Lecture 7

Linear independence (Section 1.7)
(will return to Sec. 1.6)

Today's lecture: linear independence (key concept in linear alg.)

Def^n and Examples

Def^n A set of vectors $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly independent if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_p = \vec{0} \quad (*)$$

has only the trivial sol^n $x_1 = x_2 = \cdots = x_n = 0$

If $(*)$ has a non-trivial sol^n (some $x_i \neq 0$), then $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is linearly dependent

Ex Show $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ 2020 \end{bmatrix}$ are linearly independent

Look at soln's to

$$x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2020 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2020 x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Leftrightarrow$ $x_1 = 0 = x_2$ ← only the trivial soln

Ex Show $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$ are linearly dependent

Note $x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 6 \\ 10 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix} 1 & 1 & 6 \\ 0 & 2 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 6 & 0 \\ 0 & 2 & 10 & 0 \end{array}\right] \sim \begin{bmatrix} \textcircled{1} & 1 & 6 & 0 \\ 0 & \textcircled{1} & 5 & 0 \end{bmatrix}$$ ← x_3 is free! ⇒ infinite # of soln's

$$\sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 5 & 0 \end{bmatrix}$$

$x_1 = -x_3$

$x_2 = -5x_3$

So all sol^{n}s have form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix}$

So, a specific non-trivial sol^n when $x_3 = 1$ gives $\begin{bmatrix} -1 \\ -5 \\ 1 \end{bmatrix}$

Key observations Determining if $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is linearly independent is equivalent to determining if the homog. s.l.e.

$$[\vec{v}_1 \cdots \vec{v}_p]\vec{x} = \vec{0} \text{ has only trivial sol}^n$$

Def^n Given a matrix $A = [\vec{a}_1 \cdots \vec{a}_n]$, columns of A are linearly independent if $\{\vec{a}_1, \ldots, \vec{a}_n\}$ is linearly independent

columns of A are linear independent
$\iff [\vec{a}_1 \cdots \vec{a}_n : \vec{0}]$ has only trivial sol^n
$\iff A\vec{x} = \vec{0}$ has only trivial sol^n

Small sets of vectors and linear independence

Thm Let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$

1. If $\vec{0} \in S$, then S is linearly dependent.
2. If $p = 1$, and $\vec{v}_1 \neq \vec{0}$, then S linearly independent.
3. If $p = 2$, $\vec{v}_1, \vec{v}_2 \neq \vec{0}$, and $\vec{v}_2 \neq c \cdot \vec{v}_1$ for any $c \in \mathbb{R}$, then S linearly independent.

Proof (1) Suppose $\vec{0} \in S$, and say $\vec{v}_p = \vec{0}$. Then

$$0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_{p-1} + 1 \cdot \vec{v}_p = \vec{0}$$

is a non-trivial sol^n. So S is linearly dependent.

(2) If $S = \{\vec{v}_1\}$, then $x_1 \vec{v}_1 = \vec{0}$ and $\vec{v}_1 \neq 0$, has only the sol^n $x_1 = 0$ (i.e. trivial sol^n)

(3) (Proof by contradiction)
Suppose there was a non-trivial sol^n
$a\vec{v}_1 + b\vec{v}_2 = \vec{0}$, a and b not both zero

Assume $a \neq 0$. If $b = 0$, then $a\vec{v}_1 = \vec{0}$. This implies $\vec{v}_1 = \vec{0}$, but not allowed!

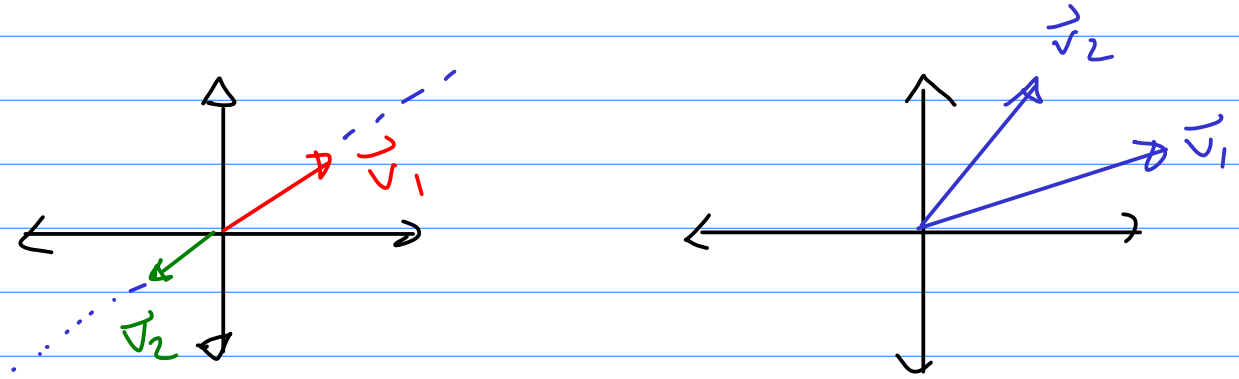
If $b \neq 0$, $a\vec{v}_1 = -b\vec{v}_2 \Rightarrow \vec{v}_1 = \left(\frac{-b}{a}\right)\vec{v}_2$, but not allowed!

