NAME: SOLUTIONS

STUDENT NUMBER: ______________

MATH 1XX3 (CALCULUS FOR MATH AND STATS II) MIDTERM 1

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DAY CLASS

DURATION OF MIDTERM: 50 minutes

MIDTERM February 11, 2015

THIS MIDTERM PAPER INCLUDES 5 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

SPECIAL INSTRUCTION

- Only the McMaster Standard Calculator (Casio fx 991) is allowed.
- All solutions should be written on the exam.
- The total number of points is 40. **Do all of the 8 questions.**
- For multiple choice questions, full marks will be received for a questions with the correct circled answer. For incorrect answers, part marks are based upon work provided.
- Use the back of pages if you need more room.

Page	Possible	Received
2	10	
3	10	
4	10	
5	10	
Total	40	

1. **[5 pts]** Which equation does the function $y = e^{-6x}$ satisfy?

$\boxed{\text{a. } y'' - y' - 42y = 0}$ b. $y'' - y' + 42y = 0$ c. $y'' + y' - 42y = 0$ d. $y'' + y' + 42y = 0$

$y = e^{-6x}$

$y' = -6e^{-6x}$

$y'' = 36e^{-6x}$

Satisfies (a) b/c

$$\underbrace{(36e^{-6x})}_{y''} - \underbrace{(-6e^{-6x})}_{y'} - 42\underbrace{(e^{-6x})}_{y} = 0$$

2. **[5 pts]** Solve the differential equation $4y' + 5e^{x+y} = 0$.

a. $y = \frac{5}{4}e^x - C$ $\boxed{\text{b. } y = -\ln\left(\frac{5}{4}e^x - C\right)}$ c. $y = \ln\left(\frac{5}{4} + \frac{5}{4}Ce^x\right) - x$ d. $y = \ln\left(\frac{5}{4}x\right) - x$

This eqn is separable:

$$4\frac{dy}{dx} = -5e^x e^y \Rightarrow \frac{dy}{e^y} = -\frac{5}{4}e^x\,dx$$

$$\Rightarrow \int e^{-y}\,dy = \int -\frac{5}{4}e^x\,dx \Rightarrow -e^{-y} = -\frac{5}{4}e^x + C \Rightarrow e^{-y} = \frac{5}{4}e^x - C$$

So $\ln(e^{-y}) = -y = \ln\left(\frac{5}{4}e^x - c\right)$

$$\Rightarrow y = -\ln\left(\frac{5}{4}e^x - c\right)$$

3. **[5 pts]** Solve the differential equation $y' = 15 + 5y + 3x + xy$:

a. $y = -3 + Ce^{\frac{x^2}{2} - 5x}$.   b. $y = -3 + Ce^{x^2 - 5x}$. $\boxed{\text{c. } y = -3 + Ce^{\frac{x^2}{2} + 5x}.}$

d. $y = -3 + Ce^{\frac{x^2}{2} + 5x^2}$ e. $y = -3 + Ce^{x^2 + 5x}$

Eq'n is linear, so use integrating factor

$y' + (-5-x)y = 15+3x$ Set $I(x) = e^{\int -5-x\,dx} = e^{-5x-\frac{x^2}{2}}$

So $\int e^{-5x-\frac{x^2}{2}} y' + e^{-5x-\frac{x^2}{2}}(-5-x)y \, dx = \int e^{-5x-\frac{x^2}{2}}(15+3x)\,dx$

$\left(e^{-5x-\frac{x^2}{2}} y\right) = (-3)e^{-5x-\frac{x^2}{2}} + C$

$\Rightarrow y = \dfrac{-3e^{-5x-\frac{x^2}{2}} + C}{e^{-5x-\frac{x^2}{2}}} = -3 + \dfrac{C}{e^{-5x-\frac{x^2}{2}}} = \boxed{-3 + Ce^{5x+\frac{x^2}{2}}}$

4. **[5 pts]** Find the equation of the tangent line to the curve at the point corresponding to the given value of the parameter.

$$x = \cos\theta + \sin 2\theta + 8 \quad y = \sin\theta + \cos 2\theta + 8 \quad \text{when} \quad \theta = \pi.$$

a. $y = \frac{x}{2} + 2$ b. $y = \frac{2}{x}$ c. $y = \frac{x}{2} + \frac{3}{2}$ $\boxed{\text{d. } y = \frac{25}{2} - \frac{x}{2}}$ e. None of the above.

The pt on the curve is $(x,y) = (\cos\pi + \sin 2\pi + 8, \sin\pi + \cos 2\pi + 8) = (7,9)$

The derivative dy/dx is given by

$\dfrac{dy}{dx} = \dfrac{dy/d\theta}{dx/d\theta} = \dfrac{\cos\theta - 2\sin 2\theta}{-\sin\theta + 2\cos 2\theta} \Rightarrow @\ \theta = \pi \quad \dfrac{dy}{dx} = \dfrac{\cos\pi - 2\sin 2\pi}{-\sin\pi + 2\cos 2\pi} = \dfrac{-1}{2}$

So, eqn of the line is $(y-9) = \frac{-1}{2}(x-7) \Rightarrow y = \frac{-1}{2}x + \frac{7}{2} + 9$

$\boxed{= \frac{-1}{2}x + \frac{25}{2}}$

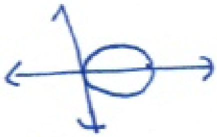
5. **[5 pts]** Alice wants to show her love to Bob by sending him a valentine on her graphing calculator. Which of following curves should Alice use to send Bob a picture of a heart?

a. $r = 4\cos\theta$. $\boxed{\text{b. } r = 1 - \sin\theta.}$ c. $r = \cos 3\theta$.

