Lecture 22 | 4.3 Linear independent sets and bases

Recall two important concepts:
- linear independence
- span

today's lecture: a basis (combines these concepts)

Basis

Def^n Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_p\}$ is a **basis** for H if

1. $\mathcal{B}$ is linearly independent
2. $\mathcal{B}$ spans H, i.e. $H = \text{span}\{\vec{b}_1, \ldots, \vec{b}_p\}$

Note: If $H = V$, call $\mathcal{B}$ a basis for V

Ex $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$

1. $\mathcal{E}$ is linearly independent since $c_1\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix} + c_2\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix} + \cdots + c_n\begin{bmatrix}0\\\vdots\\1\end{bmatrix}$

$$= \begin{bmatrix}c_1\\\vdots\\c_n\end{bmatrix} = \begin{bmatrix}0\\0\\\vdots\\0\end{bmatrix} \iff c_1 = c_2 = \cdots = c_n = 0$$

$\mathcal{E}$ spans $\mathbb{R}^n$ since for any $\vec{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$

$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + \cdots + a_n\vec{e}_n \in \text{span}\{\vec{e}_1, \ldots, \vec{e}_n\}$

Call $\mathcal{E} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ STANDARD BASIS for $\mathbb{R}^n$

Ex STANDARD BASIS for P_n is $\{1, t, t^2, \ldots, t^n\}$

Ex $M_{2\times 2} = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid a, b, c, d \in \mathbb{R} \right\}$

(all 2x2 matrices)

What would be a basis for $M_{2\times 2}$?

Soln $M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

$M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

This is STANDARD BASIS for $M_{2\times 2}$

1. linearly independent since

$$c_1 M_1 + c_2 M_2 + c_3 M_3 + c_4 M_4 = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Leftrightarrow c_1 = c_2 = c_3 = c_4 = 0$$

2. Spanning set of $M_{2\times 2}$ since

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = aM_1 + bM_2 + cM_3 + dM_4 \in \text{span}\{M_1, \ldots, M_4\}$$

Remark Idea generalizes to $M_{m\times n}$

Ex Show $S=\{1-3t+2t^2,\ 1+t+4t^2,\ 1-7t\}$ is **not** a basis for $\mathbb{P}_2$

Need to show either (A) S not linearly independent
(B) S does not span $\mathbb{P}_2$

Show (A) Consider the equation:

$$c_1(1-3t+2t^2)+c_2(1+t+4t^2)+c_3(1-7t)$$
$$=(c_1+c_2+c_3)+(-3c_1+c_2-7c_3)t+(2c_1+4c_2)t^2$$
$$=0+0t+0t^2$$

$\Leftrightarrow$ SLE
$$\begin{bmatrix}1&1&1\\-3&1&-7\\2&4&0\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Compute $\det\begin{vmatrix}1&1&1\\-3&1&-7\\2&4&0\end{vmatrix}=0-14-12-(2)-(-28)-0$
$=0$

matrix not invertible $\Rightarrow$ $A\vec{x}=\vec{0}$ has a nontrivial sol^n
$\Rightarrow$ polynomials linearly dependent
$\Rightarrow$ S is not basis

Thm (Spanning Set Thm) Let $S=\{\vec{v}_1,\dots,\vec{v}_p\}$ be a set in V and set $H=\text{span}\{\vec{v}_1,\dots,\vec{v}_p\}$

1. If $\vec{v}_k$ is a linear combination of remaining vectors of S, then the set formed by removing $\vec{v}_k$ from S still spans H.

2. If $H\neq\{\vec{0}\}$, some subset of S is a basis of H

Big idea: basis "effecient" way to describe H
- smallest number of vectors needed to span H
- largest number of vectors that is linearly independent

Ex $H=\text{span}\{1-3t+2t^2, 1-t+4t^2, 1-7t\}\subseteq\mathbb{P}_2$

By above, not linearly independent. In fact

$$1-7t=2(1-3t+2t^2)+(-1)(1-t+4t^2)$$

By thm, $H=\text{span}\{1-3t+2t^2, 1-t+4t^2\}$

Since $1-t+4t^2$ is not a multiple of $1-3t+2t^2$, they are lin. indep. So this is a **basis** for H

Bases of Nul(A) and Col(A)

Bases for these subspaces in reduced row echelon form

Ex $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} \textcircled{1} & 0 & 6 & 5 \\ 0 & \textcircled{2} & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ x_3, x_4 free

pivot columns (of A) ↔ pivot columns (of the RREF)

Bases for Nul(A): We saw how to get a spanning set

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6x_3 - 5x_4 \\ -5/2 x_3 - 3/2 x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix}$$

$$\Rightarrow \text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Fact This procedure actually produces a basis.
i.e. vectors are linearly independent

E.g. $c_1 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -5 \\ -3/2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} * \\ * \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow c_1 = c_2 = 0$

spots of free variables

Bases for Col(A)

Recall $\text{Col}(A) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$

By spanning set theorem, need to "throw out" lin. dep. vectors!

Thm Pivot columns of A is a basis for Col(A).

Ex $\text{Col}(A) = \text{span}\left\{\begin{bmatrix} -2 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \\ 8 \end{bmatrix}\right\}$

WARNING use the columns of A, not the columns of the reduced row echelon form!

Key points
* bases
* Spanning set theorem