Name: ______________________

Student Number: Solutions

Math 1B03 C01 (Linear Algebra I) Final Exam (VERSION 1)

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Day Class

Duration of Exam: 2.5 hours

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This examination paper includes 21 pages and 38 multiple choice questions printed on both sides of the page. The questions are on pages 2 through 21. Scrap paper is available for rough work. You are responsible for ensuring that your copy of the paper is complete. Bring any discrepancies to the attention of the invigilator.

Special Instructions

1. Select the one correct answer to each question and ENTER THAT ANSWER INTO THE SCAN CARD PROVIDED USING AN HB PENCIL.
2. You are required to submit this booklet along with your answer sheet. HOWEVER, NO MARKS WILL BE GIVEN FOR THE WORK IN THIS BOOKLET. Only the answers on the scan card count for credit. Each question is worth 1 mark. The final is graded out of 38. There is no penalty for incorrect answers.
3. **NO CALCULATORS ALLOWED.**

Computer Card Instructions:

IT IS YOUR RESPONSIBILITY TO ENSURE THAT THE ANSWER SHEET IS PROPERLY COMPLETED. YOUR TEST RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.

- Print your name, student number, course name, and the date in the space provided at the top of Side 1 (red side) of the form. Then the sheet **MUST** be signed in the space marked SIGNATURE.
- You are writing **VERSION 1**; indicate this in the version column.
- Mark your student number in the space provided on the sheet on Side 1 **and fill the corresponding bubbles underneath**. If you have a student number that is 7 digits, begin your student number with 00 (two zeroes).
- Mark only **ONE** choice (A, B, C, D, E) for each question.
- Begin answering questions using the first set of bubbles, marked "1".

1. Suppose that the augmented matrix of system of linear equations has been placed into the following reduced row echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & 0 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

The set of solutions for this systems is described by

(a) $\begin{aligned} x_1 &= -3 \\ x_2 &= 0 \\ x_3 &= 1 \\ x_4 &= 0 \\ x_5 &= 2 \end{aligned}$

(b) $\boxed{\begin{aligned} x_1 &= -3-2q-5r \\ x_2 &= q \\ x_3 &= 1+r \\ x_4 &= r \\ x_5 &= 2 \end{aligned}}$

(c) $\begin{aligned} x_1 &= 3+2q+5r \\ x_2 &= q \\ x_3 &= 1-r \\ x_4 &= r \\ x_5 &= 2 \end{aligned}$

(d) $\begin{aligned} x_1 &= 2q+5r \\ x_2 &= q \\ x_3 &= -1+r \\ x_4 &= r \\ x_5 &= -2 \end{aligned}$

Note that x_2 and x_4 are free
Also $x_3 = 1 + x_4$ and $x_5 = 2$
So the answer must be (b)

2. Consider the following system of linear equations:

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 1 \\ 2x_1 + 2x_2 + 2x_3 + 2x_4 &= 2 \\ 2018x_1 + 2018x_2 + 2018x_3 + 2018x_4 &= 2018 \end{aligned}$$

How many solutions does it have? Matrix form

(a) 0

(b) 1

(c) 4

(d) 2018

(e) $\boxed{\text{Infinitely many}}$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 2 \\ 2018 & 2018 & 2018 & 2018 & 2018 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$\leftarrow$ free variable, So infinite # of sol^{n}s

3. Let

$$A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 & -5 \\ 3 & x \end{bmatrix}.$$

What are all the real values of x so that $AB = BA$.

(a) $x = -5$

(b) $x = 1$ or $x = -1$.

