Lecture 28 — 5.1 Eigenvalues & Eigenvectors

Today's goal: introduce eigenvalues & eigenvectors

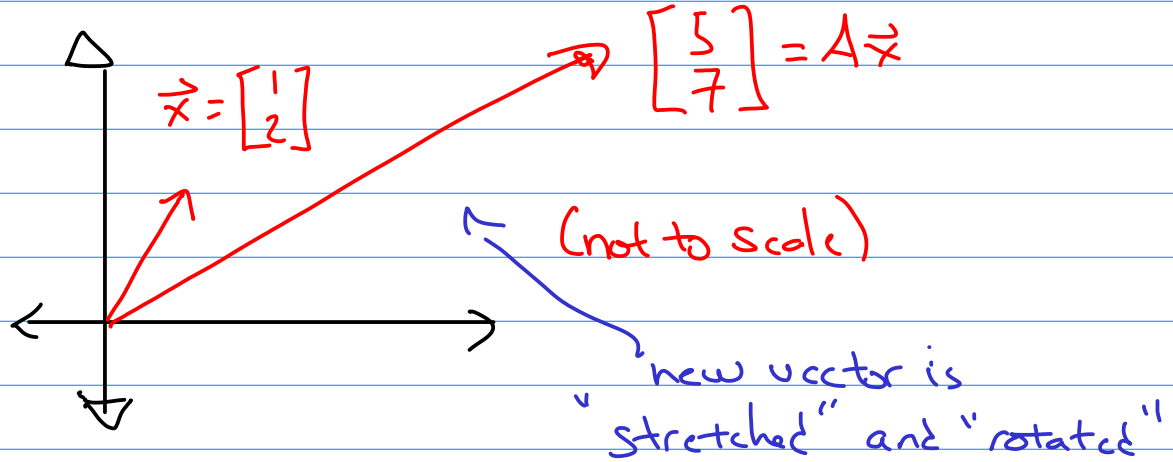
Eigenvalues & Eigenvectors

Let A be an $n \times n$ square matrix. Then A defines a linear transformation

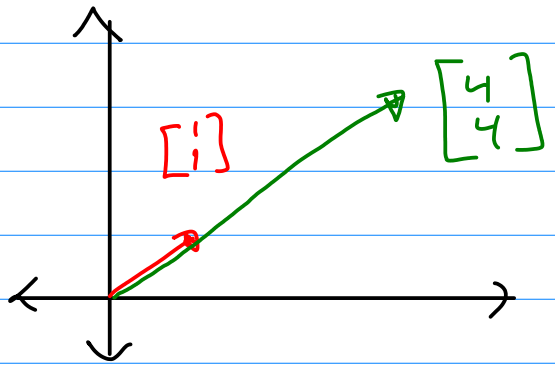
$$T: \mathbb{R}^n \longrightarrow \mathbb{R}^n \text{ by } T(\vec{x}) = A\vec{x}$$

$T(\vec{x})$ takes $\vec{x}$ to a new vector either by stretching and/or rotating the vector

Ex $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ $\vec{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $A\vec{x} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$



Repeat for $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad A\vec{x} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$



A only "stretches" $\vec{x}$

Want to capture "stretch"

Def^n An eigenvector of A is a non-zero vector $\vec{x}$ such that

$$A\vec{x} = \lambda\vec{x} \quad \text{for some scalar } \lambda.$$

The scalar λ is an eigenvalue and $\vec{x}$ is the corresponding eigenvector.

Ex $\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvalue of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ since

$$\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and $\lambda = 4$ corresponding eigenvalue

Ex Show $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 \\ 6 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 6 \end{bmatrix} = -3\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$ ← $\lambda = -3$ is the eigenvalue

Ex Show $\lambda = 3$ is an eigenvalue of $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$ and find corresponding eigenvector

Soln Need to show $A\vec{x} = 3\vec{x}$ has a non trivial soln

Note $A\vec{x} = 3\vec{x} \iff A\vec{x} = (3I_2)\vec{x}$

$$\iff A\vec{x} - 3I_2\vec{x} = \vec{0}$$
$$\iff (A - 3I_2)\vec{x} = \vec{0}$$

Need to show $(A - 3I_2)\vec{x} = \vec{0}$ has a non-trivial soln

$$(A - 3I_2) = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 0 \\ 2 & -4 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ ← x_2 is free, so non trivial soln exists!

$\Rightarrow$ $x_1 = 2x_2$ All soln's have the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{with } x_2 \in \mathbb{R}$$

If $x_2 \neq 0$, then $x_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is an eigenvector.

So $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (i.e., take $x_2 = 1$) is an eigenvector of $\lambda = 3$

CHECK: $\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ✓ ☺

Eigenspace

λ is an eigenvalue $\iff (A - \lambda I_n)\vec{x} = \vec{0}$ has a nontrivial sol^n

$(A - \lambda I_n)$ is a matrix! Use matrix language:

$$\text{Nul}(A - \lambda I_n) = \{ \vec{x} \mid (A - \lambda I_n)\vec{x} = \vec{0} \}$$

Thus, λ is an eigenvalue
$\iff \text{Nul}(A - \lambda I_n) \neq \{\vec{0}\}$ (contains more than $\vec{0}$)
$\iff \dim \text{Nul}(A - \lambda I_n) \geq 1$
$\iff A - \lambda I_n$ has a free variable

Def^ $\text{Nul}(A-\lambda I_n)$ is the eigenspace of A corresponding to λ

(it contains all eigenvectors corresponding to λ and $\{\vec{0}\}$.)

Fact
- $\text{Nul}(A-\lambda I_n)$ is a subspace of $\mathbb{R}^n$
- $\dim \text{Nul}(A-\lambda I_n)$ = # of free variables in $A-\lambda I_n$

Properties

Why care about eigenvalues & eigenvectors?

Thm A is not invertible $\iff \lambda = 0$ is an eigenvalue

Proof:

$\lambda = 0$ is an eigenvalue $\iff A\vec{x} = 0\vec{x} = \vec{0}$ has a nontrivial soln

$\iff A$ is not invertible

□

Thm If $\vec{v}_1, \ldots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of an $n \times n$ matrix A, then $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is linearly independent

Consequence An $n \times n$ matrix A has at most n distinct eigenvalues

Why? Eigenvectors $\{\vec{v}_1, \ldots, \vec{v}_r\} \subseteq \mathbb{R}^n$, and $\mathbb{R}^n$ can have at most n linearly independent vectors

Q How do we find eigenvalues?

A General case $\Rightarrow$ next class

... special case below

Thm The eigenvalues of a triangular (upper or lower) matrix are the entries on the diagonal.

Proof (3×3 case only)

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$, then $A - \lambda I_3 =$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}$$

So $(A - \lambda I_3)\vec{x} = \vec{0}$ has a nontrivial sol^n

$\Leftrightarrow$ $A - \lambda I_3$ has a free variable

$\Leftrightarrow$ $\lambda = a_{11}, a_{22}$, or a_{33}

Key ideas: eigenvalues & eigenvectors

ha!