MATH 1B03: Midterm 2 - VERSION 1
Instructor: Adam Van Tuyl
Date: November 7, 7:30PM
Duration: 75 min.

Name: SOLUTIONS ID #: ______________

Instructions:

This test paper contains **20** multiple choice questions printed on both sides of the page. The questions are on pages 2 through 10. Scrap paper is available for rough work. YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE PAPER IS COMPLETE. BRING ANY DISCREPANCIES TO THE ATTENTION OF THE INVIGILATOR.

Select the one correct answer to each question and ENTER THAT ANSWER INTO THE SCAN CARD PROVIDED USING AN HB PENCIL. You are required to submit this booklet along with your answer sheet. HOWEVER, NO MARKS WILL BE GIVEN FOR THE WORK IN THIS BOOKLET. Only the answers on the scan card count for credit. Each question is worth 1 mark. The test is graded out of 20. There is no penalty for incorrect answers.

NO CALCULATORS ALLOWED.

Computer Card Instructions:

IT IS YOUR RESPONSIBILITY TO ENSURE THAT THE ANSWER SHEET IS PROPERLY COMPLETED. YOUR TEST RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.

The scanner that will read the answer sheets senses areas by their non-reflection of light. A heavy mark must be made, completely filling the circular bubble, with an HB pencil. Marks made with a pen or felt-tip marker will **NOT** be sensed. Erasures must be thorough or the scanner may still sense a mark. Do **NOT** use correction fluid.

- Print your name, student number, course name, and the date in the space provided at the top of Side 1 (red side) of the form. Then the sheet **MUST** be signed in the space marked SIGNATURE.
- Mark your student number in the space provided on the sheet on Side 1 **and fill the corresponding bubbles underneath**. Your student number **MUST** be 9 digits long. If you have a student number that is 7 digits, begin your student number with 00 (two zeroes).
- Mark only **ONE** choice (A, B, C, D, E) for each question.
- Begin answering questions using the first set of bubbles, marked "1".

1. Find the determinant of the following matrix:

$$A = \begin{bmatrix} 5 & -7 & 2 & 2 \\ 0 & 3 & 0 & -4 \\ -5 & -8 & 0 & 3 \\ 0 & 5 & 0 & -6 \end{bmatrix}.$$

(a) -20

(b) -5

(c) 5

(d) $\boxed{20}$

(e) 2018

Do cofactor factor expansion down 3rd column

$$\det(A) = (-1)^{1+3} \cdot 2 \begin{vmatrix} 0 & 3 & -4 \\ -5 & -8 & 3 \\ 0 & 5 & -6 \end{vmatrix}$$ do cofactor expansion down 1st column

$$= 2 \cdot \left((-1)^{2+1} \cdot (-5) \begin{vmatrix} 3 & -4 \\ 5 & -6 \end{vmatrix} \right)$$

$$= 2((-1)(-5)(-18+20))$$

$$= 2 \cdot 5(2) = \boxed{20}$$

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2. Suppose that A is an $n \times n$ matrix such that $A^2 = I_n$. Which statement is true about A?

(a) $\lambda = 0$ is an eigenvalue of A.

(b) $\det(A^2) = 2\det(A)$.

(c) $\boxed{\det(A) = \pm 1.}$

(d) A is not invertible.

(e) Not enough information provided.

Note that $A \cdot A = I_n$, so $A^{-1} = A$. So (a) and (d) cannot be true. Now $1 = \det(I_n) = \det(A^2) = \det(A) \cdot \det(A) = (\det(A))^2$

So $\det(A) = \pm\sqrt{1} = \pm 1$.

(Statement (b) implies $\det(A^2) = \det(A)\det(A) = 2\det(A)$. So $\det(A) = 2$, and thus $\det(A^2) = 4 \neq \det(I_n)$)

3. Consider the matrix

$$A = \begin{bmatrix} 0 & -2 & -1 \\ 5 & 0 & 0 \\ -1 & 1 & 1 \end{bmatrix}.$$

What is $(\mathrm{adj}(A))_{2,3}$, i.e., what is the value in position $(2,3)$ of the adjoint matrix $\mathrm{adj}(A)$?

We need the cofactor

(a) -5

$$C_{3,2} = (-1)^{3+2} \det(A_{3,2}) = (-1)\begin{vmatrix} 0 & -1 \\ 5 & 0 \end{vmatrix} = (-1)(5) = \boxed{-5}$$

(b) 0

(c) 1

(d) 2

(e) 10

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4. Given the matrices

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} a & b & a \\ c & d & c \\ 2 & -1 & 3 \end{bmatrix}$$

and the fact that $\det(A) = 2018$, what is $\det(B)$?