$\boxed{\text{(c) } x = 5.}$

(d) All real numbers x

(e) No such x exists

$$\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 4 & -5 \\ 3 & x \end{bmatrix} = \begin{bmatrix} 23 & -10+5x \\ -9 & 15+x \end{bmatrix}$$

$$\begin{bmatrix} 4 & -5 \\ 3 & x \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 15 \\ 6-3x & 15+x \end{bmatrix}$$

So $-10+5x = 15 \Rightarrow x=5.$

Since $6-3(5) = -9$, $x=5$ is the only real value that works

4. Compute A if $(A^T - 3I)^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$.

$\Rightarrow A^T - 3I = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}^{-1}$

(a) $\frac{1}{4}\begin{bmatrix} 12 & -2 \\ -2 & 7 \end{bmatrix}$ $\boxed{\text{(b) } \frac{1}{3}\begin{bmatrix} 8 & 2 \\ 1 & 10 \end{bmatrix}}$ (c) $\frac{1}{4}\begin{bmatrix} 10 & -2 \\ -2 & 6 \end{bmatrix}$

(d) $\frac{1}{3}\begin{bmatrix} 5 & -2 \\ -1 & 7 \end{bmatrix}$ (e) $\frac{1}{4}\begin{bmatrix} 4 & -2 \\ -1 & 8 \end{bmatrix}$

Now $\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}^{-1} = \frac{1}{(-3)}\begin{bmatrix} 1 & -1 \\ -2 & -1 \end{bmatrix}$

So $A^T - 3I = \begin{bmatrix} -1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} -1/3 & 1/3 \\ 2/3 & 1/3 \end{bmatrix} + 3I$

$A^T = \begin{bmatrix} 8/3 & 1/3 \\ 2/3 & 10/3 \end{bmatrix} \Rightarrow A = \frac{1}{3}\begin{bmatrix} 8 & 2 \\ 1 & 10 \end{bmatrix}$

5. A matrix A is said to be **involutory** if $A = A^{-1}$. Which of the following matrices are involutory:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(a) Only B.

(b) Only D.

(c) A and B.

(d) Only C.

(e) None of them.

Note: If $A = A^{-1} \Rightarrow A \cdot A = I_2$.

So $\det(A) = \pm 1$

So only possibility is B since all other matrices have determinant $\neq \pm 1$

6. For what value of k is the matrix A **not** invertible, where

$$A = \begin{bmatrix} k & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

[Hint: What is $\det(A)$?]

(a) -2 $\boxed{\text{(b)} -1}$ (c)1 (d)2 (e) 3

By the hint, $\det(A) = k \begin{vmatrix} 2 & 0 & 0 \\ 2 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} + 1 \begin{vmatrix} 2 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}$

$= k[4-6] + [4-6] = -2k - 2.$

So ~~de~~ A not invertible $\Leftrightarrow \det(A) = 0 \Leftrightarrow \boxed{k = -1}$

7. If A, B, C and D are invertible matrices of the same size and

$$(A^TB)^{-1}CA^{-1} = D$$

which of the following must be C?

(a) $\boxed{A^TBDA}$

(b) BA^TDA

(c) $(A^T)^{-1}BDA$

(d) $A^TB^{-1}DA$

(e) $A^TBD^{-1}A$

$(A^TB)^{-1}CA^{-1} = D \Rightarrow B^{-1}(A^T)^{-1}CA^{-1} = D$

$\Rightarrow (A^T)^{-1}C = BDA$

$\Rightarrow C = A^TBDA$

8. Which one of the following statements is not equivalent to the others?

(a) A is invertible.

(b) $A\mathbf{x} = \mathbf{0}$ has a unique solution.

(c) The reduced row echelon form of A is I_n.

(d) $\lambda = 0$ is an eigenvalue of A.

(e) $\det(A) \neq 0$.

All the other statements imply A is invertible

9. The **rank** of a matrix A is the number of leading 1's in the reduced row echelon form of A. What is the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 4 & -2a & 2 \\ 1 & 4a+2 & -2a^2-1 & 2a \end{bmatrix}.$$

(a) 0

(b) 1

(c) 2

(d) 3

(e) Not enough information; answer will depend upon the value of a.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 4 & -2a & 2 \\ 1 & 4a+2 & -2a^2-1 & 2a \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 4 & -2a & 2 \\ 0 & 4a & -2a^2 & 2a \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 4 & -2a & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Regardless of the value of a, $\operatorname{rank}(A) = 2$.

10. Let $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T_A(\mathbf{e}_2) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad T_A(\mathbf{e}_3) = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \quad \text{and} \quad T_A(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

where $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are the standard basis vectors. What is the standard matrix of this linear transformation?

