Lecture 21

- Null Spaces, Column Spaces, and Linear Transformations (4.2)
- Linearly independent sets and bases (4.3)

Last time: The null space of a $m \times n$ matrix A:

$$\text{Nul}(A) = \{\vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0}\} \subseteq \mathbb{R}^n \quad \text{(subspace)}$$

Today:
- column space of A
- linearly independent sets

Column Space

Recall: If $\vec{v}_1, \ldots, \vec{v}_p$ any p vectors in $\mathbb{R}^n$,

$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is a subspace of $\mathbb{R}^n$

Def^n The column space of an $m \times n$ matrix A, denoted

$\text{Col}(A)$, is $\text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$ where $A = [\vec{a}_1 \cdots \vec{a}_n]$

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \Rightarrow Col(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \end{bmatrix} \right\}$

$$= \left\{ c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 3 \\ 6 \end{bmatrix} \mid c_1, c_2, c_3 \in \mathbb{R} \right\}$$

Thm $Col(A)$ is a subspace of $\mathbb{R}^m$

Note $Col(A) = \{ \vec{b} \mid x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b} \text{ for some } x_i \in \mathbb{R} \}$

$$= \{ \vec{b} \mid A\vec{x} = \vec{b} \text{ for some } \vec{x} \in \mathbb{R}^n \}$$

$$= \{ A\vec{x} \mid \vec{x} \in \mathbb{R}^n \}$$

In general, $\text{Col}(A) \subseteq \mathbb{R}^m$

Thm Let A be an $m \times n$ matrix

$\text{Col}(A) = \mathbb{R}^m \iff A\vec{x} = \vec{b}$ has a sol^n for all $\vec{b} \in \mathbb{R}^m$

$\iff$ A has a pivot in every row

Comparing Col(A) and Nul(A)

$\text{Col}(A)$ and $\text{Nul}(A)$ are different subspaces.
If A is $m \times n$

- $\text{Col}(A) \subseteq \mathbb{R}^m$ and $\text{Nul}(A) \subseteq \mathbb{R}^n$

(A) $\text{Nul}(A)$ defined implicitly $\Rightarrow$ given an condition a vector must satisfy, i.e. $A\vec{x} = \vec{0}$.

$\text{Col}(A)$ defined explicitly $\Rightarrow$ $\text{Col}(A) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$

(B) Typical $\vec{v} \in \text{Nul}(A)$ satisfies $A\vec{v} = \vec{0}$
Typical $\vec{v} \in \text{Col}(A)$ has property that the SLE $A\vec{x} = \vec{v}$ is consistent

(C) Given $\vec{v}$, "easy" to check if $\vec{v} \in \text{Nul}(A) \Rightarrow$ check $A\vec{v} = \vec{0}$.
"hard" to check if $\vec{v} \in \text{Col}(A) \Rightarrow$ need to solve $A\vec{x} = \vec{v}$

(D) $Nul(A) = \{\vec{0}\}$ and $Nul(A) = \{\vec{0}\}$ $\iff$
$A\vec{x} = \vec{0}$ has only trivial soln

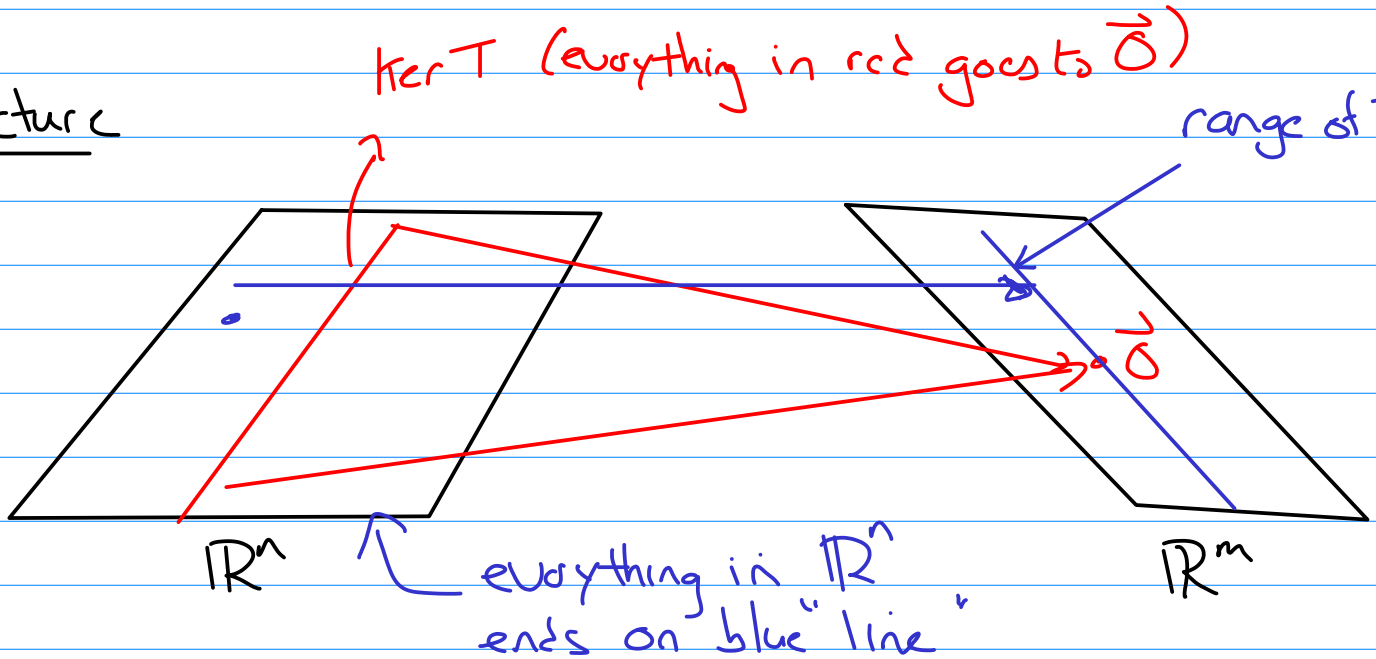
$Col(A) \subseteq \mathbb{R}^m$ and $Col(A) = \mathbb{R}^m$ $\iff$
columns of A span $\mathbb{R}^m$

