Lecture 31 5.3 Diagonalization II

Last class: A diagonalizable $\Leftrightarrow$ exists a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$

Today how to determine if A is diagonalizable

Fact: If A is an $n \times n$ matrix and has n distinct eigenvalues, then A is diagonalizable

Ex Diagonalize $A = \begin{bmatrix} 1 & 0 \\ 6 & -1 \end{bmatrix}$

Step 1 Find eigenvalues (short cut: A is triangular, so diagonal entries are eigenvalues).

$\lambda = 1, -1$ ← distinct

Step 2 Find eigenspaces

$\boxed{\lambda = 1}$ $A - I_2 = \begin{bmatrix} 0 & 0 \\ 6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$ x_2 free, $x_1 = \frac{1}{3}x_2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}t \\ t \end{bmatrix} = t\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix} \Rightarrow$ eigenspace of $\lambda = 1$ $= \text{span}\left\{\begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}\right\}$

$\boxed{\lambda = -1}$ $A - (-1)I_2 = \begin{bmatrix} 2 & 0 \\ 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ x_2 free, $x_1 = 0$

So $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ t \end{bmatrix} = t\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow$ eigenspace of $\lambda = -1$ $= \text{Span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$

Step 3 $\quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

Step 4 $\quad P = \begin{bmatrix} 1/3 & 0 \\ 1 & 1 \end{bmatrix}$

So: $A = \begin{bmatrix} 1/3 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}P^{-1}$

Check: $AP = PD$

Q. Can we diagonalize if A has $< n$ eigenvalues

A. It depends (on the size of eigenspaces!)

Algebraic multiplicity

If λ_0 is an eigenvalue of A, algebraic multiplicity of λ_0 is the exponent of $(\lambda_0 - \lambda)$ in the factorization of $\det(A - \lambda I_n)$

Ex $A = \begin{bmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{bmatrix}$ $\det(A - \lambda I_3) = (1-\lambda)^1(2+\lambda)^2$

$\lambda = 1$ eigenvalue with alg. mult $= 1$
$\lambda = -2$ eigenvalue with alg. mult $= 2$

Geometric multiplicity

If λ_0 is an eigenvalue of A, geometric multiplicity of λ_0 is

dimension of the eigenspace of λ_0
$=$ # of free variables in $A - \lambda_0 I_n$
$=$ dim Nul$(A - \lambda_0 I_n)$

Ex A as above with $\lambda = -2$

$$A - (-2)I_3 = \begin{bmatrix} 4 & 4 & 3 \\ -4 & -4 & -3 \\ 3 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 4 & 4 & 3 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow x_2 \text{ free}$$

geometric multiplicity of $\lambda = -2$ is 1

$\iff \dim \text{Nul}(A + 2I_n) = 1$

Thm For all eigenvalues λ of A

$1 \leq$ geometric mult of $\lambda \leq$ alg. mult of λ

$=$ $\dim \text{Nul}(A - \lambda I_n)$

$=$ dim of eigenspace of λ

Thm Suppose A is an $n \times n$ matrix with r distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. Then

A is diagonalizable $\iff$

geo. mult. of λ_i = alg. mult of λ_i for $i = 1, \ldots, r$

Ex previous matrix A

(geo mult of $\lambda = -2$) $= 1$
(alg. mult of $\lambda = -2$) $= 2$ not equal

Ex diagonalize $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$

Step 1 Find eigenvalues: $\det(A - \lambda I_3) = -\lambda^2(\lambda + 2)$

$\lambda = -2$ with alg. mult 1
$\lambda = 0$ with alg. mult 2.

Step 2 For each λ, find basis of eigenspace

$\boxed{\lambda = 0}$ $\quad A - 0I_3 = A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

two free variables $\Rightarrow$ (geo. mult of $\lambda = 0$) $= 2$

free variables $x_2 = s$ and $x_3 = t$
also $x_1 = x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

eigenspace of $\lambda = 0$ $= \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

$\boxed{\lambda = -2}$ $A-(-2)I_3 = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

x_3 free (geo. mul of $\lambda=-2$) $= 1 =$ alg. mult

$x_2 = 3x_3$ and $x_1 = -x_3$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 3 \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} \Rightarrow \text{eigenspace of } \lambda=-2 = \text{span}\left\{\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}\right\}$$

We <u>can</u> diagonalize

<u>Step 3</u> $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ since $\lambda=0$ has alg. mult. 2, put it on diagonal 2 times

<u>Step 4</u> $P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ ← same order as D

basis for eigenspace of $\lambda=0$

So $A = PDP^{-1}$

Fact If A is $n \times n$ w/ distinct eigenvalues $\lambda_1, \dots, \lambda_r$ and m_i = alg. mult. of λ_i for $i = 1, \dots, r$, then

$$m_1 + m_2 + \cdots + m_r = n$$

Problem A matrix A has characteristic equation

$$\lambda^2(1-\lambda)(2-\lambda)^3 = 0$$

- What is the size of A?
- How big can the eigenspace of $\lambda = 2$ be?

A1 $2 + 1 + 3 = 6 \Rightarrow A$ is 6×6 matrix

A2 $\dim \mathrm{Nul}(A - 2I_6) \leq 3$

Note If A not diagonalizable, can <u>almost</u> diagonalize using Jordan Form (not discussed!)

Key ideas * algebraic and geometric multiplicity
* diagonalization theorem