NAME: SOLUTIONS

STUDENT NUMBER: ______________

MATH 1XX3 (CALCULUS FOR MATH AND STATS II) MIDTERM 2

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DAY CLASS

DURATION OF MIDTERM: 50 minutes

MIDTERM

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THIS MIDTERM PAPER INCLUDES 6 PAGES AND 9 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

SPECIAL INSTRUCTION

- Only the McMaster Standard Calculator (Casio fx 991) is allowed.
- All solutions should be written on the exam.
- The total number of points is 40. **Do all of the 9 questions**.
- For multiple choice questions, full marks will be received for a questions with the correct circled answer. For incorrect answers, part marks are based upon work provided.
- Use the back of pages if you need more room.

Page	Possible	Received
2	8	
3	8	
4	6	
5	10	
6	8	
Total	40	

NOTE: I MADE THE EXAM

OUT OF 34

1. **[4 pts]** A ball is dropped from a height of 10 m onto a flat surface. Each time the ball hits the surface, it rebounds to 50% of its previous height. Find the total distance the ball travels (assume the ball bounces an infinite number of times).

a. 10 m b. 20 m $\boxed{\text{c. 30 m}}$ d. 40 m e. ∞ m

Picture

Ball travels $10+5+5+2.5+2.5+1.25+1.25+\ldots$ meters

Break into two geometric series

$$A = 10+5+2.5+1.25+\ldots = \sum_{n=0}^{\infty} 10\left(\tfrac{1}{2}\right)^n = \frac{10}{1-1/2} = 20$$

$$B = 5+2.5+1.25+\ldots = \sum_{n=0}^{\infty} 5\left(\tfrac{1}{2}\right)^n = \frac{5}{1-1/2} = 10$$

So $A+B=30$

2. **[4 pts]** Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \arctan n}{n^4}.$$

[Hint: What's an upper bound for $\arctan x$?]

a. conditionally convergent $\boxed{\text{b. absolutely convergent}}$ c. divergent

From hint, $0 \le \arctan x \le \frac{\pi}{2}$ for all $x \ge 0$

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n \arctan n}{n^4} \right| = \sum_{n=1}^{\infty} \frac{\arctan n}{n^4} \le \sum_{n=1}^{\infty} \frac{\pi/2}{n^4} = \frac{\pi}{2} \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^4}}_{\text{p-series with } p=4}$$

Since the p-series converges, the series converges absolutely

3. **[4pts]** Test for convergence or divergence (use any test): $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+4}$.

use alternating series test. let $b_n = \frac{1}{n+4}$

① $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n+4} = 0$

② $b_n = \frac{1}{n+4} > \frac{1}{(n+1)+4} = b_{n+1}$

So ① + ② and the AST implies series $\boxed{\text{CONVERGES}}$

4. **[4pts]** Test for convergence or divergence (use any test): $\sum_{n=2}^{\infty} \frac{5}{n(\ln n)^2}$.

use integral test $\int_2^{\infty} \frac{5}{x(\ln x)^2}\,dx = 5\int_2^{\infty} \frac{1}{x(\ln x)^2}\,dx$

$= \lim_{t\to\infty} 5\int_2^{t} \frac{1}{x(\ln x)^2}\,dx = \lim_{t\to\infty} 5\left(\frac{-1}{\ln(x)}\right)\Big]_2^{t}$

$= \lim_{t\to\infty} \frac{-5}{\ln t} + \frac{5}{\ln 2} = \frac{5}{\ln 2}$

Since $\int_2^{\infty} \frac{5}{x(\ln x)^2}\,dx$ converges, integral test implies series $\boxed{\text{CONVERGES}}$

5. **[4pts]** Test for convergence or divergence (use any test): $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$.

[Hint: You may want to use the fact that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.]

Use limit comparison test. Let $a_n = \frac{1}{n^{1+\frac{1}{n}}} = \frac{1}{n n^{1/n}}$ and $b_n = \frac{1}{n}$

Then $\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \frac{\frac{1}{n n^{1/n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n n^{1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} = 1$

Since this limit is positive, and because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges,

the limit comparison test implies $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ $\boxed{\text{DIVERGES}}$

6. **[2 pts]** What is wrong with following argument that $0 = \pi$:

$$\begin{aligned}
0 &= 0+0+0+0+\cdots \\
&= (\pi-\pi)+(\pi-\pi)+(\pi-\pi)+\cdots \quad (*) \\
&= \pi+(-\pi+\pi)+(-\pi+\pi)+(-\pi+\pi)+\cdots \\
&= \pi+0+0+\cdots = \pi
\end{aligned}$$

The problem is that step $(*)$ is incorrect.

The series $\sum_{n=1}^{\infty} (-1)^{n+1}\pi = \pi - \pi + \pi \cdots$

does not converge by the divergence test. So it cannot be equal to 0.

7. **[5 pts]** Let $C > 0$ be any positive real number. Find the radius of convergence and the interval of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{(Cx)^n}{n!}.$$

a. $R = C, I = (-C, C)$ b. $R = 0, I = [0]$ c. $R = C, I = [-C, C]$ $\boxed{\text{d. } R = \infty, I = (-\infty, \infty).}$

Let $a_n = \frac{(Cx)^n}{n!}$. By comparision test, will converge if $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$

$$\lim_{n\to\infty} \left| \frac{\frac{(Cx)^{n+1}}{(n+1)!}}{\frac{(Cx)^n}{n!}} \right| = \lim_{n\to\infty} \frac{|Cx|}{n+1} = |Cx| \lim_{n\to\infty} \frac{1}{n+1} = 0.$$

So power series will converge for $\underline{\underline{\text{all}}}$ x

So $R = \infty$ and

$I = (-\infty, \infty)$

8. **[5 pts]** The Maclaurin Series of $\sin x$ is given by

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

Use this fact to find the values of r and s for which

$$\lim_{x\to 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s \right) = 0.$$

$$\sin 3x = 3x - \frac{(3x)^3}{3!} + \frac{(3x)^5}{5!} - \frac{(3x)^7}{7!} + \cdots$$

So $\frac{\sin 3x}{x^3} = \frac{3}{x^2} - \frac{27}{3!} + \frac{3^5 x^2}{5!} - \frac{3^7 x^4}{7!} = \frac{3}{x^2} - \frac{27}{3!} + x^2[\text{other terms}]$

Thus $\lim_{x\to 0} \left(\frac{\sin 3x}{x^3} + \frac{r}{x^2} + s \right) = \lim_{x\to 0} \left(\frac{3}{x^2} + \frac{r}{x^2} - \frac{27}{3!} + s + x^2[\text{other terms}] \right)$

$= 0$ if and only if $r = -3$ and $s = \frac{27}{3!}$

So $\boxed{r = -3 \text{ and } s = \frac{27}{3!}}$

9. **[8 pts]** The following statements are all **FALSE**. For each statement, give an example to show that the statement is false.

(a) If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

Harmonic Series $\sum_{n=1}^{\infty} \frac{1}{n}$ has $\lim_{n\to\infty} \frac{1}{n} = 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(b) If $\sum_{n=1}^{\infty} a_n^2$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

$\sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^2$ converges (a p-series with $p=2$) but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

(c) A series that converges must converge absolutely.

The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges, but not absolutely

(d) If $0 \leq a_n \leq b_n$ for all $n \geq 0$ and if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{\boxed{n=1}}^{\infty} a_n$ also diverges. * *

Use the example of (b) $\frac{1}{n^2} \leq \frac{1}{n}$ for all n

and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, but $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

THE END