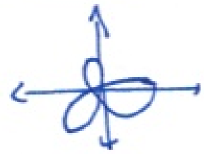
Bob responds by sending Alice a graph to show his love for her is infinite. Which equation does Bob need to use to get an infinity symbol?

a. $r = \theta$ with $\theta \geq 0$. b. $r = \frac{1}{2} + \sin\theta$. $\boxed{\text{c. } r^2 = \cos 2\theta.}$ [Hint: pay attention to domain!]

$r = 4\cos\theta$

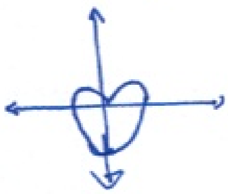


$r = \cos 3\theta$

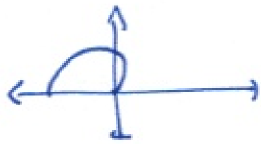


$r = \frac{1}{2} + \sin\theta$

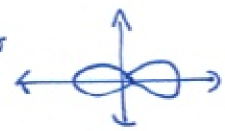
$r = 1 - \sin\theta$



$r = \theta$



$r^2 = \cos 2\theta$



domain $-\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}$

$\frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$

Since $\cos 2\theta$ must be positive

6. **[~~4~~ 5 pts]** Find the area of the region enclosed by one loop of the curve

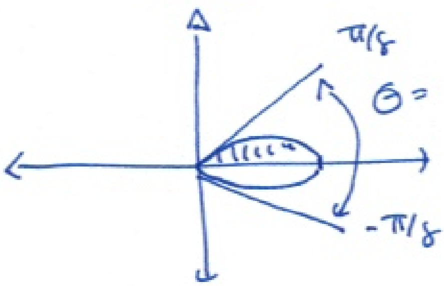
$$r = 7\cos 4\theta.$$

[Hint 1: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$. Hint 2: The curve has 8 loops.]

a. $\frac{49\pi}{11}$ $\boxed{\text{b. } \frac{49\pi}{16}}$ c. $\frac{\pi}{4}$ d. $\frac{49}{\pi}$ e. $\frac{\pi}{2}$

$r = 7\cos 4\theta = 0 \iff \theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \ldots$

So, first loop looks like



Compute top half of loop

$$\text{Area} = 2\int_0^{\frac{\pi}{8}} \frac{1}{2}(7\cos 4\theta)^2\,d\theta = \int_0^{\pi/8} 49\cos^2 4\theta\,d\theta = 49\int_0^{\frac{\pi}{8}} \frac{1+\cos 8\theta}{2}\,d\theta$$

$$= \frac{49}{2}\left[\theta + \frac{\sin 8\theta}{8}\Big|_0^{\frac{\pi}{8}}\right] = \frac{49}{2}\left[\left(\frac{\pi}{8} + \frac{\sin\pi}{8}\right) - \left(0 + \frac{\sin 0}{8}\right)\right]$$

$$= \frac{49\pi}{16}$$

7. **[5 pts]** Math-phobia is a terrible disease that prevents people from doing mathematics! It is known that math-phobia will spread according the ~~the~~ logistic differential equation. In a small town of 3,000 people, 120 people have the disease at the beginning of the week, and 1,600 are afflicted by the end of the week. When will 40% of the poplutation be infected?

Because we are given that disease spreads according to $P(t) = \frac{M}{1+Ae^{-kt}}$ with $M = 3000$ and $A = \frac{3000 - P(0)}{P(0)} = \frac{3000-120}{120} = 24$

By day 7, $P(7) = 1600 = \frac{3000}{1+24e^{-k7}} \Rightarrow e^{-7k} = \left(\frac{3000}{1600} - 1\right)/24$

$\Rightarrow k = -\frac{1}{7}\ln\left(\frac{1}{24}\left(\frac{3000}{1600}-1\right)\right) = 0.473$ So $P(t) = \frac{3000}{1+24e^{-.473t}}$.

Want to find t such that

$\frac{3000}{1+24e^{-.473t}} = 0.4(3000) = 1200 \Rightarrow e^{-.473t} = \frac{1}{24}\left(\frac{3000}{1200} - 1\right)$

$\Rightarrow t = \frac{1}{-.473}\ln\left(\frac{1}{24}\left(\frac{3000}{1200}-1\right)\right) = \boxed{5.8617 \text{ days}}$

$\boxed{\frac{5.8617}{7} \text{ weeks} = 0.837 \text{ weeks}}$

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8. **[5 pts]** Consider the two polar curves $r = f(\theta)$ and $r = 2f(\theta)$. How do the lengths of these two curves compare on the interval $\alpha \leq \theta \leq \beta$? Give a reason for your answer.

$\boxed{\text{The second curve will be twice as long}}$

Length of first curve is $\int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

For second curve, $r^2 = (2f(\theta))^2 = 4f^2(\theta)$ and $\frac{dr}{d\theta} = 2f'(\theta) \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = 4f'(\theta)^2$

So, length of second curve is

$\int_\alpha^\beta \sqrt{4f(\theta)^2 + 4f'(\theta)^2}\, d\theta = \int_\alpha^\beta 2\sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta = 2$ $\times$ length of first curve

THE END