(a) -2018

If we do cofactor expansion along bottom row,

(b) 0

$$\det(B) = 2\begin{vmatrix} b & a \\ d & c \end{vmatrix} - (-1)\begin{vmatrix} a & a \\ c & c \end{vmatrix} + 3\begin{vmatrix} a & b \\ c & d \end{vmatrix}.$$

(c) 4036

(d) 2018

(e) Not enough information provided.

Now $\begin{vmatrix} a & a \\ c & c \end{vmatrix} = 0$, and $\begin{vmatrix} b & a \\ d & c \end{vmatrix} = -\begin{vmatrix} a & b \\ c & d \end{vmatrix} = -2018.$

So

$$\det(B) = 2(-2018) + 3(2018) = 2018(2(-1)+3) = \boxed{2018}$$

5. Suppose that A is an invertible matrix with $\det(A) = 6$, B is an invertible matrix with $\det(B) = 3$, and C is an invertible matrix with $\det(C) = 2$. Furthermore, each matrix has size 2×2. Find the determinant of the matrix:

$$5A^{-1}B^T\mathrm{adj}(C)$$

Because all the matrices are 2x2,

(a) 75

(b) $\frac{5}{2}$

(c) $\frac{5}{6}$

(d) $\frac{25}{2}$

$\boxed{\text{(e) } 25}$

$\det(5A^{-1}B^T\mathrm{adj}(C)) = 5^2\det(A^{-1})\det(B^T)\det(\mathrm{adj}(C))$

Now $\det(A^{-1}) = 1/\det(A) = 1/6$ and $\det(B^T) = \det(B) = 3$.

To find $\det(\mathrm{adj}(C))$, recall $C^{-1} = \frac{1}{\det(C)}\mathrm{adj}(C)$

Since $\mathrm{adj}(C)$ is 2x2, we have $\det(C^{-1}) = \left(\frac{1}{\det(C)}\right)^2 \det(\mathrm{adj}(C))$.

So $\det(\mathrm{adj}(C)) = [\det(C)]^2\det(C^{-1}) = \det(C)$.

Thus $\det(5A^{-1}B^T\mathrm{adj}(C)) = 5^2 \cdot \frac{1}{6} \cdot 3 \cdot 2 = \boxed{25}$

6. Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}.$$

(a) -3 and -7

$\boxed{\text{(b) } 3 \text{ and } -7}$

(c) -3 and 7

(d) 3 and 7

(e) 3 (with multiplicity two)

$$\begin{vmatrix} \lambda-2 & -3 \\ -3 & \lambda+6 \end{vmatrix} = (\lambda-2)(\lambda+6)-9$$
$$= \lambda^2+4\lambda-12-9$$
$$= \lambda^2+4\lambda-21$$
$$= (\lambda+7)(\lambda-3)$$

So, eigenvalues are $\lambda_1 = -7$ and $\lambda_2 = 3$

7. The matrix

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

is similar to the diagonal matrix

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

What is the matrix P that diagonalizes A?

You can simply check if $AP = PD$ for each given matrix. Alternatively, note that i^{th} column of P ~~st~~ is an eigenvector of A with respect to i^{th} diagonal entry

Alternatively, you can use the fact that $\lambda_1 = -1$ (mult 2) and $\lambda_2 = 2$ are eigenvalues to find the eigenvectors to find the matrix P (i.e. construct the sol^{n})

(a) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$

(b) $\boxed{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}}$

(c) $\begin{bmatrix} -1 & -1 & 1 \\ 0 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

(e) None of the above

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -1 & -2 \\ 0 & -1 & 0 \end{bmatrix}$$

SAME

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & -1 & -2 \\ 0 & -1 & 0 \end{bmatrix}$$

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8. Which of the following vectors are eigenvectors of the matrix

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

For each vector given, compute $A\vec{x}$, to determine if the resulting vector is a scalar multiple of original vector.

(a) $\begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (c) $\boxed{\begin{bmatrix} 2018 \\ 2018 \\ 0 \end{bmatrix}}$ (d) $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ (e) All of them

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 2018 \\ 2018 \\ 0 \end{bmatrix} = \begin{bmatrix} 2018 \\ 2018 \\ 0 \end{bmatrix}$$

9. The following commands are inputted into Matlab:

```
a=@(i,j) i*j;
A = zeros(2,4);
for i=1:2
    for j=1:4
        A(i,j)=a(i,j);
    end
end
A
```

Creates a 2x4 matrix when entry a_{ij} is $i \times j$

What is the final output?