$Nul(A)$, $Col(A)$ and linear transformations

Defn Given a linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$

kernel of T = $\ker T$ $= \{\vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0}\} \subseteq \mathbb{R}^n$
range of T = range T $= \{T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m$

Picture



Thm Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation with standard matrix A, i.e. $T(\vec{x}) = A\vec{x}$.

Then (1) $\ker T = \{\vec{x} \mid \vec{0} = T(\vec{x}) = A\vec{x}\} = \text{Nul}(A)$

(2) $\text{range } T = \{T(\vec{x}) = A\vec{x} \mid \vec{x} \in \mathbb{R}^n\} = \text{Col}(A)$

Thm Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation w/ standard matrix A.

(1) T is one-to-one $\iff A\vec{x} = \vec{0}$ has only trivial sol^n
$\iff \text{Nul}(A) = \{\vec{0}\}$

(2) T is onto $\iff \text{range } T = \text{Col}(A) = \mathbb{R}^m$
$\iff$ columns of A span $\mathbb{R}^m$

Linearly independent sets

In lecture 7 (go review!) we introduced linear independence in $\mathbb{R}^n$. Extend to any vector space.

Def^n An indexed set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in V is linearly independent if the vector equation

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \vec{0}$$

has only the trivial sol^n $c_1 = c_2 = \cdots = c_p = 0$

If there is a nontrivial sol^n, say there is a linear dependence relation among $\vec{v}_1, \ldots, \vec{v}_p$.

Ex (1) Same def^n as in $\mathbb{R}^n$

So standard basis vectors $\vec{e}_1, \ldots, \vec{e}_n$ linearly independent

$$\vec{0} = c_1\vec{e}_1 + \cdots + c_n\vec{e}_n = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \iff c_i = 0$$

Ex (2) Let $P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R}\}$

Show $p_1(t) = -3+4t^2$ $\quad p_2(t) = 5 - t + t^2$ $\quad p_3(t) = 1 + t + 3t^2$

linearly independent.

Need to show the only soln to

$$c_1 p_1(t) + c_2 p_2(t) + c_3 p_3(t) = 0 \iff c_1 = c_2 = c_3 = 0$$

$$c_1(-3+4t^2) + c_2(5 - t + t^2) + c_3(1 + t + 3t^2)$$

$$= (-3c_1 + 5c_2 + c_3) + (-c_2 + c_3)t + (4c_1 + c_2 + 3c_3)t^2$$

$$= 0 + 0t + 0t^2$$

SLE $\quad \begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Fact $\quad \begin{vmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 1 & 3 \end{vmatrix} = 9 + 20 + 0 - (-4) - (-3) = 36 \neq 0$

determinant of coefficient matrix $\neq 0$

$\Rightarrow$ coefficient matrix is invertible

$\Rightarrow$ SLE has only trivial soln $c_1 = c_2 = c_3 = 0$

$\Rightarrow$ our vectors $p_1(t), p_2(t), p_3(t)$ linearly independent.

Key ideas · Col(A)
· linear transformations & Nul(A)/Col(A)
· linear independence in any vector space