MATH 3GR3 - Soln's (Final Exam

PART A

1. (i) $\mathbb{Z}_2 \times \mathbb{Z}_2$
   (ii) D_5
   (iii) $M_{2,2}(\mathbb{R})$ ← all 2x2 matrices
   (iv) $E = \{2n \mid n \in \mathbb{Z}\}$
   (v) $\mathbb{Z}$

2. (i) False
   (ii) True
   (iii) FALSE
   (iv) TRUE
   (v) FALSE

3. $$(ab^2)^{-1}(c^2a^{-1})^{-1}(c^2b^2d)(a^2d)^{-2}a$$
$$= b^{-2}a^{-1}ac^{-2}c^2b^2d\,d^{-2}a^{-2}a$$
$$= b^{-2}b^2d^{-1}a^{-1} = \boxed{d^{-1}a^{-1}}$$

4. (i) $\sigma = (1234)(5\,10)(789)$
   (ii) $|\sigma| = \operatorname{lcm}(4,2,3) = 12$

5. Let H be any subgp of G. Need to show $gHg^{-1} \subseteq H$ for all $g \in G$. Fix some $g \in G$ and let $t \in gHg^{-1}$. So $t = ghg^{-1}$ for some $h \in H$. Because ~~g~~ G is abelian
$$t = ghg^{-1} = (gg^{-1})h = e \cdot h = h \in H.$$
So $gHg^{-1} \subseteq H$. Thus every subgp is normal in G.

6. 2017 is prime so $\mathbb{Z}_{2017}$ is a domain (and field!)

7. (i) Note that $f(a)=0$ if a is $0,2,4$ and $f(b)=1$ if $b=1,3,5$. So, need to check cases when a,b odd or even.

If a,b both even

$$f(a)+f(b) = 0+0 = f(a+b)$$
$$f(a)f(b) = 0\cdot 0 = 0 = f(ab)$$

If a even & b odd

$$f(a)+f(b) = 0+1 = f(a+b)$$
$$f(ab) = 0 \cdot = 0\,1 = f(a)f(b)$$

Same if a odd & b even

If a & b odd

$$f(a)+f(b) = 1+1 = 0 = f(a+b)$$
$$f(a)f(b) = 1\cdot 1 = 1 = f(ab)$$

(ii) $\text{Kernel}(f) = \{0,2,4\}$

8.

$$\begin{array}{r|l} & 4x+3 \\ \hline 2x+4 & 3x^2+2x+1 \\ & \underline{3x^2+x} \\ & \quad x+1 \\ & \quad \underline{x+2} \\ & \quad -1 = 4 \end{array}$$

$2 \in \mathbb{Z}_5$, so $2^{-1}=3$

So $2^{-1}\cdot 3 = 3\cdot 3 = 4$

So $(3x^2+2x+1) = (2x+4)(4x+3)+4$ in $\mathbb{Z}_5[x]$

$$= 8x^2+6x+16x+12+4$$
$$= 8x^2+22x+16 = 3x^2+2x+1$$

9_3

9. Apply Eisenstein's Criterion with $p=3$. Since $3 \mid 15$, $3 \mid 9$, $3 \mid -3$, $3 \mid 6$, $3 \nmid 5$, and $3^2 \nmid 15$, by Eisenstein's Criterion, the polynomial is irreducible

Part B

1. Check subgp prop.

(identity) $e \in C(H)$ since $ehe^{-1} = h$ for all $h \in H$.

(closure). Let $a, b \in C(H)$. So $aha^{-1} = h$ and $bhb^{-1} = h$ for all $h \in H$. Then

$$abh(ab)^{-1} = abhb^{-1}a^{-1} = aha^{-1} \quad (\text{since } bhb^{-1} = h)$$
$$= h. \quad (\text{since } aha^{-1} = h.$$

So $ab \in C(H)$

(inverse). Suppose $a \in C(H)$. So $aha^{-1} = h$ for all $h \in H$.
Since H is a subgp, $h^{-1} \in H$ for any $h \in H$. So

$$ah^{-1}a^{-1} = h^{-1} \quad \text{for all } h \in H.$$

But then

$$h^{-1} = a^{-1}h^{-1}a \quad \text{for all } h \in H.$$

So $(h^{-1})^{-1} = (a^{-1}h^{-1}a)^{-1} = a^{-1}ha$ for all $h \in H$

But this means $a^{-1} \in C(H)$.

