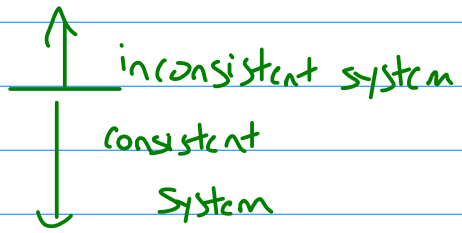
Lecture 2 Gaussian Elimination I

(see Section 1.2)

Last time: A s.l.e. may have:

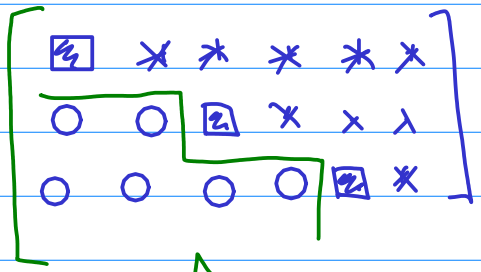
1. no sol^n's
2. exactly one sol^n
3. infinite number of sol^n's



This lecture: procedure to solve s.l.e = Gaussian elimination

(connection to s.l.e. next lecture)

Basic idea: manipulate the augmented matrix so has form:



■ = nonzero element

* = any number

"staircase of zeros"

Def^n A rectangular matrix is in echelon form if

1. all nonzero rows are above any rows of all zeroes
2. each leading entry (left most nonzero entry) is in a column to the right of the leading entry in the row above.
3. all entries in a column below leading entry are zero.

Ex

$$\begin{bmatrix} \underline{1} & 2 & -1 & -3 \\ 0 & \underline{6} & -6 & 3 \\ 0 & 0 & 0 & \underline{15} \end{bmatrix}$$

echelon form
not row echelon form

The matrix is in row echelon form if

4. each leading entry is a 1

Ex $\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ row echelon form

not reduced row echelon form

The matrix is in reduced row echelon form if

5. each leading 1 is the only nonzero entry in its column.

Ex $\begin{bmatrix} \underline{1} & 0 & 1 & 0 \\ 0 & \underline{1} & -1 & 0 \\ 0 & 0 & 0 & \underline{1} \end{bmatrix}$ reduced row echelon form

Gaussian elimination: procedure to put matrix into (reduced row) echelon form using 3 rules:

- multiple a row by a nonzero constant
- interchange two rows
- add a constant multiple of one row to another

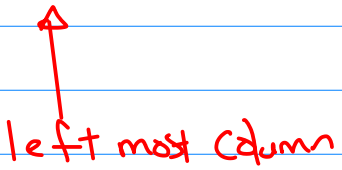
} called "elementary row operations"

Example of Procedure

Step 0 Start with a matrix

$$\begin{bmatrix} 0 & 2 & -2 & 1 \\ 1 & 2 & -1 & -3 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

Step 1 Begin with leftmost column



Step 2 Interchange top row (if necessary) so a nonzero entry is at top of its column

$$\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \\ 1 & -1 & 2 & 3 \end{bmatrix} \leftarrow \text{swap}$$

Step 3 use elementary row operations to create zeroes in rest of the column

$$\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \\ 0 & -3 & 3 & 6 \end{bmatrix} \leftarrow \text{row}(1) \times (-1) + \text{row}\,3$$

SCRAP

row 1 $\times(-1)$: $-1 \quad -2 \quad 1 \quad 3$

$+$ row 3 : $1 \quad -1 \quad 2 \quad 3$

$0 \quad -3 \quad 3 \quad 6$

Step 4 once all entries below leading entry are 0, ignore top row. Repeat procedure on remaining submatrix

$$\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 2 & -2 & 1 \\ 0 & -3 & 3 & 6 \end{bmatrix} \begin{matrix} \\ \to \text{row}2 \times 3 \\ \to \text{row}3 \times 2 \end{matrix} \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \\ 0 & -6 & 6 & 12 \end{bmatrix} \xrightarrow[\text{row}3]{\text{row}2+} \begin{bmatrix} \underline{1} & 2 & -1 & -3 \\ 0 & \underline{6} & -6 & 3 \\ 0 & 0 & 0 & \underline{15} \end{bmatrix}$$

Stop when you reach echelon form

Step 5 If you need row echelon form, multiply each row by $1/a$ where a is the leading entry

$$\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{row } 2 \times 1/6 \\ \leftarrow \text{row } 3 \times 1/15 \end{matrix}$$

Step 6 If you want reduced row echelon form, begin with right most leading 1 & create 0s in the columns above it. Then move left.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow \text{row } 3 \times (3) + \text{row } 1 \\ \leftarrow \text{row } 3 \times (-\tfrac{1}{2}) + \text{row } 2 \\ \\ \end{matrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \leftarrow \text{row } 2 \times (-2) \\ \quad + \text{row } 1 \\ \\ \end{matrix}$$

r.r.e.f.

Def^ Two matrices A and B are **row equivalent** if B can be obtained from A via elementary row operations

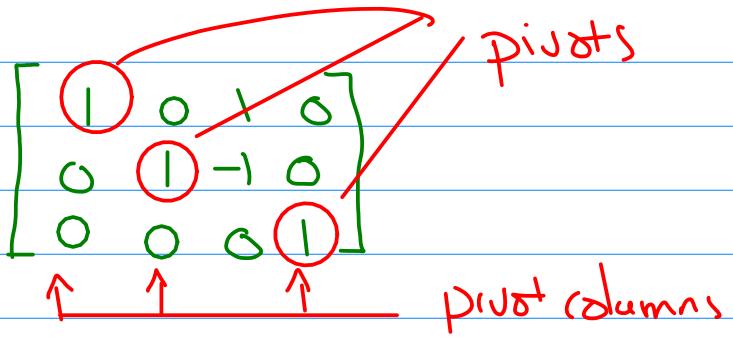
Thm (Uniqueness of Reduced Row Echelon Form)
Every matrix is row equivalent to a unique reduced row echelon matrix

Ex $\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \\ 6 & 0 & 0 & 15 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ← row equivalent

Although (row) echelon forms not unique, leading entries in same spot:

$$\begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 6 & -6 & 3 \\ 0 & 0 & 0 & 15 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 & -3 \\ 0 & 1 & -1 & 1/2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Def^n A pivot position in a matrix A is a location in A that corresponds to a leading 1 in reduced row echelon form of A
A pivot column is a column that contains a pivot



Fact: many results in linear alg can be translated into results about pivots

key ideas: * (reduced row) echelon form

CRUCIAL: you must master Gaussian Elimination

next lecture: use Gaussian Elimination to solve S.L.E.