NAME: Solutions

STUDENT NUMBER: ______________

MATH 1XX3 (CALCULUS FOR MATH AND STATS II) MIDTERM 2

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DAY CLASS

DURATION OF MIDTERM: 50 minutes

MIDTERM

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THIS MIDTERM PAPER INCLUDES 6 PAGES AND 10 QUESTIONS. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCY TO THE ATTENTION OF YOUR INVIGILATOR.

SPECIAL INSTRUCTION

- Only the McMaster Standard Calculator (Casio fx 991) is allowed.
- All solutions should be written on the exam.
- The total number of points is 37. **Do all of the 10 questions.**
- Use the back of pages if you need more room.

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1. **[4pts]** Test for convergence or divergence (use any test): $\sum_{n=1}^{\infty} \cos^2(n\pi)$.

For all $n \geq 1$, $\cos(n\pi) = \pm 1$, so $a_n = \cos^2(n\pi) = 1$.

Since $\lim_{n\to\infty} a_n \neq 0$, series $\boxed{\text{diverges}}$ by the divergence test

2. **[4 pts]** Test for convergence or divergence (use any test): $\sum_{n=1}^{\infty} \frac{\sqrt{n}+2017}{n^4}$.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}+2017}{n^4} = \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^4} + \sum_{n=1}^{\infty} \frac{2017}{n^4} = \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{3\frac{1}{2}}}}_{A} + 2017 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^4}}_{B}$$

A is a p-series with $p = 3\frac{1}{2} > 1$, so it converges

B is a p-series with $p = 4 > 1$, so B converges.

Since A & B converge, so does A+B.

i.e $\sum_{n=1}^{\infty} \frac{\sqrt{n}+2017}{n^4}$ $\boxed{\text{converges}}$

3. **[4pts]** Test for convergence or divergence (use any test): $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1}$

Compare with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. Let $a_n = \dfrac{e^n+1}{ne^n+1}$ and $b_n = \dfrac{1}{n}$

Then $\displaystyle\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{e^n+1}{ne^n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \left| \frac{n(e^n+1)}{n(e^n+1/n)} \right| = \lim_{n\to\infty} \left| \frac{e^n+1}{e^n+1/n} \right| = 1 > 0$

Since $\displaystyle\lim_{n\to\infty} \left|\frac{a_n}{b_n}\right| > 0$ and since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the limit comparison test implies $\displaystyle\sum_{n=1}^{\infty} a_n$ $\boxed{\text{diverges}}$

4. **[4 pts]** Test for convergence or divergence (use any test): $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+10n}$

Let $b_n = \dfrac{1}{3+10n}$. Then

$$\left.\begin{array}{l} ① \displaystyle\lim_{n\to\infty} \frac{1}{3+10n} = 0 \\ ② \; b_n = \dfrac{1}{3+10n} \geq \dfrac{1}{3+10(n+1)} = b_{n+1} \end{array}\right\} \xrightarrow{\text{Alternating Series test}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+10n} \; \boxed{\text{Converges}}$$

5. **[5 pts]** If the nth partial sum of a series $\sum_{n=1}^{\infty} a_n$ is

$$s_n = 4 - n3^{-n}$$

find a_n and $\sum_{n=1}^{\infty} a_n$. Recall $S_n = a_1 + a_2 + \cdots + a_n$

So $S_{n-1} = a_1 + a_2 + \cdots + a_{n-1}$. Thus $S_n - S_{n-1} = a_n$

Hence $a_n = \left(4 - \frac{n}{3^n}\right) - \left(4 - \frac{(n-1)}{3^{n-1}}\right) = -\frac{n}{3^n} + \frac{(n-1)}{3^{n-1}} = \frac{-n + 3(n-1)}{3^n} = \boxed{\frac{2n-3}{3^n}}$

By def$^{\underline{n}}$ $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} (a_1 + \cdots + a_n) = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(4 - \frac{n}{3^n}\right) = \boxed{4}$

6. **[2 pts]** The following statement is **FALSE**:

If a series $\sum_{n=1}^{\infty} a_n$ is a convergent series, then it is absolutely convergent.

Explain why the statement is false, or give an example to show why the statement is false.

The alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ is a counter example

because it converges, but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So, it does

not diverge absolutely.

$\boxed{\begin{array}{l}\text{Alt sol}^{n}\ f(x)=\cos x \\ \text{So } f''(x)=-\cos x \\ \text{So } f(x)+f''(x)=0\end{array}}$

7. **[5 pts]** Show that the power series

$$f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

satisfies the differential equation $f''(x) + f(x) = 0$.

Note $f'(x) = 0 - \frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \frac{8x^7}{8!} \dots = \cancel{\sum_{n=0}^{\infty}}$

So $f''(x) = \frac{-2}{2!} + \frac{4 \cdot 3x^2}{4!} - \frac{6 \cdot 5x^4}{6!} + \frac{8 \cdot 7x^6}{8!} - \dots$

The coeff of x^{2n} in $f''(x)$ is $\frac{(-1)^{n+1}(2n+2)(2n+1)x^{2n}}{(2n+2)!} = \frac{(-1)^{n+1}x^{2n}}{(2n)!}$

So $f''(x) = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n}}{(2n)!}$. But then $f''(x)+f(x)$

$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1}x^{2n}}{(2n)!} + \frac{(-1)^n x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{2n!}\underbrace{((-1)^{n+1}+(-1)^n)}_{=0} = 0$

8. **[2 pts]** The following statement is **FALSE**:

The Ratio Test can be used to determine if the series $\sum_{n=1}^{\infty} \frac{1}{n^{2017}}$ converges.

Explain why the statement is false, or give an example to show why the statement is false.

If we apply the ratio test, $a_n = \frac{1}{n^{2017}}$ and $a_{n+1} = \frac{1}{(n+1)^{2017}}$

Then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{n^{2017}}{(n+1)^{2017}} = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{2017} = 1$ ← Since the limit is 1, the ratio test gives us no information about convergence

9. **[5pts]** Find the radius of convergence of the series

$$\sum_{n=0}^{\infty} \frac{(n!)^2}{(2n)!} x^n.$$

Based upon your answer, make a prediction for the radius of convergence for the series

Find radius of convergence.

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n.$$

$$\lim_{n\to\infty} \left| \frac{\frac{((n+1)!)^2}{(2n+2)!} x^{n+1}}{\frac{(n!)^2}{(2n)!} x^n} \right| = \lim_{n\to\infty} \left| \frac{((n+1)!)^2}{(n!)^2} \frac{(2n)!}{(2n+2)!} x \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2}{(2n+2)(2n+1)} x \right| = \lim_{n\to\infty} \left| \frac{n^2+2n+1}{4n^2+3n+2} \right| |x|$$

$$= \left| \frac{1}{4} x \right|.$$

by ratio test

So series will converge if $\left|\frac{1}{4}x\right| < 1 \iff |x| < 4$, so

Radius of convergence is $\boxed{R=4}$

In the general case, radius of convergence will be $\boxed{R = k^k}$

10. **[2 pts]** The following statement is **FALSE**:

> There is a powers series of the form $\sum_{n=0}^{\infty} c_n(x-2)^n$ with an interval of convergence $[0, 10)$.

Explain why the statement is false, or give an example to show why the statement is false.

This power series is centered at $x=2$. The interval of convergence should have 2 as its midpoint, but it has 5.

THE END