Lecture 36

5.9 Markov Chains (Section 4.9 in 5th Ed)

Today: special type of discrete dynamical system: Markov Chains

Setup

Defn • A vector with nonnegative entries that add to 1 is probability vector

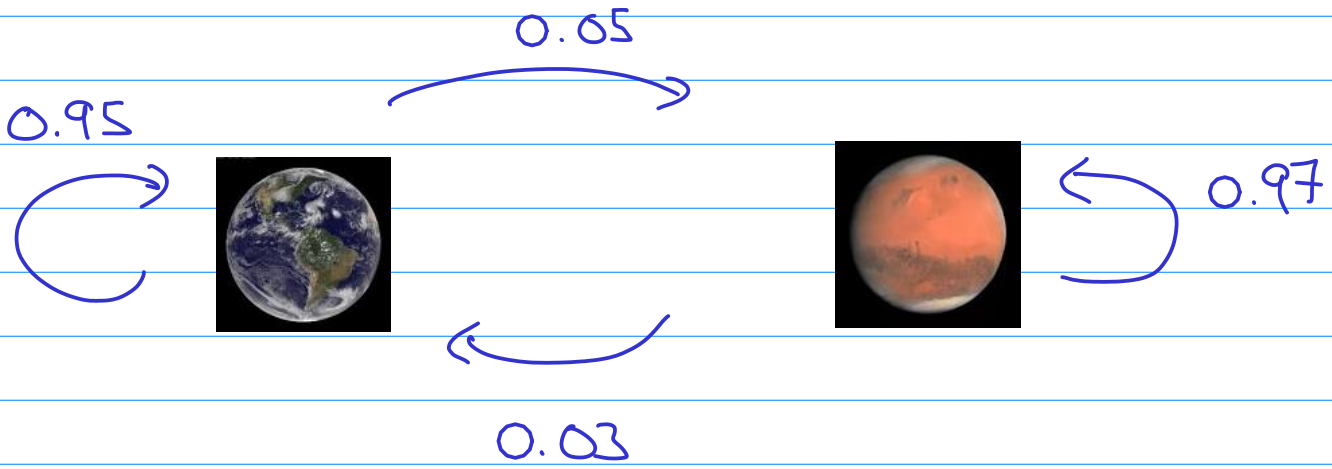
• A stochastic matrix is a square matrix whose columns are probability vectors

• A Markov Chain is a sequence of probability vectors $\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots$ and a stochastic matrix P such that

$$\vec{x}_1 = P\vec{x}_0, \quad \vec{x}_2 = P\vec{x}_1, \quad \vec{x}_3 = P\vec{x}_2, \ldots$$

Note special case of a dynamical system! (new conditions on $\vec{x}_0$ and P)

Ex (revisited) Migration between planets:



$$P = \begin{matrix} \text{From Earth} & \text{Mars} \\ \end{matrix} \quad \text{/To}$$

$$\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{matrix} \text{Earth} \\ \text{Mars} \end{matrix}$$

Columns of P probability vectors $\Rightarrow$ P stochastic matrix

probability vector

Q If $\vec{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$, $\vec{x}_1 = P\vec{x}_0$, $\vec{x}_2 = P\vec{x}_1, \ldots$ is a Markov Chain

What is $\vec{x}_k$ as $k \to \infty$?

A. Last class, used diagonalization to get

$$\vec{x}_k \to \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix} \quad \text{as} \quad k \to \infty$$

Can get this answer via different approach since P is a stochastic matrix

Observation $\begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$

* makes sense $\Rightarrow$ equilibrium for migration

* implies $\lambda = 1$ eigenvalue with $\begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$ is the associated eigenvector that is also a probability vector

Thm If P is a stochastic matrix, then $\lambda = 1$ is an eigenvalue

Proof The columns of P sum to 1, so rows of P^T sum to 1. Let $e = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

Then $P^T e = e$

(e.g. $P = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \Rightarrow P^T = \begin{bmatrix} 0.95 & 0.05 \\ 0.03 & 0.97 \end{bmatrix}$

$P^T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$)

So $\lambda = 1$ is an eigenvalue of P^T. Now

$$P^T - \lambda I_n = (P - \lambda I_n)^T$$

Since $\det(P - \lambda I_n) = \det\left((P - \lambda I_n)^T\right)$

$$= \det(P^T - \lambda I_n)$$

So P and P^T have same eigenvalues. So $\lambda = 1$ is an eigenvalue of P ∎

Steady-state vector

Def^n If P is a stochastic matrix, then a steady-state vector (or equilibrium vector) for P is the probability vector $\vec{q}$ such that

$$P\vec{q} = \vec{q}$$

Fact $\vec{q}$ steady state vector $\Leftrightarrow$ $\vec{q}$ is an eigenvector of $\lambda = 1$ that is also a probability vector

Ex Find steady-state vector for population model

Find eigenvector for $\lambda = 1$

$$P - 1 \cdot I_2 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix}$$

Solve $(P - 1I_2)\vec{x} = \vec{0}$ $\quad \begin{bmatrix} -0.05 & 0.03 \\ 0.05 & -0.03 \end{bmatrix} \sim \begin{bmatrix} 0.05 & -0.03 \\ 0 & 0 \end{bmatrix}$

x_2 free $\Rightarrow$ $x_1 = \frac{0.03}{0.05} x_2 = \frac{3}{5} x_2$

eigenspace of $\lambda = 1$

So all sol'n's $\left\{ \begin{bmatrix} 3/5 \\ 1 \end{bmatrix} t \;\middle|\; t \in \mathbb{R} \right\}$

need to pick the vector in this set that is also a probability vector

I.e $3/5\,t + t = 1 \iff 8/5\,t = 1 \Rightarrow t = 5/8$

So $\vec{q} = \begin{bmatrix} 3/5 \\ 1 \end{bmatrix} 5/8 = \begin{bmatrix} 3/8 \\ 5/8 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$ ← same answer from yesterday!

Def^n A stochastic matrix P is regular if some power P^k contains only strictly positive entries.

Ex $P = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$ ← regular $k=1$

Thm If P is an $n \times n$ regular stochastic matrix, then P has a unique steady-state vector $\vec{q}$

Also, given any probability vector $\vec{x}_0$, the Markov Chain
$\vec{x}_1 = P\vec{x}_0$, $\vec{x}_2 = P\vec{x}_1$, ...
converges to $\vec{q}$.

Note This means we can start with any probability vector $\vec{x}_0$ and end up at $\vec{q}$.

Key ideas: Markov chain
Probability vector
Steady-state vector