(a) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 2 \\ 1 & 3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 1 & 3 \\ 4 & 2 & 2 \\ 4 & 3 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$ (e) $\begin{bmatrix} -3 & 1 & 3 \\ -3 & 2 & 2 \\ -3 & 3 & 1 \end{bmatrix}$

The matrix is $A = [\,T(\vec{e}_1)\ \ T(\vec{e}_2)\ \ T(\vec{e}_3)\,]$.

To find $T(\vec{e}_1)$, we have $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = T_A(\vec{e}_1 + \vec{e}_2 + \vec{e}_3) = T_A(\vec{e}_1) + T_A(\vec{e}_2) + T_A(\vec{e}_3)$

$= T_A(\vec{e}_1) + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$

So $T_A(\vec{e}_1) = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix}$. So sol^n is (e)

11. Given the matrices

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} d & 2a & g-3d \\ e & 2b & h-3e \\ f & 2c & i-3f \end{bmatrix}$$

and the fact that $\det(A) = 2$, what is $\det(B)$?

(a) -4 (b) -2 (c) $-\frac{1}{2}$ (d) 2 (e) 4

we need to use row operations to change A to B

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{\text{row 1} \times 2} \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ g & h & i \end{bmatrix} \xrightarrow{\text{row 2}\times(-3) + \text{row 3}} \begin{bmatrix} 2a & 2b & 2c \\ d & e & f \\ g-3d & h-3e & i-3f \end{bmatrix} \xrightarrow{\text{swap row 1 and 2}} \begin{bmatrix} d & e & f \\ 2a & 2b & 2c \\ g-3d & h-3e & i-3f \end{bmatrix}$$

$\xrightarrow{\text{transpose}} B$

So, keeping track of row op, we have

$$\det(B) = -2\det(A) = -2(2) = \boxed{-4}$$

12. Let A be a 2018×2018 lower triangular matrix. All the diagonal entries of A are $\frac{2}{i}$. What is the determinant of A?

(a) $\frac{1}{2}$ (b) $\frac{2^{2018}}{i}$ (c) $2^{2018}i$ (d) -2^{2018} (e) $-2^{2018}i$

$$A = 2018\left\{\underbrace{\begin{bmatrix} 2/i & & 0 \\ & \ddots & \\ * & & 2/i \end{bmatrix}}_{2018}\right.$$

Because A is triangular, $\det(A) = \left(2/i\right)^{2018} = 2^{2018}\left(1/i\right)^{2018}$

Now $1/i = -i$. So $(-i)^{2018} = (-i)^{4\cdot 504}(-i)^2 = 1(i^2) = -1$

So $\det(A) = -2^{2018}$

13. The following two commands are entered into Matlab:

```
A = [4 2 1 0 5; 0 1 2 1 0; 0 0 -2 0 10; 0 0 0 3 1; 0 0 0 0 1]
det(A)
```

What is the output?

$\boxed{\text{(a) } -24}$ (b) −7 (c) 0 (d) 7 (e) 50

$$A = \begin{bmatrix} 4 & 2 & 1 & 0 & 5 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 0 & 10 \\ 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad \det(A) = 4(1)(-2)(3)(1) = \boxed{-24}$$

14. The following two commands are entered into Matlab:

```
A = [0 1 1; 1 0 1; 1 1 0]
e = eig(A)
```

returns eigenvalues of $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

What is the output?

(a) a column vector containing -1.0000, -1.0000, -2.0000

(b) a column vector containing 1.0000, 1.0000, 2.0000

(c) a column vector containing 1.0000, -1.0000, -2.0000

$\boxed{\text{(d) a column vector containing -1.0000, -1.0000, 2.0000}}$

(e) a column vector containing 2018, 2018, 2018

$$\lambda I_3 - A = \begin{bmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{bmatrix} \qquad \det(\lambda I_3 - A) = \lambda^3 - 1 - 1 - \lambda - \lambda - \lambda = \lambda^3 - 3\lambda - 2$$

Note that 1 is not a root, so answer not b
Note that −2 " " " " " " a or c
2018 not a root. So answer is d

15. Which of the following vectors

$$1. \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \quad 2. \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad 3. \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

are eigenvectors for the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}.$$

(a) All of them (b) Only 1. (c) Only 1 and 2 (d) Only 2 and 3
$\boxed{\text{(e) Only 1 and 3}}$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \\ 10 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \leftarrow \text{eigenvector}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 7 \end{bmatrix} \leftarrow \text{not an eigenvector}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \leftarrow \text{eigenvector}$$

16. For a 3×3 matrix A, you are given the following information:

$$A\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 2 \end{bmatrix}, \quad A\begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ -8 \end{bmatrix}, \quad A\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}$$

What diagonal matrix is A similar to?

