Lecture 32 | 5.4 Eigenvectors & Linear Transformations

Last time Diagonalization $A = PDP^{-1}$

Today pull together many ideas (bases, $\mathcal{B}$-coordinates, linear transformations) to understand connection between diagonalization & linear transformations

Setup V is a vector space with $\dim V = n$ and basis
$$\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$$

W is a vector space with $\dim W = m$ and basis
$$\mathcal{C} = \{\vec{c}_1, \ldots, \vec{c}_m\}$$

Ex $V = \mathbb{P}_2$ and $\mathcal{B} = \{1, t, t^2\}$

$W = \mathbb{P}_1$ and $\mathcal{C} = \{1, t\}$

Def$^{\underline{n}}$ A linear transformation is a function $T: V \to W$ between vector spaces such that

- $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$
- $T(c\vec{u}) = c\,T(\vec{u})$

Ex $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_1$ given by $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$

e.g $T(3+8t+10t^2) = 8 + 20t$ ↑ differentiation

Recall If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, there is a matrix A such that $T(\vec{x}) = A\vec{x}$

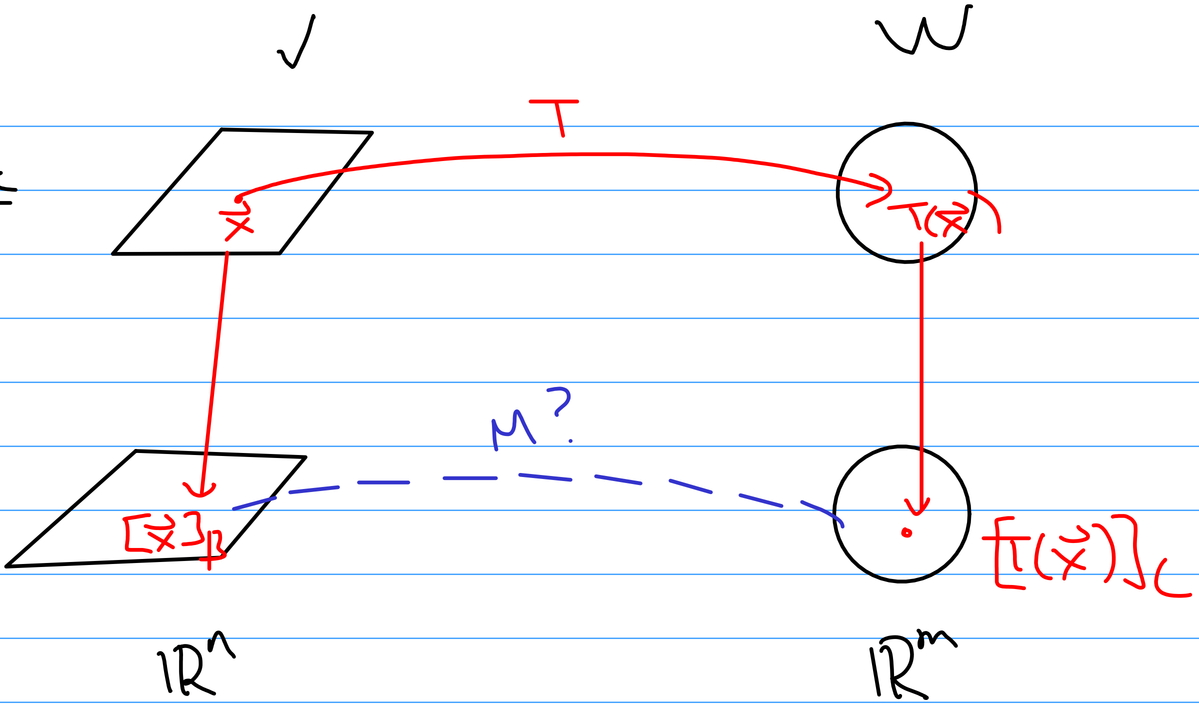
$\boxed{Q}$ Does something similar hold for general vector spaces?

