Lecture 34 — 5.5 Complex Eigenvalues/vectors

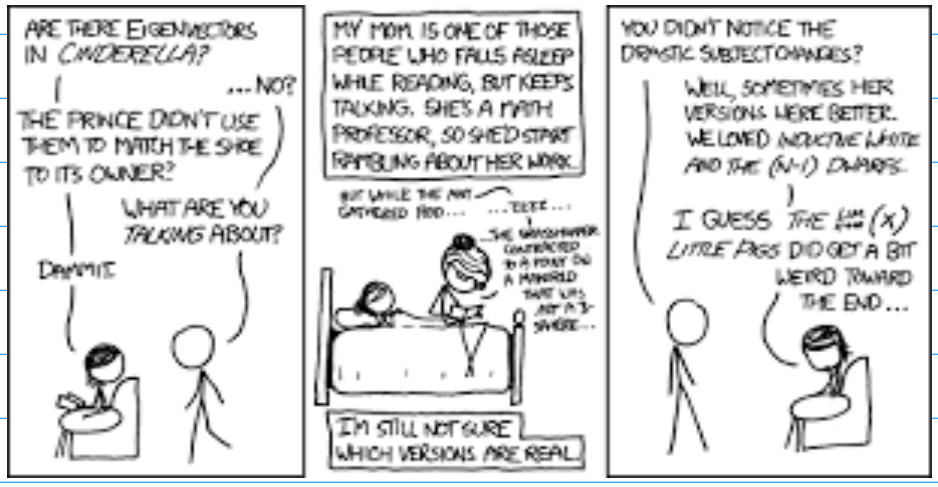
Today: Geometry of 2×2 matrices with complex eigenvalues

Complex eigenvalues and eigenvectors

Def^n If A is an $n \times n$ complex matrix, a complex scalar λ is a complex eigenvalue if there is a nonzero vector $\vec{v} \in \mathbb{C}^n$ such that

$$A\vec{v} = \lambda\vec{v}.$$

Call $\vec{v}$ the complex eigenvector



← ha!

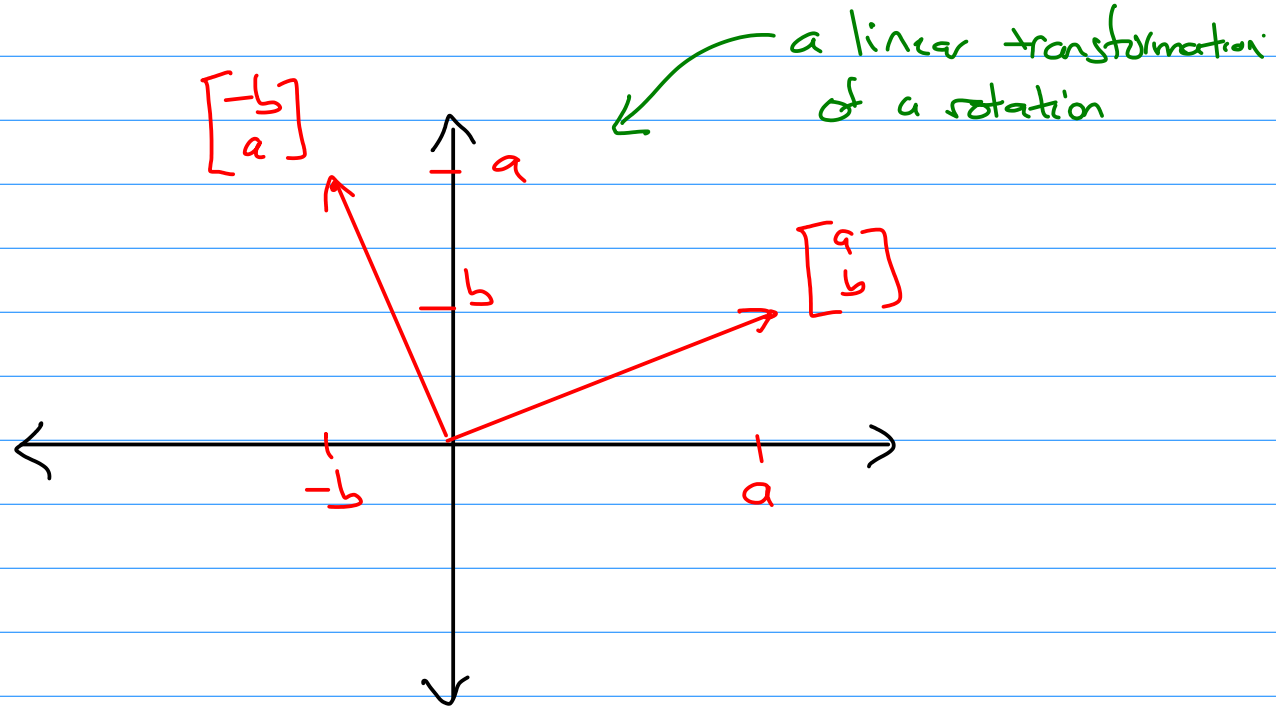
Recall over $\mathbb{R}^n$, eigenvectors correspond to "stretching"