(a) $\boxed{\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$ (b) $\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(d) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ (e) Not enough information given

This info tells you $\lambda = 2$, $\lambda = -2$, $\lambda = 3$ are eigenvalues. These are the diagonal entries

17. Let

$$A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

What is the eigenvalue associated to the eigenvector $\mathbf{x} = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}$?

(a) 2 $\boxed{\text{(b) } 2+i}$ (c) $2-i$ (d) i (e) $4+i$

$$\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} -1+i \\ 1 \end{bmatrix} = \begin{bmatrix} -1+i-2 \\ -1+i+3 \end{bmatrix} = \begin{bmatrix} -3+i \\ 2+i \end{bmatrix}$$

Eigenvalue is $2+i$

18. Given $z_1 = 4\left(\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right)$ and $z_2 = 2\left(\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)\right)$ compute $\frac{z_1}{z_2}$.

$\boxed{\text{(a) } 2(\cos\left(\frac{2\pi}{9}\right) + i\sin\left(\frac{2\pi}{9}\right))}$

(b) $4(\cos\left(\frac{4\pi}{6}\right) + i\sin\left(\frac{4\pi}{6}\right))$

(c) $\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

(d) $\cos(\pi) + i\sin(\pi)$

(e) $\cos\left(\frac{\pi}{9}\right) + i\sin\left(\frac{\pi}{9}\right)$

$$z_1/z_2 = 4/2\left(\cos(\pi/3 - \pi/9) + i\sin(\pi/3 - \pi/9)\right)$$
$$= 2\left(\cos\left(2/9\,\pi\right) + i\sin\left(\tfrac{2}{9}\pi\right)\right)$$

19. On the continent of Pangea, Tyrannosaurus Rexes and Velociraptors battle for resources. Suppose that their populations are modeled by the differential equations

$$\begin{aligned} y_1' &= 4y_1 - 5y_2 \\ y_2' &= -2y_1 + y_2 \end{aligned}$$

where $y_1 = y_1(t)$ is the size of the T. Rex population at time t and $y_2 = y_2(t)$ is the Velociraptor population at time t. If one of the solutions is

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = 3 \begin{bmatrix} -5 \\ 2 \end{bmatrix} e^{at} + 2018 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{bt},$$

then what is the value of a and b?

(a) $a = -1$ and $b = 6$

(b) $a = -6$ and $b = 1$

(c) $a = -6$ and $b = -1$

(d) $a = 6$ and $b = 1$

(e) $a = 6$ and $b = -1$

The a and b are eigenvalues of $\begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}$

$$\det\left(\begin{bmatrix} \lambda-4 & 5 \\ 2 & \lambda-1 \end{bmatrix}\right) = (\lambda-4)(\lambda-1)-10 = \lambda^2-5\lambda+4-10$$
$$= \lambda^2-5\lambda-6 = (\lambda-6)(\lambda+1)$$

So $\lambda = 6$ and $\lambda = -1$. Now we need to check which is a and which is b. The $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ is an eigenvector associated to a.

$$\begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ 2 \end{bmatrix} = \begin{bmatrix} -30 \\ 12 \end{bmatrix} = 6\begin{bmatrix} -5 \\ 2 \end{bmatrix} \leftarrow \text{eigenvector of } \lambda = 6$$

So $\boxed{a=6 \text{ and } b=-1}$

20. Let θ be the angle between $(1,0,2,1)$ and $(1,2,0,1)$ in $\mathbb{R}^4$. What is $\cos(\theta)$?

(a) $\frac{1}{2}$ $\boxed{\text{(b) } \frac{1}{3}}$ (c) 2 (d) 0 (e) None of the above.

