Lecture 16 — 3.1 Introduction to Determinants

Today's lecture: introduce determinants
(can be used to determine if a matrix has an inverse)

Recall If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \frac{1}{ad-cb}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

The number $ad-cb$ is the determinant

A^{-1} exists $\iff \det A \neq 0$

Goal: Introduce determinants for $n \times n$ matrices

Defn If A is an $n \times n$ matrix, A_{ij} denotes the matrix with row i, column j removed

Ex $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 2 & 4 & 6 & 8 \\ 10 & 20 & 30 & 40 \end{bmatrix}$ $A_{11} = \begin{bmatrix} \cancel{1} & \cancel{2} & \cancel{3} & \cancel{4} \\ \cancel{1} & 0 & 1 & 0 \\ \cancel{2} & 4 & 6 & 8 \\ \cancel{10} & 20 & 30 & 40 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 4 & 6 & 8 \\ 20 & 30 & 40 \end{bmatrix}$

$A_{23} = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 10 & 20 & 40 \end{bmatrix}$

Def^n For $n \geq 2$, the determinant of an $n \times n$ matrix $A = [a_{ij}]$ is the sum

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n}$$

$$= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det (A_{1j})$$

Note Formula is recursive. To find the determinant of an $n \times n$ matrix, need the determinant of an $(n-1) \times (n-1)$ matrix. And so on

Ex Find $\det(A) = \det\left(\begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 4 \\ 1 & 2 & 3 \end{bmatrix}\right)$ Alt. notation $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 4 \\ 1 & 2 & 3 \end{vmatrix}$

$$\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 4 \\ 1 & 2 & 3 \end{vmatrix} = 1 \cdot \begin{vmatrix} 0 & 4 \\ 2 & 3 \end{vmatrix} - 2 \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= 1 \cdot (0 \cdot 3 - 2 \cdot 4) - 2(3 \cdot 3 - 4 \cdot 1) + 1(3 \cdot 2 - 1 \cdot 0)$$

$$= 1(-8) - 2(5) + 1 \cdot 6 = \boxed{-12}$$

In def^{n} of $\det(A)$, used top row of A. In fact

Thm The determinant of an $n \times n$ matrix can be computed using an expansion down any row or column

Precisely, using row i

$$\det(A) = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \cdots + (-1)^{i+n} a_{in} \det A_{in}$$

using column j

$$\det(A) = (-1)^{1+j} a_{1j} \det A_{1j} + (-1)^{2+j} a_{2j} \det A_{2j} + \cdots + (-1)^{n+j} a_{nj} \det A_{nj}$$

① $C_{ij} = (-1)^{(i+j)} \det A_{ij}$ is called $(i,j)^{th}$ cofactor

② formulas are row/column expansions

Advantage: simplify calculations by picking rows or columns with lots of zeroes

Ex Find $\begin{vmatrix} 1 & 2 & 1 \\ 3 & 0 & 4 \\ 1 & 2 & 3 \end{vmatrix}$ ← cofactor expansion down 2nd column

$$\det(A) = (-1)^{1+2} \cdot 2 \begin{vmatrix} 3 & 4 \\ 1 & 3 \end{vmatrix} + (-1)^{2+2} \cdot 0 \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + (-1)^{3+2} \cdot 2 \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}$$

$$= (-1) \cdot 2(5) + 0 + (-1) \cdot 2(1) = \boxed{-12}$$

Ex Find determinant of $A = \begin{bmatrix} 3 & 2 & 7 & 9 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Step 1 cofactor down first column

$$\det(A) = (-1)^{1+1} \cdot 3 \begin{vmatrix} 1 & 4 & 6 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{vmatrix} + (-1)^{1+2} \cdot 0 \det A_{21} + (-1)^{1+3} \cdot 0 \det A_{31} + (-1)^{1+4} \cdot 0 \cdot \det A_{41}$$

$= 0$

Step 2 Let $B = A_{11} = \begin{bmatrix} 1 & 4 & 6 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$ ← do cofactor expansion down first column

$$\det B = (-1)^{1+1} \cdot 1 \begin{vmatrix} -1 & -2 \\ 0 & 2 \end{vmatrix} + 0 + 0$$

$$= (-1)^{2} \cdot 1 (-2) = (-2) = 1(-1)(2)$$

$$\det(A) = (-1)^{2} \cdot 3 \cdot 1 \cdot (-1)(2) = 3 \cdot 1 \cdot (-1) \cdot 2$$

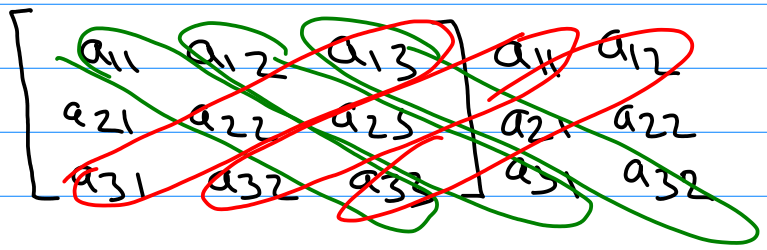
Thm If A is a triangular matrix, then

$$\det A = a_{11}a_{22}\cdots a_{nn}$$

diagonal entries

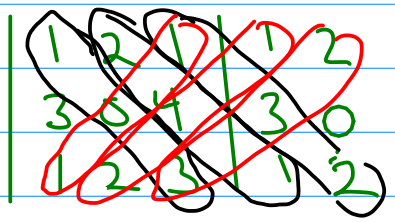
Short cut for 3×3 matrices

To compute $\det(A)$ of 3×3 matrix can use



$$\det(A) = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{11} - a_{33}a_{21}a_{12}$$

Ex



$$\det(A) = 1\cdot 0\cdot 3 + 2\cdot 4\cdot 1 + 1\cdot 3\cdot 2 - 1\cdot 0\cdot 1 - 2\cdot 4\cdot 1 - 3\cdot 3\cdot 2$$

$$= \boxed{-12}$$

$$\begin{vmatrix} 3 & 0 & 4 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} \begin{matrix} 3 & 0 \\ 1 & 2 \\ 1 & 2 \end{matrix}$$

$\det A = 18 + 0 + 8 - 8 - 6 - 0 = \boxed{12}$

Note swapping rows changes $\det(A)$: more details are coming!

WARNING! This "trick" only works for 3x3 matrices

OCTAVE $\det(A)$

Key points * determinant formula