Recall If $\vec{x} \in V$, $\vec{x} = r_1 \vec{b}_1 + \dots + r_n \vec{b}_n$

$$\Rightarrow [\vec{x}]_B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

If $\vec{x} \in W$, $\vec{x} = s_1 \vec{c}_1 + \dots + s_m \vec{c}_m \Rightarrow [\vec{x}]_C = \begin{bmatrix} s_1 \\ \vdots \\ s_m \end{bmatrix}$

Picture



Our question: Does there exist a matrix M such that

$$[T(\vec{x})]_C = M[\vec{x}]_B \;??$$

Because T is a linear transformation

$$T(\vec{x}) = T(r_1\vec{b}_1 + \dots + r_n\vec{b}_n) = r_1 T(\vec{b}_1) + \dots + r_n T(\vec{b}_n)$$

So

$$[T(\vec{x})]_C = [r_1 T(\vec{b}_1) + \dots + r_n T(\vec{b}_n)]_C$$

$$= r_1[T(\vec{b}_1)]_C + r_2[T(\vec{b}_2)]_C + \dots + r_n[T(b_n)]_C$$

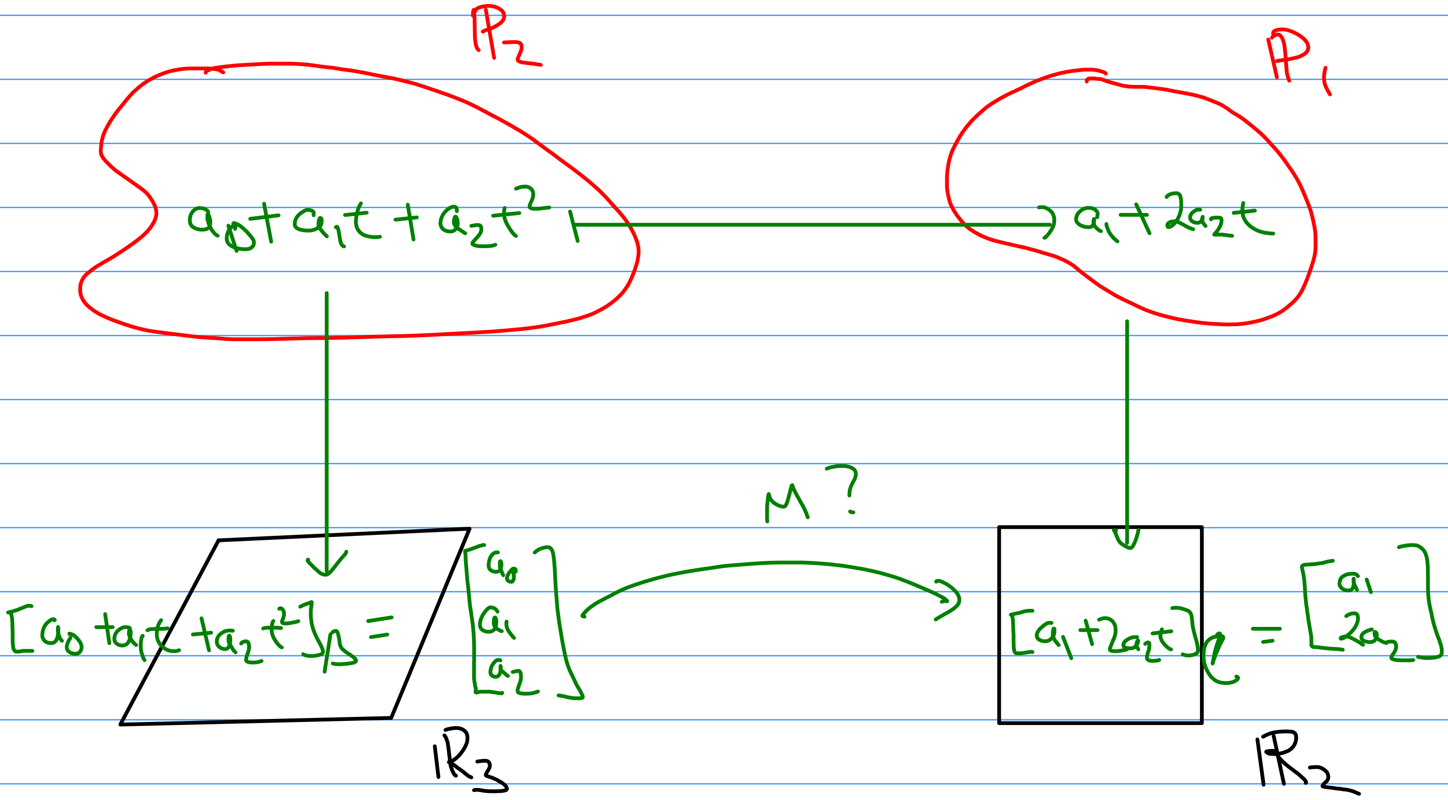
$$= \begin{bmatrix}[T(\vec{b}_1)]_C & [T(\vec{b}_2)]_C & \cdots & [T(\vec{b}_n)]_C\end{bmatrix} \underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}}_{[\vec{x}]_B}$$