Recall $\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right) \Rightarrow \cos(\theta) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{1+0+0+1}{\sqrt{1^2+2^2+1^2}\sqrt{1^2+2^2+1^2}} = \frac{2}{6}$

$\boxed{=\frac{1}{3}}$

21. Suppose $\mathbf{u} = (1, u_2)$ and $\mathbf{v} = (v_1, 4)$ are two vectors in $\mathbb{R}^2$. If $2\mathbf{u} - 3\mathbf{v} = (-7, -8)$, what are the values of u_2 and v_1?

(a) $u_2 = 2018$ and $v_1 = 2018$

(b) $u_2 = 4$ and $v_1 = 2$

$\boxed{\text{(c) } u_2 = 2 \text{ and } v_1 = 3}$

(d) $u_2 = -2$ and $v_1 = 1$

(e) Not enough information.

Want to solve

$2\begin{bmatrix}1\\u_2\end{bmatrix} - 3\begin{bmatrix}v_1\\4\end{bmatrix} = \begin{bmatrix}-7\\-8\end{bmatrix}$

So $2-3v_1=-7 \Rightarrow v_1 = 3$

$2u_2-12=-8 \Rightarrow u_2 = 2$

22. Let $\mathbf{u} = (1,1,1)$ and $\mathbf{a} = (1,2,2)$. What is $\text{proj}_{\mathbf{a}}\mathbf{u}$?

(a) $(0,0,0)$ (b) $\frac{5}{9}(1,1,1)$ $\boxed{\text{(c) } \frac{5}{9}(1,2,2)}$ (d) $\frac{5}{3}(1,1,1)$ (e) $\frac{5}{3}(1,2,2)$

$\text{proj}_{\vec{a}}\vec{u} = \frac{\vec{u}\cdot\vec{a}}{\|\vec{a}\|^2}\vec{a} = \left(\frac{1+2+2}{1^2+2^2+2^2}\right)(1,2,2) = \boxed{\frac{5}{9}(1,2,2)}$

23. What is the area of the triangle in $\mathbb{R}^2$ with vertices at $(2,0)$, $(3,4)$, and $(-1,2)$?

(a) 2 (b) 5 (c) 4 (d) 14 $\boxed{\text{(e) 7}}$

Translate so $(2,0)$ is at the origin: $(2,0) \to (0,0)$
$(3,4) \to (1,4)$
$(-1,2) \to (-3,2)$

Then area of triangle $= \frac{1}{2}\left|\det\begin{bmatrix} 1 & 4 \\ -3 & 2 \end{bmatrix}\right| = \frac{1}{2}|14| = \boxed{7}$

24. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors of $\mathbb{R}^n$, and consider the following three expressions:

$$(i) \quad (2018\mathbf{u}) \cdot (\mathbf{v} \cdot \mathbf{w}) \quad (ii) \quad (\mathbf{u} \cdot \mathbf{v}) - 2018 \quad (iii) \quad 2018\|\mathbf{v} \times \mathbf{u}\|.$$

Which of these expressions makes sense mathematically?

(a) (i) and (ii) only.
(b) (ii) only.
$\boxed{\text{(c) } (ii) \text{ and } (iii) \text{ only.}}$
(d) (iii) only.
(e) All of them.

$(2018\vec{u}).(\vec{v}\cdot\vec{w})$ ← not defined
$\underbrace{2018\vec{u}}_{\text{vector}}$ $\underbrace{\vec{v}\cdot\vec{w}}_{\text{number}}$

$(\vec{u}\cdot\vec{v}) - 2018$ ← defined
$\underbrace{\vec{u}\cdot\vec{v}}_{\text{number}}$ $\underbrace{2018}_{\text{number}}$

$2018\|\vec{u}\times\vec{v}\|$ ← defined
$\underbrace{\|\vec{u}\times\vec{v}\|}_{\text{number}}$

25. Let $L_{2,2}$ be the set of all 2×2 invertible matrices with the usual scalar multiplication, but with a new vector addition operation given by:

$$\text{if } \mathbf{u} = A \in L_{2,2} \text{ and } \mathbf{v} = B \in L_{2,2}, \text{ then } \mathbf{u}+\mathbf{v} = AB.$$

(That is, our "addition" in $L_{2,2}$ is matrix multiplication.) Recall the following axiom of a vector space V:

> **Axiom 4.** There is an object $\mathbf{0}$ in V, called the zero vector for V, such that $\mathbf{0}+\mathbf{u} = \mathbf{u}+\mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in V.

