Lecture 19 — 4.1 Vector Spaces & Subspaces

Today's lecture: Introduce vector spaces (abstracting the properties of $\mathbb{R}^n$, so $\mathbb{R}^n$ special case of more general theory)

Def^n (vector space) A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition & multiplication of scalars (real numbers) subject to axioms

1. For all $\vec{u}, \vec{v} \in V$, $\vec{u} + \vec{v} \in V$
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
3. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
4. There is a zero vector $\vec{0} \in V$ such that $\vec{u} + \vec{0} = \vec{u}$
5. For each $\vec{u} \in V$, there is a vector $-\vec{u} \in V$ such that $\vec{u} + (-\vec{u}) = \vec{0}$
6. For all $\vec{u} \in V$ and $c \in \mathbb{R}$, $c\vec{u} \in V$
7. $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$
8. $(c+d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$

To show a set is a vector space

1. identify the set of vectors
2. identify addition and scalar multiplication operations in V
3. show 1 & 6 are true (the "closure" of the operation)
4. check the rest of your axioms

Ex $\mathbb{R}^n$ with $n \geq 1$ is a vector space

- $\mathbb{R}^n = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mid u_i \in \mathbb{R} \right\}$

- addition $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{bmatrix}$

scalar mult $c \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$

- operations satisfy 1 & 6 since new vectors still in $\mathbb{R}^n$
- $\mathbb{R}^n$ satisfies rest of axioms (Section 1.3)

Ex ① (Zero vector spaces) $V = \{0\}$ ← only contains zero vector

- addition: $0 + 0 = 0$
  scalar mult $c0 = 0$
- operations satisfy 1 & 6
- rest of axioms satisfied (easy to check)

Ex ② (2×2 matrices)

$$M_{2\times 2} = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \;\middle|\; a_i \in \mathbb{R} \right\} \leftarrow \text{all } 2\times 2 \text{ matrices}$$

- addition $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$

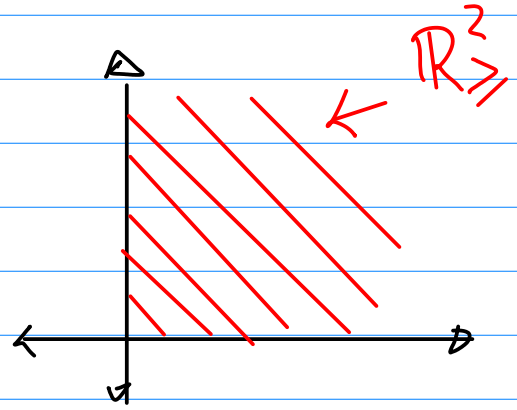
  scalar mult $c\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} = \begin{bmatrix} ca_1 & ca_2 \\ ca_3 & ca_4 \end{bmatrix}$

- operations satisfy 1 & 6 (operation produces 2×2 matrix)
- rest of axioms Section 2.1

Ex (3) $m \times n$ matrices $M_{m \times n}$ + same reasoning

Ex (4) (not a vector space)

$$\mathbb{R}^2_{\geq} = \{(u_1, u_2) \mid u_1, u_2 \in \mathbb{R},\ u_1, u_2 \geq 0\}$$



addition

$(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2)$

Scalar mult.

$c(u_1, u_2) = (cu_1, cu_2)$

Not a vector space because axiom 5 fails

$(5,6) \in \mathbb{R}^2_{\geq}$ but no $(v_1, v_2) \in \mathbb{R}^2_{\geq}$ such that

$$(5,6) + (v_1, v_2) = (0,0)$$

$$(5 + v_1, 6 + v_2)$$

[to solve this, $v_1 = -5$, $v_2 = -6$, but $(-5,-6) \notin \mathbb{R}^2_{\geq}$]

Ex $P_n = \{a_0 + a_1 t + a_2 t + \cdots + a_n t^n \mid$ all polynomials of degree $\le n$ with $a_i \in \mathbb{R}\}$

$P_2 = \{a_0 + a_1 t + a_2 t^2 \mid a_i \in \mathbb{R}\}$

Can make P_n a vector space:

- the "vectors" are polynomials
- addition:

  If $p(t) = a_0 + a_1 t + \cdots + a_n t^n$ and

  $q(t) = b_0 + b_1 t + \cdots + b_n t^n$,

  then $p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + \cdots + (a_n + b_n)t^n$
- scalar mult.

  $c\,p(t) = ca_0 + ca_1 t + \cdots + ca_n t^n$
- Need to check 10 axioms (see text)

Note $\vec{0} \in P_n$ is $\vec{0} = 0 = 0 + 0t + \cdots + 0t^n$ ← zero polynomial

Ex $p(t) = 1 + 2t^2$ and $q(t) = -1 + 2t + 3t^2 \in P_2$

$p(t) + q(t) = (1 + (-1)) + (0 + 2)t + (2 + 3)t^2 = 2t + 5t^2$

Subspaces

Def^n A subspace of a vector space V is a subset $H \subseteq V$ such that H is also a vector space

Fact Let V be a vector space. A subset $H \subseteq V$ is a subspace if

1. $\vec{0} \in H$ (the zero vector of V also belongs to H)
2. if $\vec{u}, \vec{v} \in H$, then $\vec{u}+\vec{v} \in H$
3. if $\vec{u} \in H$ and $c \in \mathbb{R}$, $c\vec{u} \in H$

Ex $H = \{\vec{0}\}$ is a subspace of V

Proof Check 3 conditions

1. Clear that $\vec{0} \in \{\vec{0}\}$
2. For any $\vec{u}, \vec{v} \in H$, $\vec{u} = \vec{v} = \vec{0}$. Since $\vec{0}+\vec{0} = \vec{0}$, $\vec{u}+\vec{v} \in H$
3. If $\vec{u} \in H$ and $c \in \mathbb{R}$, then $\vec{u} = \vec{0}$, and $c\vec{u} = c \cdot \vec{0} = \vec{0} \in H$.

Ex For any vector space V, V is a subspace of itself

Def^n For any vector space V, $\{\vec{0}\}$ and V called the trivial subspaces

Ex Consider $Q = \{a_0 + a_1 t + a_2 t^2 \mid a_0 = 0\} \subseteq \mathbb{P}_2$

This is a subspace; we check only condition 2

If $p(t), q(t) \in Q$, $p(t) = 0 + a_1 t + a_2 t^2$ and $q(t) = 0 + b_1 t + b_2 t^2$

So $p(t) + q(t) = 0 + (a_1 + b_1)t + (a_2 + b_2)t^2$.

↑ a zero constant term

So $p(t) + q(t) \in Q$.

Ex $H = \{a_0 + a_1 t + a_2 t^2 \mid a_i \text{ an integer}\} \subseteq \mathbb{P}_2$

Explain why not a subspace

Sln H is not closed under scalar multiplication

e.g. $1 + t + t^2 \in H$ and $\frac{1}{2} \in \mathbb{R}$

but $\frac{1}{2}(1 + t + t^2) = \frac{1}{2} + \frac{1}{2}t + \frac{1}{2}t^2 \notin H$

Key points * vector spaces & examples
* subspaces and how to check