Ex Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and consider the linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(\vec{x}) = A\vec{x}$.

Show geometrically that A has no real eigenvalues

For any point $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$, $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$

Picture:



The matrix A has no real eigenvalues, since every point is sent to a new direction

Indeed $\det(A-\lambda I_2) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2+1 = 0$

$\iff \lambda = \pm i$

BIG IDEA If $T:\mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation given by $T(\vec{x}) = A\vec{x}$ and A has complex eigenvalues, then T corresponds to a rotation.

Computing eigenvectors

conjugate of $z = a+bi \implies \overline{z} = a-bi$

Useful fact: Let A be an $n \times n$ real matrix. If λ is an eigenvalue of A with eigenvector $\vec{x}$, then $\overline{\lambda}$ is an eigenvalue of A with eigenvector $\overline{\vec{x}}$.

Proof Given $A\vec{x} = \lambda\vec{x}$. So

$$\overline{A\vec{x}} = \overline{A}\,\overline{\vec{x}} = \overline{\lambda}\,\overline{\vec{x}} = \overline{\lambda\vec{x}}$$

Since A is real, $\overline{A} = A$. So $A\overline{\vec{x}} = \overline{\lambda}\,\overline{\vec{x}}$, i.e. $\overline{\lambda}$ is an eigenvalue with eigenvector $\overline{\vec{x}}$. ∎

Problem Find eigenvectors of $A=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$

Recall, eigenvalues $\lambda = i, -i$

$\lambda = i$

$$A - i\lambda = \begin{bmatrix}-i & -1\\ 1 & -i\end{bmatrix} \sim \begin{bmatrix}1 & -i\\ 0 & 0\end{bmatrix} \quad \text{so } x_2 \text{ free}$$

$x_1 - x_2 i = 0 \iff x_1 = i x_2$

eigenvalue is $\vec{v} = \begin{bmatrix} i x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$ $\Rightarrow$ an eigenvector of $\lambda = i$ is $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda = -i$ by thm $\overline{\vec{v}} = \begin{bmatrix} \overline{i} \\ \overline{1} \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ is the

eigenvector of $\lambda = -i$

Ex $A = \begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}$ eigenvalue $\lambda = 2+i$ with eigen vector $\vec{x} = \begin{bmatrix} 2 \\ -1-i \end{bmatrix}$

$\Rightarrow \overline{\lambda} = 2-i$ is an eigenvalue with eigenvector $\begin{bmatrix} 2 \\ -1+i \end{bmatrix}$

Special case of 2×2 matrices

Thm If $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with a, b real,

then eigenvalues are $a \pm bi$

If $\lambda = a + bi = |\lambda|(\cos\theta + i\sin\theta)$, then (polar form)

$$C = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

stretching — rotating by θ

Proof (first part only)

$$\det(C - \lambda I_2) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 + b^2 = 0$$

So $(a-\lambda)^2 = -b^2 \Rightarrow a - \lambda = \pm bi \Rightarrow a \pm ib = \lambda$

Ex $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ with $a=0$, $b=1$

Eigenvalues are $\lambda = 0 \pm 1i = \pm i$ ← as before

$\lambda = 0 + i \Rightarrow \lambda = |\lambda|(\cos(\pi/2) + i\sin(\pi/2))$
$= 1(\cos(\pi/2) + i\sin(\pi/2))$

So $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix}$

"stretch" by $\times 1$ ↑ rotate by $\pi/2 = 90°$ ↑

General case of 2x2 matrices

If A is a real 2×2 matrix with complex eigenvalues, has a nice decomposition.

