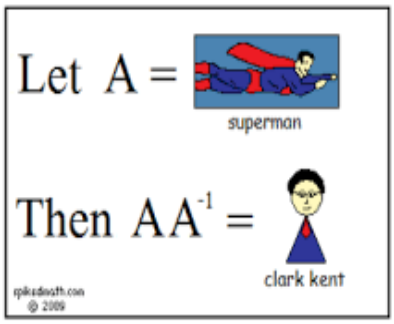
Lecture 13

Invertible matrices II (Section 2.2)
Characterization of invertible matrices (Section 2.3)

Last time: • Procedure to find A^{-1}

Today • elementary matrices & justify the procedure
• characterize invertible matrices

Use linear algebra to find the identity of superman.



Joke ↑ Ha! Ha!

Elementary Matrices

Observation

$$A=\begin{bmatrix}2&3&1\\1&0&4\\0&1&0\end{bmatrix} \xrightarrow{\text{row op}} \text{row } 2\times 6 \xrightarrow{} \begin{bmatrix}2&3&1\\6&0&24\\0&1&0\end{bmatrix}$$

Same as

$$\begin{bmatrix}1&0&0\\0&6&0\\0&0&1\end{bmatrix}\begin{bmatrix}2&3&1\\1&0&4\\0&1&0\end{bmatrix}=\begin{bmatrix}2&3&1\\6&0&24\\0&1&0\end{bmatrix}$$

Def^n A matrix E is an elementary matrix if it can be formed from I_n by a single row operation

Ex

$$E=\begin{bmatrix}1&0&0\\0&6&0\\0&0&1\end{bmatrix}$$ ← row 2 of I_3 multiplied by 6

$$E=\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}$$ ← swap rows 2 & 3 of I_3

$$E=\begin{bmatrix}1&0&5\\0&1&0\\0&0&1\end{bmatrix}$$ ← add 5 × row 3 + row 1 of I_3

Ex $\begin{bmatrix}1&2&3\\1&0&1\\1&0&2\end{bmatrix} \xrightarrow{\text{row3}\times 5+\text{row1}} \begin{bmatrix}6&2&13\\1&0&1\\1&0&2\end{bmatrix}$

Same as $\begin{bmatrix}1&0&5\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1&2&3\\1&0&1\\1&0&2\end{bmatrix}=\begin{bmatrix}6&2&13\\1&0&1\\1&0&2\end{bmatrix}$

Consequence each elementary row operation can be represented by multiplication by an elementary matrix.

Fact Every elementary matrix has an inverse which is also an elementary matrix (just reverse the row operation)

Ex $E = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← row 3 × 5 + row 1 $\quad E^{-1} = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ row 3 × (−5) + row 1 of I_3

$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← row 2 × 6 $\quad E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ← row 2 × 1/6

Thm An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n

Proof ($\Rightarrow$) Suppose A is invertible.

Last class, $A\vec{x} = \vec{b}$ has a sol^n for all $\vec{b} \in \mathbb{R}^n$.

So A has a pivot in every row. But then A has a pivot column since A is $n \times n$. So A reduces to I_n

($\Leftarrow$) Can reduce A to I_n via basic row operations

$$A \sim A_1 \sim A_2 \sim \cdots \sim A_t = I_n$$

Each operation can be represented by multiplication by an elementary matrix

$$A_1 = E_1 A \quad A_2 = E_2 A_1, \ldots, \quad A_t = E_t A_{t-1} = I_n$$

So

$$I_n = E_t E_{t-1} \cdots E_1 A$$

Each E_i invertible, so E_i^{-1} exists. Then

$$(E_1^{-1} E_2^{-1} \cdots E_{t-1}^{-1} E_t^{-1})(E_t E_{t-1} \cdots E_1 A) = E_t^{-1} \cdots E_1^{-1} I_n$$

$$\Rightarrow A = E_1^{-1} \cdots E_t^{-1}$$

So A is invertible since it is a product of invertible matrices

Consequence: Note that

$$A = E_1^{-1} \cdots E_t^{-1} \Rightarrow (A)^{-1} = (E_1^{-1} \cdots E_t^{-1})^{-1}$$

$$\Rightarrow A^{-1} = (E_t^{-1})^{-1} (E_{t-1}^{-1})^{-1} \cdots (E_1^{-1})^{-1}$$

$$= E_t E_{t-1} \cdots E_1$$

So operations that change A to I_n, i.e

$$I_n = E_t \cdots E_1 A$$

change I_n to A^{-1}, i.e.

$$A^{-1} = E_t E_{t-1} \cdots E_1 I_n$$

Justifies procedure!

(Invertible Matrix Theorem) Let A be a square $n \times n$ matrix. The following are equivalent (all true or false)

(a) A is an invertible matrix ↩ just proved!
(b) A is row equivalent to I_n
(c) A has n pivot positions
(d) The equation $A\vec{x} = \vec{0}$ has only the trivial sol^n
(e) The columns of A form an independent set
(f) The linear transf. $T(\vec{x}) = A\vec{x}$ is one-to-one
(g) The equation $A\vec{x} = \vec{b}$ has at least one sol^n for all $\vec{b} \in \mathbb{R}^n$
(h) columns of A span $\mathbb{R}^n$ ↑ saw last class
(i) The linear transf. $T(\vec{x}) = A\vec{x}$ maps onto $\mathbb{R}^n$
(j) There is a matrix C such that $CA = I_n$
(k) There is a matrix D such that $AD = I_n$
(l) A^T is invertible.

Ex Is $A = \begin{bmatrix} 5 & 0 & 0 \\ -3 & -7 & 0 \\ 2 & 6 & 2020 \end{bmatrix}$ invertible?

Look at $A\vec{x} = \vec{0}$. Corresponding SLE

$$\left.\begin{aligned} 5x_1 &= 0 \\ -3x_1 - 7x_2 &= 0 \\ 2x_1 + 6x_2 + 2020x_3 &= 0 \end{aligned}\right\}$$

only sol^n is $x_1 = x_2 = x_3 = 0$

trivial sol^n

So, A is invertible since (d) is true

Thm Let A be an $n \times n$ matrix

1. If B is a square-matrix such that $BA = I_n$, then $B = A^{-1}$
2. If B is a square-matrix such that $AB = I_n$, then $B = A^{-1}$

Note By defn, B is the inverse of A if $BA = I_n$ and $AB = I_n$. Thm says you only need to check one.

Proof of 2

Let $\vec{x}_0$ be any soln to $B\vec{x}_0 = \vec{0}$. Then

$$\vec{x}_0 = I_n \vec{x}_0 = (AB)\vec{x}_0 = A(B\vec{x}_0) = A\vec{0} = \vec{0}$$

So only soln to $B\vec{x} = \vec{0}$ is trivial soln. By classification theorem, B is invertible. Then

$$A = A(BB^{-1}) = (AB)B^{-1} = I_n B^{-1} = B^{-1}.$$

So $A = B^{-1} \iff A^{-1} = (B^{-1})^{-1} = B$ ∎

Key ideas:
- elementary matrices
- justification of procedure
- invertible matrix thm.