What element of $L_{2,2}$ is the zero vector with respect to the operation of addition as defined above?

(a) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\boxed{\text{(b) } I_2}$ (c) A^{-1} (d) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (e) There is no such element.

Since $AI_2 = I_2A = A$

26. Consider the following set

(x)(*)

$$W = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \;\middle|\; a, b, d \in \mathbb{R} \right\}.$$

That is, W is a subset of the vector space $M_{2,2}$ (all 2×2 matrices) consisting of all the upper triangular matrices. Which statement is true about W?

$\boxed{\text{(a) } W \text{ is a subspace of } M_{2,2}.}$

(b) W is closed under addition, but not closed under scalar multiplication.

(c) W is closed under scalar multiplication but not closed under addition.

(d) W is not closed under scalar multiplication and not closed under addition.

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

2. If $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} \in W$, then $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + \begin{bmatrix} d & e \\ 0 & f \end{bmatrix} = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix} \in W$

3. If $k \in \mathbb{R}$, $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \in W$, $k\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} = \begin{bmatrix} ka & kb \\ 0 & kc \end{bmatrix} \in W$

27. Consider the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 4 \\ -2 \\ 6 \end{bmatrix}.$$

Which of the following statements is true about $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$?

(a) S is linearly independent and $\text{span}(S) = \mathbb{R}^3$.

(b) S is linearly independent and $\text{span}(S) \neq \mathbb{R}^3$.

(c) S is not linearly independent and $\text{span}(S) = \mathbb{R}^3$.

(d) S is not linearly independent and $\text{span}(S) \neq \mathbb{R}^3$.

Note that if $A = [\vec{u}_1, \vec{u}_2\ \vec{u}_3] = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 0 & -2 \\ 0 & 1 & 6 \end{bmatrix}$,

then $\det A = 0 + 0 + 4 - 0 + 2 - 6 = 0$. So $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are not linearly independent.

They cannot span $\mathbb{R}^3$ since any spanning set must contain at least 3 linearly independents since $\dim \mathbb{R}^3 = 3$.

So sol^n is (d)

28. Let

$$\mathbf{p}_1(t) = 1 + x \quad \mathbf{p}_2(t) = 1 + x^2 \quad \mathbf{p}_3(t) = x + x^2.$$

The set $\mathcal{B} = \{\mathbf{p}_1(t), \mathbf{p}_2(t), \mathbf{p}_3(t)\}$ is a basis for the vector space $\mathbb{P}_2$, the set of all polynomials of degree at most two.

Let $\mathbf{p}(t) = 2 - x + x^2$. Find the coordinates of $\mathbf{p}(t)$ with respect to the basis $\mathcal{B}$.

(a) $(1, -1, 2)$ $\boxed{\text{(b) } (0, 2, -1)}$ (c) $(0, 2, 2)$ (d) $(2, -1, 0)$ (e) $(-1, 1, 3)$

Want k_1, k_2, k_3 such that

$$k_1(1+x) + k_2(1+x^2) + k_3(x+x^2) = 2 - x + x^2$$

$$\Leftrightarrow \begin{array}{l} k_1 + k_2 = 2 \\ k_1 + k_3 = -1 \\ k_2 + k_3 = 1 \end{array} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{array}\right]$$

$$\Leftrightarrow (k_1, k_2, k_3) = (0, 2, -1)$$

29. Let $V = \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a subspace of $\mathbb{R}^4$ where

$$\mathbf{v}_1 = \begin{bmatrix}1\\2\\2\\0\end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix}3\\1\\2\\0\end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix}-2\\1\\0\\1\end{bmatrix}.$$

The Gram-Schmidt process can find an orthogonal basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ of V. What is $\mathbf{u}_3$?

