5.3 Diagonalization I

Today: introduce Similarity & diagonalization

Similarity

Let A, B be $n \times n$ matrices. Say A is **similar** to B if there exists an invertible matrix P such that
$$B = P^{-1}AP \iff A = PBP^{-1}$$

Remark: If A is similar to B, then B is similar to A

why? Given $B = P^{-1}AP$ Let $Q = P^{-1}$. Then
$$Q^{-1}BQ = (P^{-1})^{-1}BP^{-1} = PBP^{-1} = A$$

Thm If A and B are similar, then A and B have the same characteristic polynomial (and thus, same eigenvalues)

Proof Let $B = P^{-1}AP$ with P invertible. Then
$$B - \lambda I_n = P^{-1}AP - \lambda I_n = P^{-1}AP - \lambda P^{-1}P$$
$$= P^{-1}(A - \lambda I_n)P$$

$$\det(B - \lambda I_n) = \det(P^{-1}(A - \lambda I_n)P)$$
$$= \det(P^{-1}) \det(A - \lambda I_n) \det(P)$$
$$= \det(A - \lambda I_n)$$

Consequence Given A, want B which is similar whose eigenvalues are easier to compute.

Diagonalization

Easiest matrix to find eigenvalues $\Rightarrow$ diagonal matrix

$\underline{\text{Def}^n}$ A is $\underline{\text{diagonalizable}}$ if A is similar to a diagonal matrix D, i.e., exists an invertible matrix P such that
$$D = P^{-1} A P \qquad \Longleftrightarrow \qquad A = PDP^{-1}$$

$$\underset{\underset{A}{||}}{P} \qquad\qquad \underset{\underset{A}{||}}{P^{-1}}$$

$\underline{\text{Ex}}$ $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$\hookrightarrow$ eigenvalues of A $\lambda = 1, -1$

Note: A diagonalizable $\Rightarrow$ diagonal values of D are eigenvalues of A

Application fast way to compute A^k

Fact If $D = \begin{bmatrix} d_1 & & & \bigcirc \\ & d_2 & & \\ & & \ddots & \\ \bigcirc & & & d_n \end{bmatrix}$, then $D^k = \begin{bmatrix} d_1^k & & & \bigcirc \\ & d_2^k & & \\ & & \ddots & \\ \bigcirc & & & d_n^k \end{bmatrix}$

Thm If A is diagonalizable, i.e $A = PDP^{-1}$, then
$$A^k = PD^k P^{-1}$$

Proof $A^k = (PDP^{-1})^k =$

$= \underbrace{(PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})}_{k \text{ times}}$

$= PD(P^{-1}P)D(P^{-1}P) \cdots (P^{-1}P)DP^{-1}$

$= P\underbrace{DD\cdots D}_{k}P^{-1} = PD^k P^{-1}$ ▣

Ex $\begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}^{100} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^{100} \end{bmatrix}^{100} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Finding diagonalization

Q1. When is A diagonalizable?
Q2. If A is diagonalizable, how to find P and D?

A1. (Diagonalization Theorem)
  A diagonalizable $\iff$ A has n linearly independent
                          eigenvectors.

Fact: If an $n \times n$ matrix A has n distinct eigenvalues,
      then A diagonalizable

Proof: Let $\lambda_1, \dots, \lambda_n$ be n distinct eigenvalues of A.
    Let $\vec{v}_1, \dots, \vec{v}_n$ be the corresponding eigenvectors.
    These n eigenvectors are linearly independent
    (Thm 2, Sec 5.1). So, we have n linearly
    indep. eigenvectors, so A diagonalizable   ☒

A2. (Example of procedure if n distinct eigenvalues)

$$ A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix} $$

Step 1 Find all eigenvalues

$$A - \lambda I_3 = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{bmatrix}$$

$\Rightarrow \det(A-\lambda I_3)$
$= (2-\lambda)(1-\lambda)(-1-\lambda)$

Eigenvalues are
$\lambda = 2, 1, -1$ ← all distinct, can proceed.

Step 2 For each eigenvalue, find an eigenvector

$\boxed{\lambda = 2}$ $A - 2I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & 3 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

So, x_3 is free, $x_2 = x_3$ and $x_1 = x_3$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{Since } x_3 = t \in \mathbb{R}$$

Eigenspace of $\lambda = 2$ $\left\{ t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

basis of eigenspace

$\boxed{\lambda = 1}$ eigenspace $\left\{ t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\boxed{\lambda = -1}$ eigenspace $\left\{ t \begin{bmatrix} 0 \\ 1/3 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 0 \\ 1/3 \\ 1 \end{bmatrix} \right\}$

Step 3   Construct D from eigenvalues

$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

↑ Keep same order as eigenvalues

Step 4 Construct P from eigenvectors

$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1/3 \\ 1 & 1 & 1 \end{bmatrix}$

← **check!**

Then $AP = PD$ $\iff$ $A = PDP^{-1}$ ← this is the diagonalization

Q Suppose A has $< n$ distinct eigenvalues. Can it be diagonalized?

A It depends! (details are in next lecture)

Key ideas: Similarity
diagonalization
procedure to find diagonalization
with n distinct eigenvalues