Thm Let A be a real 2×2 matrix with complex eigenvalue $\lambda = a - bi$ and associated eigenvector $\vec{v} \in \mathbb{C}^2$. Then

$A = PCP^{-1}$ where $P = [\mathrm{re}(\vec{v}) \ \mathrm{im}(\vec{v})]$
and $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

↑ a factorization of A

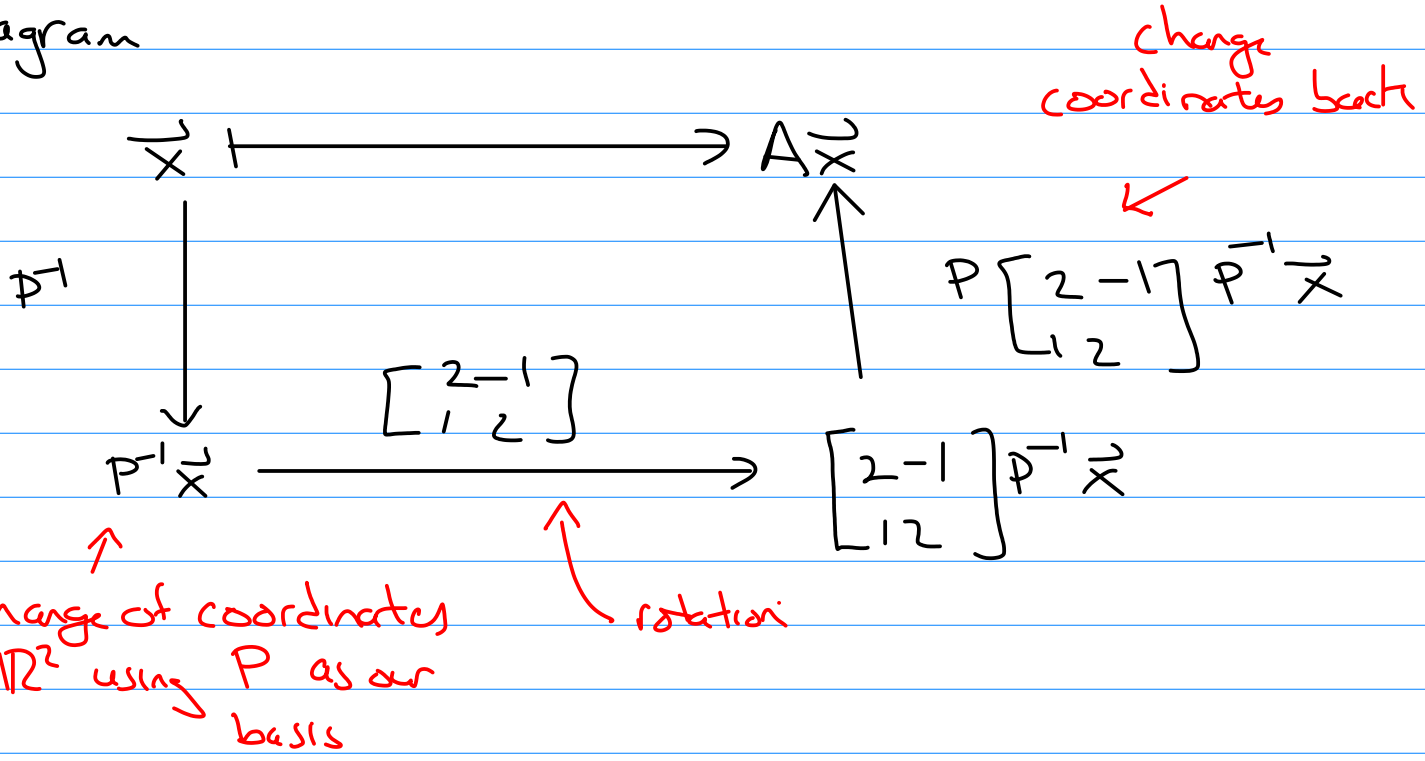
$a-bi$

Ex $A=\begin{bmatrix}1 & -2\\ 1 & 3\end{bmatrix}$ $\lambda=2-i$ and $\vec{v}=\begin{bmatrix}2\\ -1+i\end{bmatrix}$

$P=[re(\vec{v})\ im(\vec{v})]=\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix}$

So $A=\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & -1\\ 1 & 2\end{bmatrix}\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix}^{-1}$

Diagram



Key idea: 2×2 real matrices with complex eigenvalues $\Rightarrow$ rotations.