject to the constraint function $x^2 + y^2 = 1$. Use Lagrange multipliers to find the extreme values. [This is Exercise 3 of Section 14.8]

(Do not worry about
this question.
Topic only covered in 2016)

**Bonus. [1 pts]** What was your favourite topic in the course, and why?

All of it :)

THE END