(a)
```
1    2    3    4
2    4    6    8
```

(b)
```
1    2
2    4
3    6
4    8
```

(c) 2018

(d)
```
1    2    3    4
2    3    4    5
```

(e)
```
1   -2    3   -4
2    3   -4    5
```

We need to solve

$$\begin{vmatrix} \lambda-a & -4 \\ 3 & \lambda-7 \end{vmatrix} = (\lambda-1)(\lambda-5)$$

So $(\lambda-a)(\lambda-7)+12 = \lambda^2-6\lambda+5$

Hence

$\lambda^2-(7+a)\lambda+7a+12 = \lambda^2-6\lambda+5$

So $7+a=6$ and $7a+12=5$.

We have $\boxed{a=-1}$

10. The matrix

$$\begin{bmatrix} a & 4 \\ -3 & 7 \end{bmatrix}$$

has eigenvalues $\lambda = 1$ and $\lambda = 5$. What is a?

(a) $a = -7$.

(b) $a = 1$.

(c) $a = -1$.

(d) $a = 7$.

(e) Not enough information provided.

Alt sol^n For any 2x2 matrix A, $\text{trace}(A) = \lambda_1+\lambda_2$. So $a+7 = 1+5$. Thus $\boxed{a=-1}$

$1/i = -i$ and $(-i)^4 = 1$

11. What is

$$\left(\frac{1}{i}\right)^{2018}$$

Since $2018 = 4 \cdot 504 + 2$

(a) 0 (b) $-i$ $\boxed{(c)\ -1}$ (d) i (e) 1

$(-i)^{2018} = \left[(-i)^4\right]^{504}(-i)^2 = (1)^{504} \cdot (-i)^2$

$= i^2 = \boxed{-1}$

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12. You are given two complex numbers written in polar coordinate form:

$$z_1 = \sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$$
$$z_2 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)).$$

What is z_1/z_2? (Note: the following numbers are not in polar coordinate form.)

$\boxed{(a)\ 1+i}$ In Polar form

(b) $-1+i$ $z_1/z_2 = \sqrt{2}/1\left(\cos\left(3\pi/4 - \frac{\pi}{2}\right) + i\sin\left(3\pi/4 - \frac{\pi}{2}\right)\right)$

(c) $\sqrt{2}+i$

(d) $\sqrt{2}-i$ $= \sqrt{2}\left(\cos(\pi/4) + i\sin(\pi/4)\right)$

(e) $2+i$ Now $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, So $z_1/z_2 = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\,1/\sqrt{2}\right)$

$= \boxed{1+i}$

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13. The matrix A is a 3×3 matrix such that A is invertible. Which of the following cannot be the characteristic polynomial of A?

(a) $\lambda^3 + 3\lambda^2 + 3\lambda + 4$

(b) $\lambda^3 + 6\lambda^2 + 3\lambda + 3$

(c) $\lambda^3 + 10\lambda^2 + 3\lambda + 2$

(d) $\lambda^3 + 15\lambda^2 + 3\lambda + 1$

$\boxed{(e)\ \lambda^3 + 21\lambda^2 + 3\lambda}$

For this equation, $\lambda = 0$ is a root. So, this cannot be the char eq since A is invertible, so $\lambda = 0$ is not an eigenvalue (or root of the characteristic polynomial)

14. Brontosauruses and tyrannosaurus rexes fight for resources on the continent of Pangaea. Suppose that the population of each species is modeled by the system of differential equations

$$\begin{aligned} y_1' &= 4y_1 - 5y_2 \\ y_2' &= -2y_1 + y_2 \end{aligned}$$

where y_1 is the size of the brontosaurus population and y_2 is the size of the tyrannosaurus rex population at time x. Which equation models the brontosaurus's population? [Below, the c_1 and c_2 are constants that depend upon the initial population sizes.]