So, $\vec{v}_1$ and $\vec{v}_2$ must be linearly independent. $\square$

Geometry

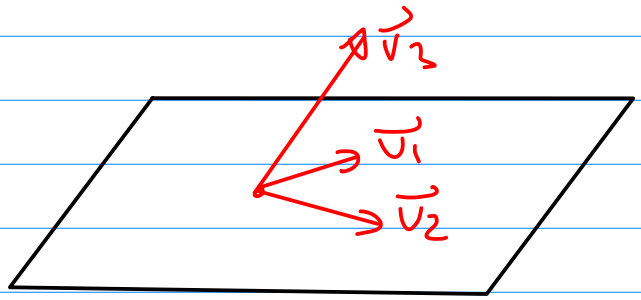
In $\mathbb{R}^2$ or $\mathbb{R}^3$, if $\vec{v}_1, \vec{v}_2$ linearly dependent $\Rightarrow$ on same line

if $\vec{v}_1, \vec{v}_2$ linearly independent $\Rightarrow$ point in different directions



In $\mathbb{R}^3$, $\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly dependent $\Rightarrow$ all in same plane or same line

$\vec{v}_1, \vec{v}_2, \vec{v}_3$ linearly independent $\Rightarrow$ no 3-vectors in same plane



Thm (more than 2 vectors) Let $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ with $\vec{v}_1 \neq \vec{0}$.

Then S is linearly dependent if and only if exists some $j > 1$ such that $v_j \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{j-1}\}$

Proof Note that statement has form "A if and only if B"

Need to show "if A, then B" and "if B, then A".
We will show

"If $v_j \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{j-1}\}$, then S linearly dependent"

If $v_j \in \text{Span}\{\vec{v}_1, \ldots, \vec{v}_{j-1}\}$, then

$$\vec{v}_j = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{j-1}\vec{v}_{j-1} \quad \text{for some } c_1, \ldots, c_{j-1} \neq 0$$

But then

$$\vec{0} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_{j-1}\vec{v}_{j-1} + (-1)\vec{v}_j + 0\vec{v}_{j+1} + \cdots + \vec{0}\cdot\vec{v}_p.$$

A nontrivial soln, so, vectors linearly dependent ∎

Thm If $S = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a set of vectors in $\mathbb{R}^n$ with $p > n$, then S is linearly dependent.

Proof Vector equation $x_1\vec{v}_1 + \cdots + x_p\vec{v}_p = \vec{0}$ equivalent to SLE

$$[\vec{v}_1 \cdots \vec{v}_p]\begin{bmatrix} x_1 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \iff \left\{ \begin{bmatrix} \vec{v}_1 \cdots \vec{v}_p & \vdots & 0 \\ & \vdots & 0 \\ & \vdots & 0 \end{bmatrix} \right. \begin{matrix} \\ n \\ \\ \end{matrix}$$

(p columns)

Since $p > n$, cannot have a pivot in each column

$\Rightarrow$ SLE has a free variable

$\Rightarrow$ SLE has an infinite # of sol^{n}s

$\Rightarrow$ vectors are linearly dependent $\square$

Ex $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}6\\10\end{bmatrix}\right\}$ linear dependent since $p=3>2=n$

CAUTION If $S=\{\vec{v}_1,\ldots,\vec{v}_p\}$ in $\mathbb{R}^n$ is a set of vectors with $p \leq n$, then S may or may not be linearly independent.

Ex $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}2\\2\\2\end{bmatrix}\right\}$ in $\mathbb{R}^3$ is not linearly independent

since $1\cdot\begin{bmatrix}1\\1\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}2\\2\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

Ex $\left\{\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ in $\mathbb{R}^3$ is linearly independent

Key ideas: * def^n of linear independence
* connection to homog s.l.e.
* special cases of linear dependence and linear independence.