(a) $\begin{bmatrix}2\\-1\\0\\0\end{bmatrix}$ (b) $\begin{bmatrix}3\\1\\2\\0\end{bmatrix}$ (c) $\begin{bmatrix}2018\\2018\\2018\\2018\end{bmatrix}$ (d) $\begin{bmatrix}2\\-1\\0\\1\end{bmatrix}$ $\boxed{(e) \begin{bmatrix}0\\0\\0\\1\end{bmatrix}}$

$\vec{u}_1 = \vec{v}_1$

$$\vec{u}_2 = \vec{v}_2 - \frac{\vec{u}_1 \cdot \vec{v}_2}{\|\vec{u}_1\|^2} u_1 = \begin{bmatrix}3\\1\\2\\0\end{bmatrix} - \frac{(3+2+4)}{1^2+2^2+2^2}\begin{bmatrix}1\\2\\2\\0\end{bmatrix}$$

$$= \begin{bmatrix}3\\1\\2\\0\end{bmatrix} - \begin{bmatrix}1\\2\\2\\0\end{bmatrix} = \begin{bmatrix}2\\-1\\0\\0\end{bmatrix}$$

$$\vec{u}_3 = \vec{v}_3 - \frac{\vec{u}_1 \vec{v}_3}{\|u_1\|^2}\vec{u}_1 - \frac{\vec{u}_2 \cdot v_3}{\|\vec{u}_2\|^2}\vec{u}_2$$

$$= \begin{bmatrix}-2\\1\\0\\1\end{bmatrix} - 0 - \frac{(-5)}{[2^2+(-1)^2]}\begin{bmatrix}2\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}-2\\1\\0\\1\end{bmatrix} + \begin{bmatrix}2\\-1\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

30. A $n \times n$ matrix A is called **orthogonal** if $A^{-1} = A^T$. Consider the following statements about orthogonal matrices:

(1) The n column vectors of A are all orthogonal with each other.

(2) For every orthogonal matrix A, $\det(A) = 1$ or -1.

(3) For any two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, $\mathbf{u} \cdot \mathbf{v} = (A\mathbf{u}) \cdot (A\mathbf{v})$.

Which statements are true? [Hint: Note that $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T\mathbf{v}$ if we view $\mathbf{u}$ and $\mathbf{v}$ as column vectors.]

(a) (1) and (2) only

(b) (1) and (3) only

(c) (2) and (3) only

$\boxed{\text{(d) All are true.}}$

(e) None are true.

Write A as $A = [\vec{c}_1\ \vec{c}_2 \cdots \vec{c}_n]$

Then $AA^{-1} = AA^T = \cancel{I_n}\ A^TA = I_n$

So $\begin{bmatrix} \vec{c}_1^{\,T} \\ \vec{c}_2^{\,T} \\ \vdots \\ \vec{c}_n^{\,T} \end{bmatrix} [\vec{c}_1 \cdots \vec{c}_n] = I_n$ This means $\vec{c}_i^{\,T} c_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

But $\vec{c}_i^{\,T} c_j = 0 \iff \vec{c}_i \cdot \vec{c}_j = 0 \iff$ columns are orthogonal.

So 1 is true

For 2, note that $A^TA = I_n \Rightarrow 1 = \det(I_n) = \det(A^TA)$
$= \det(A^T)\det(A)$
$= \det(A)^2$.

So $\det(A)^2 = 1 \Rightarrow \det(A) = \pm 1$. So 2 true

For 3, $(A\vec{u}) \cdot (A\vec{v}) = (A\vec{u})^T A\vec{v} = \vec{u}^T A^T A \vec{v} = \vec{u}^T \vec{v} = \vec{u} \cdot \vec{v}$

since $A^TA = I$

So true

31. The matrix A is row equivalent to the matrix B:

$$A = \begin{bmatrix} 1 & 2 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ -1 & 0 & -3 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

x_3, x_4 free

What is a basis for Nul(A)?