$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$. We need to diagonalize this matrix

(a) $y_1 = -5c_1e^{-x} + c_2e^{6x}$

(b) $y_1 = c_1e^{6x}$

(c) $\boxed{y_1 = -5c_1e^{6x} + c_2e^{-x}}$

(d) $y_1 = 2c_1e^{6x} + c_2e^{-x}$

(e) $y_1 = 2018e^{6x} + 2018e^{-x}$

$\begin{vmatrix} \lambda-4 & 5 \\ 2 & \lambda-1 \end{vmatrix} = (\lambda-4)(\lambda-1)-10 = \lambda^2 - 5\lambda - 6 = (\lambda-6)(\lambda+1)$

Find eigenvectors

$\boxed{\lambda=6}$ $\begin{bmatrix} 2 & 5 \\ 2 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & 5 \\ 0 & 0 \end{bmatrix}$ So x_2 free and $2x_1 = -5x_2$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} -5 \\ 2 \end{bmatrix}$

$\boxed{\lambda=-1}$ $\begin{bmatrix} -5 & 5 \\ 2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$

x_2 free $x_1 = x_2$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\therefore$ $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = c_1\begin{bmatrix} -5 \\ 2 \end{bmatrix}e^{6x} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-x}$

is a sol^n to $A\vec{y} = y'$.

We want y_1 which is

$y_1 = -5c_1e^{6x} + c_2e^{-x}$

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15. This question continues the previous question. Suppose that at time $x = 0$, we have $y_1(0) = 6,000$ and $y_2(0) = 20,000$. What is the tyrannosaurus rex population at time $x = 1$?

We need to solve for c_1 and c_2 at $x=0$

(a) 14000

(b) 2018

(c) $2000e^6$

(d) $-10000e^6 + 16000e^{-1}$

(e) $\boxed{4000e^6 + 16000e^{-1}}$

$c_1\begin{bmatrix} -5 \\ 2 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6000 \\ 20000 \end{bmatrix} \Rightarrow c_1 = 2000, \ c_2 = 16000$

Thus $y_2 = 4,000e^{6x} + 16000e^{-x}$

So at $x=1$, we get

$\boxed{4000e^6 + 16000e^{-1}}$

16. Which of the following statements are true?

   (1) If A is an $n \times n$ matrix, then the determinant of A is the product of the entries on the main diagonal of the matrix. FALSE (only true for special matrices)

   (2) If B is the matrix that results when two rows of A are interchanged, then $\det(B) = -\det(A)$. TRUE

   (a) (1) is false and (2) is false.

   (b) (1) is true and (2) is false.

   (c) (1) is false and (2) is true.

   (d) (1) is true and (2) is true.

17. Which of the following statements are true?

   (1) If A is an invertible matrix, then $\det(A)A^{-1} = \operatorname{adj}(A)$. TRUE

   (2) If A is an $n \times n$ matrix, then a nonzero vector $\mathbf{x}$ in $\mathbb{R}^n$ is called an eigenvector if there is a scalar λ such that $A\mathbf{x} = \lambda\mathbf{x}$. TRUE

   (a) (1) is false and (2) is false.

   (b) (1) is true and (2) is false.

   (c) (1) is false and (2) is true.

   (d) (1) is true and (2) is true.

18. Which of the following statements are true?

   (1) If λ is an eigenvalue of the matrix A, then the matrix $(\lambda I - A)$ is invertible. FALSE $(\lambda I - A)\vec{x} = \vec{0}$ has non trivial soln

   (2) If λ is an eigenvalue of the matrix A, then λ is also an eigenvalue of the reduced row echelon form of A. FALSE eigenvalues change if you use row operations

   (a) (1) is false and (2) is false.

   (b) (1) is true and (2) is false.

   (c) (1) is false and (2) is true.

   (d) (1) is true and (2) is true.

19. Which of the following statements are true? FALSE Should be less than or equal

    (1) For every eigenvalue of a matrix A, the geometric multiplicity of the eigenvalue is greater than or equal to the algebraic multiplicity of the eigenvalue.

    (2) If A is an $n \times n$ matrix with n distinct eigenvalues, then A is diagonalizable.

    TRUE

    (a) (1) is false and (2) is false.

    (b) (1) is true and (2) is false.

    (c) (1) is false and (2) is true.

    (d) (1) is true and (2) is true.

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20. Which of the following statements are true?

    different equations can satisfy same D.E.

    (1) If $\mathbf{x}' = A\mathbf{x}$ and $\mathbf{y}' = A\mathbf{y}$, then $\mathbf{x} = \mathbf{y}$. FALSE

    (2) There is a real 3×3 matrix with no real eigenvalues. FALSE

    (a) (1) is false and (2) is false.

    (b) (1) is true and (2) is false.

    (c) (1) is false and (2) is true.

    (d) (1) is true and (2) is true.

The characteristic poly has degree 3 $\lambda^3 + a\lambda^2 + b\lambda + c$ where a, b, c are real. Any cubic polynomial has at least one real root.

END OF TEST PAPER