NAME: SOLUTIONS

STUDENT NUMBER: ______________

MATH 1XX3 (CALCULUS FOR MATH AND STATS II) MIDTERM 1
ADAM VAN TUYL, MCMASTER UNIVERSITY

DAY CLASS

DURATION OF MIDTERM: 50 minutes

MIDTERM

February 13, ~~2016~~ 2017

THIS MIDTERM PAPER INCLUDES 6 PAGES AND 8 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

SPECIAL INSTRUCTION

- Only the McMaster Standard Calculator (Casio fx 991) is allowed.
- All solutions should be written on the exam.
- The total number of points is 40. **Do all of the 8 questions.**
- For multiple choice questions, full marks will be received for a questions with the correct circled answer. For incorrect answers, part marks are based upon work provided.
- Use the back of pages if you need more room.

Page	Possible	Received
2	10	
3	10	
4	10	
5	5	
6	5	
Total	40	

1. **[5 pts]** To evaluate the following anti-derivative,

$$\int \frac{1}{(x^2+1)(x^2+4)}dx$$

we need to use partial fractions. Which of following is the correct partial fraction equation that needs to be solved?

(a) $\frac{A}{x^2+1}+\frac{B}{x^2+4}$.

(b) $\frac{A}{x^2+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$.

(c) $\boxed{\frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+4}}$. ← x^2+1 and x^2+4 irreducible, so cannot factor more. Both have degree 2, so numerator must have form a_1x+a_2

(d) $\frac{Ax}{x^2+1}+\frac{Bx}{x^2+4}$.

2. **[5 pts]** Solve the differential equation $\frac{dy}{dx}+\frac{1}{3}y=1$.

This is a first order linear d.e. Integrating factor $P(x)=e^{\int\frac{1}{3}dx}=e^{\frac{1}{3}x}$

Multiply both sides by $P(x)$:

$$e^{1/3x}\frac{dy}{dx}+\frac{1}{3}e^{1/3x}y=e^{1/3x}$$ ← integrate both sides

$$e^{1/3x}y=\int e^{1/3x}dx = 3e^{1/3x}+C$$

Solve for y

$$y=\frac{1}{e^{1/3x}}\left[3e^{1/3x}+C\right]=\boxed{3+Ce^{-\frac{1}{3}x}}$$

3. **[5pts]** Consider the initial value problem $\frac{dy}{dx} = 3x^2y^2$ with $y(0) = \frac{1}{3}$. Use Euler's method to estimate the value of $y(1)$ with step size $h = \frac{1}{2}$. Does your answer overestimate or underestimate the correct answer? Justify your answer

Eulers method: $x_i = x_{i-1} + h$

$y_i = y_{i-1} + hF(x_{i-1}, y_{i-1})$. Let $F(x,y) = 3x^2y^2$

Step i	x_i	y_i
0	0	$1/3$
1	$1/2$	$1/3 + \frac{1}{2}F(0, 1/3) = 1/3$
2	1	$1/3 + \frac{1}{2}F(\frac{1}{2}, \frac{1}{3}) = \frac{1}{3} + \frac{1}{2}\cdot 3\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2 = \frac{1}{3} + \frac{1}{24} = \frac{9}{24} = \boxed{\frac{3}{8}}$

So $\boxed{y(1) \approx 3/8}$

This answer underestimates the answer. This is b/c on the interval $(0,1)$, when $y \geq 1/3$, $F(x,y) \geq 0$. So ther derivative grows faster than our estimate. Eg we us slope $m=0$ to estimate on the first interval, but the slope is positive on interval $(0, 1/2)$.

** 4. **[5 pts]** Find the arc-length of the curve given by $x = \frac{1}{2}t^2$, and $y = \frac{1}{9}(6t+9)^{3/2}$ from $t = 0$ to $t = 4$

(a) 1 (b) 28 (c) 42 $\boxed{\text{(d) 20}}$ (e) 2017

$\frac{dx}{dt} = t$ $\quad \frac{dy}{dt} = \frac{1}{9}\cdot\frac{3}{2}(6t+9)^{1/2}\cdot 6 = (6t+9)^{1/2}$

So Arclength $= \int_0^4 \sqrt{t^2 + [(6t+9)^{1/2}]^2}\,dt = \int_0^4 \sqrt{t^2 + 6t + 9}\,dt = \int_0^4 \sqrt{(t+3)^2}\,dt$

$= \int_0^4 (t+3)\,dt = \frac{t^2}{2} + 3t \Big|_0^4 = \left[\frac{16}{2} + 3\cdot 4\right] - 0 = 8 + 12 = \boxed{20}$

5. **[5 pts]** Consider the family of curves described by the polar equation $r = a\cos(\theta)$ where $a > 0$ is any positive real number. Which statement describes the family of curves described by this equation?

   (a) A circle of radius $\frac{a^2}{4}$ with center at $(\frac{a}{2}, 0)$.

   * (b) A circle of radius a whose center is the origin.

   (c) A circle of radius $\sqrt{a}$ whose center is $(0, a)$.

   (d) A circle of radius $\frac{a}{2}$ whose center is $(\frac{a}{2}, 0)$

   (e) A curve with a loops.