$x_1 = -3x_3 + 2x_4$

$x_2 = 2x_3 - 3x_4.$

(a) $\boxed{\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \\ -0 \end{bmatrix}}$ (b) $\begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \end{bmatrix}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

So sol^n

(d) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3r + 2s \\ 2r - 3s \\ r \\ s \end{bmatrix}$$

So basis $\begin{bmatrix} -3 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

32. Using A and B as in the Question 31, what is a basis for Row(A), the column space.

(a) $\begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 1 \\ -0 \end{bmatrix}$ (b) $\boxed{\begin{bmatrix} 1 \\ 0 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 3 \end{bmatrix}}$ (c) $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (e) $\begin{bmatrix} 1 \\ 2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 2 \end{bmatrix}$

We need the rows of A that have a leading 1:

33. Let $\mathbf{u}$ and $\mathbf{v}$ be orthogonal vectors in $\mathbb{R}^3$. Which of the following statements are true:

    (1) $\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\} = \mathbb{R}^3$.

    (2) The set $\{\mathbf{u}, \mathbf{v}, \mathbf{u} \times \mathbf{v}\}$ is a linearly independent set of vectors in $\mathbb{R}^3$.

    Both true

    (a) (1) is false and (2) is false.

    (b) (1) is true and (2) is false.

    (c) (1) is false and (2) is true.

    (d) (1) is true and (2) is true.

    Since $\vec{u} \times \vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$, the three vectors are linearly independent. But any three linearly independent vectors in $\mathbb{R}^3$ form a basis, so they span $\mathbb{R}^3$

34. Which of the following statements are true?

    (1) If W is a subset of a vector space V such that the zero vector $\mathbf{0}$ belongs to W, then W is a subspace of V. FALSE Need other two criteria to be true

    (2) For any vector space V, the set $\{\mathbf{0}\}$ is a subspace of V. TRUE

    (a) (1) is false and (2) is false.

    (b) (1) is true and (2) is false.

    (c) (1) is false and (2) is true.

    (d) (1) is true and (2) is true.

35. Which of the following statements are true?

    (1) Any basis for $\mathbb{R}^{2018}$ contains exactly 2018 vectors. TRUE

    (2) If $\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_t\} = V$, then $\{\mathbf{v}_1, \ldots, \mathbf{v}_t\}$ are linearly independent in V.

    FALSE. The vectors in the span need not be linearly independent

    (a) (1) is false and (2) is false.

    (b) (1) is true and (2) is false.

    (c) (1) is false and (2) is true.

    (d) (1) is true and (2) is true.

36. Which of the following statements are true?

(1) If $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is an orthonormal basis for a vector space V, then $||\mathbf{u}_i|| = 1$ for all $i = 1, \ldots, n$. TRUE

(2) Every orthogonal basis of a vector space V is also an orthonormal basis for V.

FALSE (the vectors in an orthogonal basis need not have length 1)

(a) (1) is false and (2) is false.

(b) (1) is true and (2) is false.

(c) (1) is false and (2) is true.

(d) (1) is true and (2) is true.

37. Which of the following statements are true?

(1) If λ an eigenvalue of a 4×4 matrix A, then the geometric multiplicity of λ equals $(4 - \text{rank}(\lambda I_4 - A))$. TRUE

(2) If A is an 2018×2018 matrix such that $A\mathbf{x} = \mathbf{0}$ has exactly one solution, then $\text{rank}(A) = 2018$.

geo mult of $\lambda =$ dim Nul$(\lambda I_4 - A)$

But

dim Nul$(\lambda I_4 - A) = 4 -$ rank$(\lambda I_4 - A)$

TRUE. $A\vec{x} = \vec{0} \Rightarrow$ Null$(A) = 0$

So rank(A) + Null$(A) = 2018 \Rightarrow$ rank$(A) = 2018$

(a) (1) is false and (2) is false.

(b) (1) is true and (2) is false.

(c) (1) is false and (2) is true.

(d) (1) is true and (2) is true.

38. Who was your favourite Math 1B03 instructor?

(a) Adam

(b) Adam Van Tuyl

(c) Dr. Adam

(d) Professor Van Tuyl

(e) All of the above **[This is the correct answer!]**

Any answer is okay

END OF TEST PAPER