Def^{n} The matrix T relative to bases $\mathcal{B}$ and C

$$M = \big[\ [T(\vec{b}_1)]_C \ \ [T(\vec{b}_2)]_C \ \cdots \ [T(\vec{b}_n)]_C\ \big]$$

m×n matrix

Ex $T: \mathbb{P}_2 \longrightarrow \mathbb{P}_1$ given by $T(a_0 + a_1 t + a_2 t^2) = a_1 + 2a_2 t$



$\mathbb{P}_2$
$\mathbb{P}_1$
$a_0 + a_1 t + a_2 t^2 \longmapsto a_1 + 2a_2 t$
M ?
$[a_0 + a_1 t + a_2 t^2]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$
$[a_1 + 2a_2 t]_C = \begin{bmatrix} a_1 \\ 2a_2 \end{bmatrix}$
$\mathbb{R}_3$
$\mathbb{R}_2$

$$M = \Big[\, [T(\vec{b}_1)]_C \quad [T(\vec{b}_2)]_C \quad [T(\vec{b}_3)]_C \,\Big]$$

$$\vec{b}_1 = 1 \Longrightarrow T(\vec{b}_1) = T(1) = 0 \Rightarrow \underline{0} + \underline{0}t \Rightarrow [T(1)]_C = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{b}_2 = t \Longrightarrow T(\vec{b}_2) = T(t) = 1 \Rightarrow \underline{1} + \underline{0}t \Rightarrow [T(t)]_C = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{b}_3 = t^2 \Longrightarrow T(\vec{b}_3) = T(t^2) = 2t \Rightarrow \underline{0} + \underline{2}t = [T(t^2)]_C = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

So $M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

__Ex__

$$\begin{tikzcd}
17 + 3t + 12t^2 \arrow[r, "T"] \arrow[d, "{[\ ]_{\mathcal{B}}}"] & 3 + 24t \arrow[d, "{[\ ]_C}"] \\
\begin{bmatrix} 17 \\ 3 \\ 12 \end{bmatrix} \arrow[r] & \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 17 \\ 3 \\ 12 \end{bmatrix} = \begin{bmatrix} 3 \\ 24 \end{bmatrix}
\end{tikzcd}$$