$r^2 = ar\cos\theta \Rightarrow x^2+y^2 = ax \Rightarrow x^2 - ax + y^2 = 0$

$\Rightarrow \left(x^2 - ax + \frac{a^2}{4}\right) + y^2 = \frac{a^2}{4} \Rightarrow \left(x-\frac{a}{2}\right)^2 + y^2 = \left(\frac{a}{2}\right)^2$ ← a circle of radius $\left(\frac{a}{2}\right)$ centered at $\left(\frac{a}{2}, 0\right)$

6. **[5 pts]** Suppose that the polar curve $r = f(\theta)$ encloses an area as the curve is traced out for $\theta = 0$ to $\theta = 2\pi$. How does the size of this area compare to the area enclosed by the curve $r = 2017f(\theta)$? Give a reason for your answer.

Let $A = \int_0^{2\pi} \frac{1}{2} f(\theta)^2 \, d\theta$ be the area enclosed ~~traced~~ out by $r = f(\theta)$.

The Area enclosed by $r = 2017 f(\theta)$ is

$$\int_0^{2\pi} \frac{1}{2} (2017 f(\theta))^2 \, d\theta = (2017)^2 \int_0^{2\pi} \frac{1}{2} f(\theta)^2 \, d\theta = (2017)^2 A.$$

That is, it is $(2017)^2$ times larger than the area enclosed by first curve.

7. **[5pts]** Aliens have captured a number of humans in order to eat them! Every year they harvest a fixed number of people. The aliens are using the *harvesting model equation*

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right) - 15$$

to model the number of humans. Here, $P(t)$ is the population of humans at time t. (Note that the model looks like the logistic equation with a "harvesting" factor at the end.) What are the equilibrium solutions? What is the minimum number of humans the aliens should capture so they have a steady supply of food?

equilibrium sol^{n} $\iff \frac{dP}{dt} = 0 \iff \underbrace{0.08P - \frac{0.08P^2}{1000} - 15 = 0}_{\text{this is a quadratic eqn.}}$

Solve for P.

$$P = \frac{-0.08 \pm \sqrt{(0.08)^2 - 4(-15)(-\frac{0.08}{1000})}}{2(-\frac{0.08}{1000})} = \frac{-0.08 \pm 0.04}{2(-0.08)}(1000)$$

$$= 500[1 \pm \tfrac{1}{2}] = \boxed{250 \text{ or } 750}$$

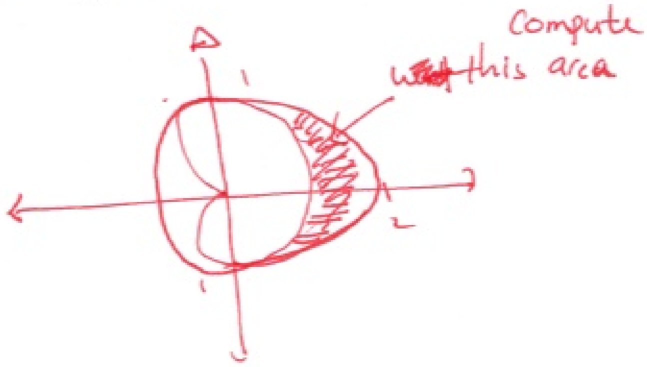
$\therefore$ Equilibrium sol^{n}s are $P = 250$ or 750

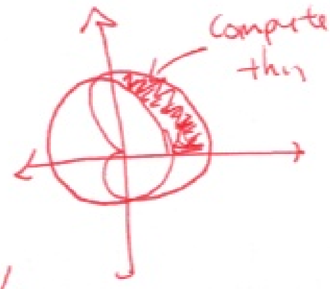
The minimum # of humans to be captured is 250. If they capture less than 250, the $\frac{dP}{dt} < 0$, so population will decrease b/c of harvesting.

(Note if $250 < P < 750$, then $\frac{dP}{dt} > 0$, so the pop will increase even if the aliens harvest people)

8. **[5 pts]** Find the area inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$. [HINT 1: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$. HINT 2: Sketching a picture may help]

Picture





Note ① Easier to compute half the area and then double

Note 2 $1+\cos\theta = 1 \iff \cos\theta = 0 \iff \theta = \pi/2, 3\pi/2$.

So

$$\text{Area} = 2\int_0^{\frac{\pi}{2}} \frac{1}{2}\left((1+\cos\theta)^2 - 1^2\right)d\theta = \int_0^{\frac{\pi}{2}} 1+2\cos\theta+\cos^2\theta - 1 \, d\theta$$

only computing half & then double

$$= \int_0^{\frac{\pi}{2}} 2\cos\theta + \cos^2\theta = \int_0^{\frac{\pi}{2}} 2\cos\theta + \frac{1+\cos 2\theta}{2}$$

$$= 2\sin\theta + \frac{1}{2}\theta + \frac{\sin 2\theta}{4}\Big]_0^{\frac{\pi}{2}}$$

$$= \left[2\sin\left(\tfrac{\pi}{2}\right) + \frac{1}{2}\cdot\frac{\pi}{2} + \frac{\sin\left(2\frac{\pi}{2}\right)}{4}\right] - \left[2\sin 0 + \frac{1}{2}0 + \frac{\sin 2\cdot 0}{4}\right] = \boxed{2+\frac{\pi}{4}}$$

THE END