Lecture 10 — Applications of linear systems (Section 1.6)

Today's lecture: two applications of SLE (economics & network flows)

I. Homogeneous Systems in Economics (Input-Output Model)

Setup: Divide a nation's economy into many sections: manufacturing, communications, entertainment, and so on. Assume for each sector, we know

1. its total output
2. how this output divided into other sectors

Def^ The total dollar value of a sector's output is the called the price of that output

Fact (Leontief - Nobel Prize Winner) There exists equilibrium prices that can be assigned to the outputs such that the income of each sector balances its expenses.

Ex (use linear alg to find equilibrium)

Manuf.	Elect.	Steel	(Purchased by)
0.3	0.4	0.6	Manuf.
0.3	0.1	0.2	Elect.
0.4	0.5	0.2	Steel

Let p_M = price of manufacturing (total annual output)
p_E = price of electricity
p_S = price of steel

Each row in table gives cost. E.g. row 2 implies electricity spends

$$0.3p_M + 0.1p_E + 0.2p_S$$

Since we want cost (expenses) to equal its output (price) get

$$0.3p_M + 0.1p_E + 0.2p_S = p_E$$

Each row gives a linear equation:

$$p_M = 0.3p_M + 0.4p_E + 0.6p_S \qquad -0.7p_M + 0.4p_E + 0.6p_S = 0$$

$$p_E = 0.3p_M + 0.1p_E + 0.2p_S \Rightarrow \quad 0.3p_M - 0.9p_E + 0.2p_S = 0$$

$$p_S = 0.4p_M + 0.5p_E + 0.2p_S \qquad 0.4p_M + 0.5p_E - 0.8p_S = 0$$

homog SLE

$$\begin{bmatrix} -.7 & .4 & .6 \\ .3 & -.9 & .2 \\ .4 & .5 & -.8 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1.21569 \\ 0 & 1 & -0.62745 \\ 0 & 0 & 0 \end{bmatrix}$$

p_S is free $\quad p_E = 0.62745\,p_S \quad p_M = 1.21569\,p_S$

So p_S = free $p_E = 0.62745 p_S$ $p_M = 1.21569 p_S$

$\Rightarrow$ all sol'ns have form $\vec{p} = p_S \begin{bmatrix} 1.21569 \\ 0.62745 \\ 1 \end{bmatrix}$

Meaning? If p_S = \$10,000,000, then if the output of manufacturing is \$12,156,900 and

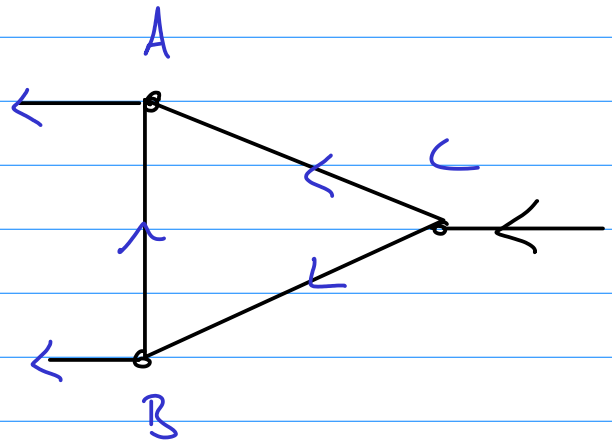
electricity is \$6,274,500,

then each sector makes exactly enough to pay off all of its costs.

II Network flows

A network consists of sets of points called junctions (or nodes) with lines or arcs called branches connecting junctions

Ex

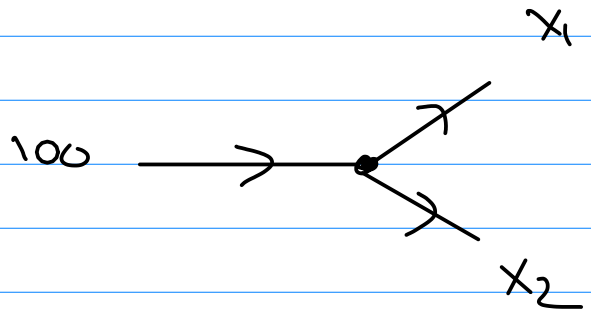


arrows denote flow direction

Basic assumption:

total flow into network = total flow out of a network
total flow into junction = total flow out of a junction

Ex



100 units in

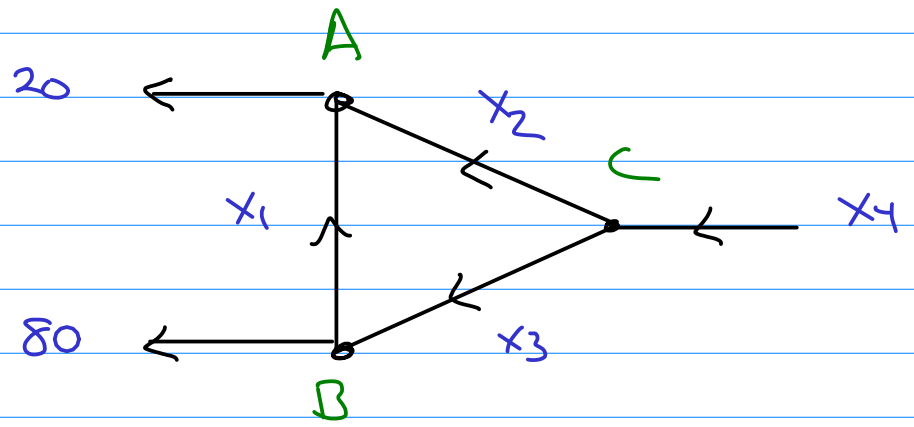
$100 = x_1 + x_2$

into out of

a linear equation

Typical Problem: Given a network with partial information, describe the flow at each branch

Ex



Set this problem up as a SLE.

Intersection	Flow-In	Flow-Out
A	x_1+x_2	20
B	x_3	$80+x_1$
C	x_4	x_2+x_3
Total network	x_4	$20+80=100$

Get a SLE:

$$\begin{aligned} x_1 + x_2 &= 20 \\ x_3 &= x_1 + 80 \\ x_4 &= x_2 + x_3 \\ x_4 &= 100 \end{aligned} \Rightarrow \left[\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 20 \\ -1 & 0 & 1 & 0 & 80 \\ 0 & -1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 100 \end{array}\right]$$

raw reduced echelon form:

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & -80 \\ 0 & 1 & 1 & 0 & 100 \\ 0 & 0 & 0 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \leftarrow x_3 \text{ is free}$$

$$\begin{aligned} x_1 &= -80 + x_3 \\ x_2 &= 100 - x_3 \\ x_4 &= 100 \end{aligned}$$

So $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -80 + x_3 \\ 100 - x_3 \\ x_3 \\ 100 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} -80 \\ 100 \\ 0 \\ 100 \end{bmatrix}$

Interpretation: · model depends upon x_3
· constraints from model. All directions one way so all $x_i \geq 0$.

$x_2 = 100 - x_3 \geq 0 \Rightarrow 100 \geq x_3$

Thus $x_1 = -80 + x_3 \geq 0 \Rightarrow x_3 \geq 80$

Hence $100 \geq x_3 \geq 80$

Key idea: linear algebra can be useful!!