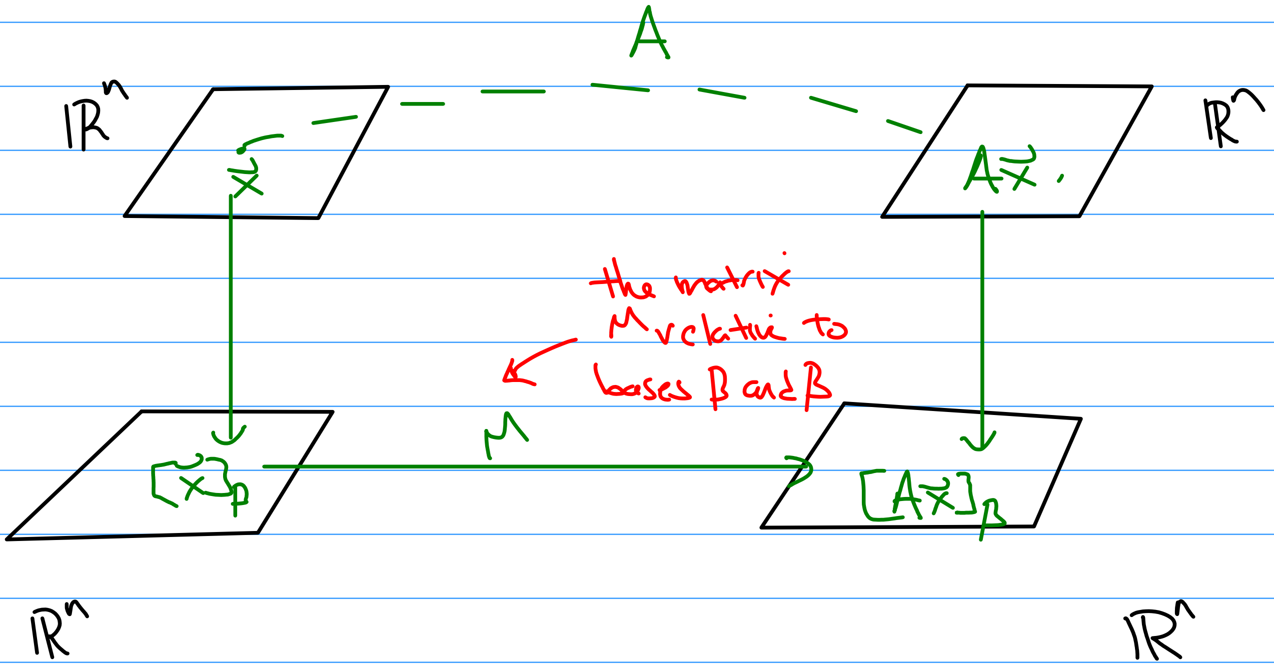
Connection with diagonal matrices

Suppose $A = PDP^{-1}$ is diagonalization

Fact $P = [\vec{b}_1 \cdots \vec{b}_n]$ is invertible with columns that eigenvectors that are linearly independent

$\Longrightarrow \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis for $\mathbb{R}^n$, called eigenvector basis of $\mathbb{R}^n$

Let $V = W = \mathbb{R}^n$ and let $\beta = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for both. Consider the linear transf $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$



the matrix M relative to bases β and β

Thm If $A = PDP^{-1}$, then $M = D$, i.e. the map is given a diagonal matrix.

Proof If $P = [\vec{b}_1 \cdots \vec{b}_n]$,

$$P[\vec{x}]_\beta = \vec{x} \iff [\vec{x}]_\beta = P^{-1}\vec{x}$$

Then $M = [[T(\vec{b}_1)]_\beta \cdots [T(\vec{b}_n)]_\beta]$

$$= [[A\vec{b}_1]_\beta \cdots [A\vec{b}_n]_\beta]$$

$$= [P^{-1}Ab_1 \cdots P^{-1}Ab_n]$$

$$= P^{-1}A[\vec{b}_1 \cdots \vec{b}_n]$$

$$= P^{-1}AP = D \quad \text{since } A = PDP^{-1} \qquad \blacksquare$$

Big idea If $A = PDP^{-1}$, then the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$

is the same as

$$T^* : \mathbb{R}^n \to \mathbb{R}^n \text{ given by } T^*(\vec{u}) = D\vec{u}$$

if we use the "correct" basis for $\mathbb{R}^n$, namely eigenvectors

Ex $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

Then

$$\begin{tikzcd}
\vec{x} \arrow[rr] \arrow[d, "P^{-1}\vec{x}"'] & & A\vec{x} \arrow[d, "P^{-1}A\vec{x} = {[A\vec{x}]_\beta}"] \\
{[\vec{x}]_\beta} \arrow[rr] & & D[\vec{x}]_\beta \\
& & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}[\vec{x}]_\beta \arrow[u, equal]
\end{tikzcd}$$

Key ideas:
- the matrix M relative to basis $\mathcal{B}$ & $\mathcal{C}$
- diagonalization & linear transformations.