2. Check subring conditions

(nonempty) $0 = 0 + 0\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$

(mult. closed) Let $a+b\sqrt{2}$ and $c+d\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. Then

$$(a+b\sqrt{2})(c+d\sqrt{2}) = ac + ad\sqrt{2} + bc\sqrt{2} + bd2$$
$$= (ac+2bd) + (ad+bc)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$$

(subtraction closed) Let $a+b\sqrt{2}, c+d\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$. Then

$$(a+b\sqrt{2}) - (c+d\sqrt{2}) = (a-c) + (b-d)\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$$

3. <u>Note.</u> If $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{R})$, then $g^{-1} = \frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Need to show $gHg^{-1} \subseteq H$ for all $g \in SL_2(\mathbb{R})$.

If $h = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$, then $ghg^{-1} = gg^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right] \in H$.

If $h = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, then $ghg^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} d/ad-cb & -b/ad-cb \\ -c/ad-cb & a/ad-cb \end{bmatrix}$

$$= \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}\begin{bmatrix} d/ad-cb & -b/ad-cb \\ -c/ad-cb & a/ad-cb \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \in H.$$

So H is normal in G.

4. Check ideal conditions

(nonempty). Since $0 \in I$ and $0 \in J$, $0 \in I \cap J$.
(closed under subtraction) Let $a, b \in I \cap J$. Then $a, b \in I$, So $a-b \in I$ since I is an ideal. Similarly, $a-b \in J$. So $a-b \in I \cap J$.
(closed under absorption) Let $a \in I \cap J$ and $r \in R$. Then $a \in I$ implies $ra \in I$ since I is an ideal. The same for J, i.e $ra \in J$. Note we also will have $ar \in I$ and $ar \in J$. So ra and ar are in $I \cap J$.

So $I \cap J$ is an ideal

#5. Because $f: G \to H$ is onto, $G/_{\ker f} \cong H$.

Since f is not 1-1, $\ker f \supsetneq \{e\}$. Since $\ker f$ is a subgp of G, $|\ker f| \mid |G| = pq$. So, since $|\ker f| \neq 1$, $|\ker f| = p, q,$ or pq.

If $|\ker f| = p$, then $|H| = |G| / |\ker f| = pq/p = q$. So $H \cong \mathbb{Z}_q$ since ~~only~~ all gps of order q a prime are cyclic.

If $|\ker f| = q$, then $|H| = |G|/|\ker f| = pq/q = p$. By same argument as above, $H \cong \mathbb{Z}_p$.

If $|\ker f| = pq$, $|H| = |G|/|\ker f| = pq/pq = 1$. So $H = \{e\}$.

In all 3 cases, H must be abelian

#6. We know $\ker f$ is an ideal of R. Since R is a field, only ideals of R are $\ker f = \langle 1 \rangle = R$ or $\ker f = \langle 0 \rangle$. In the first case, the map f is the zero map. In the second case, b/c the kernel only contains ~~$\langle 0 \rangle$~~, 0, the map is 1-1

#7 Consider the identity homom $f: \mathbb{Z} \to \mathbb{Q}$ given by $f(a) = a$. Then $f(\mathbb{Z})$ is not an ideal in $\mathbb{Q}$ because it does not have the absorption prop, e.g $1 \in f(\mathbb{Z})$, but $\frac{1}{2} \cdot 1 \notin f(\mathbb{Z})$.

$\boxed{\#8}$ Since $H = \langle g \rangle$, we have $|H| = |\langle g \rangle| = |g| = 6$. So $|G/H| = \cancel{36} \; |G|/|H| = 30/6 = 5$ by Lagrange's theorem.

Since G/H is a gp of order 5, and since 5 is prime, we must have $G/H \simeq \mathbb{Z}_5$. But this implies G/H is abelian

$\boxed{\#9}$ See the text for sol^{n}

$\boxed{\#10}$ See the text for sol^{n}

Bonus Anything!