Lecture 29 — 5.2 The Characteristic Equation

Last time: A nonzero vector $\vec{x}$ is an eigenvector of an $n \times n$ matrix A, if $A\vec{x} = \lambda\vec{x}$ for some scalar λ (called the eigenvalue)

Today: characteristic equation
finding eigenvalues and eigenvectors

I. Eigenvalues

Eigenvalues encoded in the "characteristic polynomial"

Ex Find the eigenvalues of $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$

λ is an eigenvalue $\iff A\vec{x} = \lambda\vec{x}$ has a nontrivial sol^{y}
$\iff A\vec{x} = \lambda I_2 \vec{x}$ has a nontrivial sol^{n}
$\iff (A - \lambda I_2)\vec{x} = \vec{0}$ has a nontrivial sol^{n}
$\iff \det(A - \lambda I_2) = 0$

$$A - \lambda I_2 = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 7 \\ 7 & 2-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I_n) &= (2-\lambda)(2-\lambda) - 7 \cdot 7 \\ &= 4 - 4\lambda + \lambda^2 - 49 \\ &= \lambda^2 - 4\lambda - 45 \end{aligned}$$

$$\begin{aligned} \det(A - \lambda I_n) = 0 &\iff \lambda^2 - 4\lambda - 45 = 0 \\ &\iff (\lambda - 9)(\lambda + 5) = 0 \\ &\iff \lambda = 9, \text{ or } \lambda = -5 \end{aligned}$$

Eigenvalues are $\lambda = 9, -5$

Approach generalizes...

Def^n Let A be an $n \times n$ matrix. The equation

$$\det(A - \lambda I_n) = 0$$

is the characteristic equation.

Call $\det(A - \lambda I_n)$ the characteristic polynomial

Ex Find characteristic equation of $A = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 3 & -1 \\ 0 & 6 & 0 \end{bmatrix}$

Since $A - \lambda I_3 = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 2 & 3-\lambda & -1 \\ 0 & 6 & -\lambda \end{bmatrix}$

$$\det(A - \lambda I_3) = (1-\lambda)(3-\lambda)(-\lambda) + 0 + (-12) - (6)(-1)(1-\lambda)$$
$$= \underbrace{-\lambda^3 + 4\lambda^2 - 9\lambda - 6}$$

characteristic polynomial

$-\lambda^3 + 4\lambda^2 - 9\lambda - 6 = 0$ ← characteristic equation

λ is an eigenvalue of $A \iff \lambda$ is a sol^n to $\det(A - \lambda I_n) = 0$

To find eigenvalues of A, need roots of

$-\lambda^3 + 4\lambda^2 - 9\lambda - 6 = 0$ ← Can be hard to find roots

Ex (triangular matrix $\Rightarrow$ know eigenvalues on diagonal, but can also see this via new approach)

$$A = \begin{bmatrix} 4 & -7 & 0 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ then } A - \lambda I_4 = \begin{bmatrix} 4-\lambda & -7 & 0 & 2 \\ 0 & 3-\lambda & -4 & 6 \\ 0 & 0 & 3-\lambda & -8 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I_4) &= (4-\lambda)(3-\lambda)(3-\lambda)(1-\lambda) \\ &= (4-\lambda)(3-\lambda)^2(1-\lambda) \end{aligned}$$

eigenvalues are $\lambda = 4, 3, 1$

Note: In above example, say $\lambda = 3$ has multiplicity 2 since $(3-\lambda)$ appears twice in factorization

- $\det(A - \lambda I_n)$ is a polynomial of degree n, so at most n possible eigenvalues, since at most n roots to $\det(A - \lambda I_n) = 0$

II Eigenvectors

Once you have eigenvalues, can find eigenvectors

Note: eigenvectors **not** unique. If $\vec{v}$ satisfies $A\vec{v} = \lambda\vec{v}$, then so does $A(c\vec{v}) = c(A\vec{v}) = c(\lambda\vec{v}) = \lambda(c\vec{v})$

Ex $A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$ $\lambda = 9, -5$ ← eigenvalues

$\lambda = 9$ need to find nonzero $\vec{x}$ such that

$A\vec{x} = 9\vec{x} \iff (A - 9I_2)\vec{x} = \vec{0}$

$$A - 9I_2 = \begin{bmatrix} -7 & 7 \\ 7 & -7 \end{bmatrix}$$

Solve

$$\left[\begin{array}{cc|c} -7 & 7 & 0 \\ 7 & -7 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -7 & 7 & 0 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

x_2 free, so $x_2 = t$

$x_1 - x_2 = 0 \Rightarrow x_1 = x_2 = t$

So, all sol'ns have form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

eigenvector $\begin{bmatrix} t \\ t \end{bmatrix} \iff t \neq 0$ A specific eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Eigenspace $\lambda = 9 = \{\vec{x} \mid (A - 9I_2)\vec{x} = \vec{0}\}$

$= \left\{ t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\} = \text{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\boxed{\lambda = -5}$ $\quad A - (-5)I_2 = A + 5I_2 = \begin{bmatrix} 7 & 7 \\ 7 & 7 \end{bmatrix}$

$\left[\begin{array}{cc|c} 7 & 7 & 0 \\ 7 & 7 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ x_2 free $\quad x_2 = t$

$x_1 + x_2 = 0$

$\Rightarrow x_1 = -x_2 = -t$

all sol'ns have form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

an eigenvector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Ex $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{bmatrix}$ $\det(A - \lambda I_3) = (1-\lambda)^2(2-\lambda)$

$\Rightarrow$ eigenvalues

$\lambda = 1$ (mult. is 2)

$\lambda = 2$ (mult is 1)

$\boxed{\lambda = 1}$ $A - 1I_3 = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$

Solve

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

x_2, x_3 free

$x_2 = s$ and $x_3 = t$

$x_1 - x_2 - x_3 = 0 \Rightarrow x_1 = x_2 + x_3 = s + t$

So sol'n $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s+t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ ← eigenvector if s and t not both zero

eigenspace of $\lambda = 1$ $= \operatorname{span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$ basis for eigenspace

Note # of basis elements = multiplicity of $\lambda = 1$

(not always true)

Octave:

eig(A) ← returns column vector whose entries are eigenvalues

[V,D] = eig(A) ← returns two matrices

D diagonal matrix with eigenvalues on diagonal

V columns eigenvectors in same order as D. (also scaled to have norm = 1)

Key ideas: * characteristic equation

* finding eigenvalues and eigenvectors