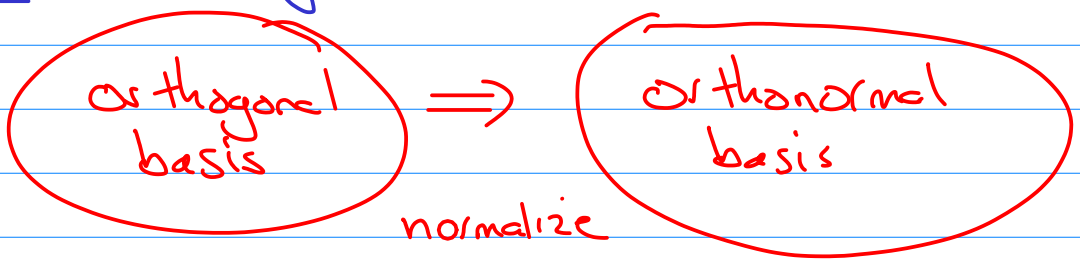
Lecture 25 Gram-Schmidt II

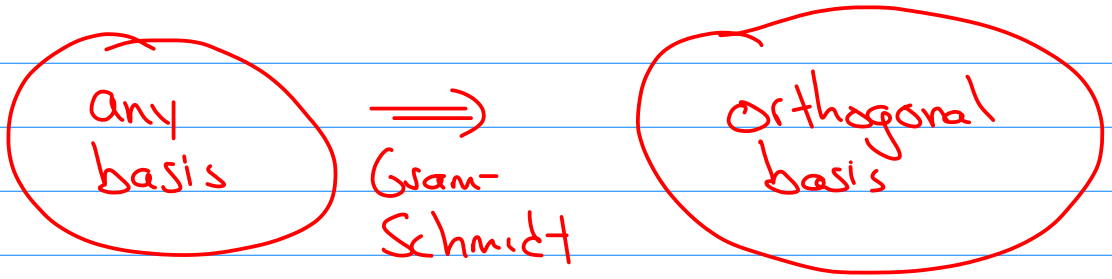
6.3 Orthogonal Projections

6.4 Gram-Schmidt Process $\langle \vec{u}, \vec{v} \rangle = \vec{u} \cdot \vec{v}$

Last time: orthogonal & orthonormal basis



Today



Gram-Schmidt Process

Given a basis $S = \{\vec{x}_1, \ldots, \vec{x}_n\}$ of $\mathbb{R}^n$, define

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1$$

note: $v_i \cdot v_i = \|\vec{v}_i\|^2$

$$\vec{v}_3 = \vec{x}_3 - \left(\frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$$

Some books write $\|\vec{v}_i\|^2$

$$\vdots$$

$$\vec{v}_n = \vec{x}_n - \left(\frac{\vec{x}_n \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{x}_n \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 - \cdots - \left(\frac{\vec{x}_n \cdot \vec{v}_{n-1}}{\vec{v}_{n-1} \cdot \vec{v}_{n-1}}\right)\vec{v}_{n-1}$$

Then $\{\vec{v}_1, \ldots, \vec{v}_n\}$ is an orthogonal basis for $\mathbb{R}^n$

Note If $\{\vec{w}_1, \ldots, \vec{w}_t\}$ is a basis for a subspace $W \subseteq \mathbb{R}^n$, then G-S applied to $\{\vec{w}_1, \ldots, \vec{w}_t\}$ produces an orthogonal basis for W.

Example Find an orthogonal basis for $W = \text{span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ where

$$\vec{w}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \qquad \vec{w}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{w}_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Soln $\vec{v}_1 = \vec{w}_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

$$\vec{v}_2 = \vec{w}_2 - \left(\frac{\vec{w}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Computational trick = can rescale so no fractions!

So $\vec{v}_2 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}$

$$\vec{v}_3 = \vec{w}_3 - \left(\frac{\vec{w}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 - \left(\frac{\vec{w}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{1}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{15}{20}\begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}$$

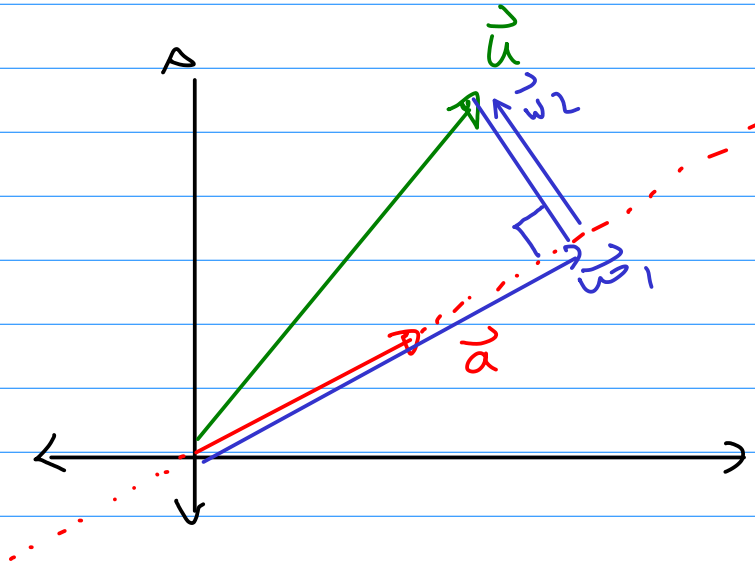
Can rescale so $\vec{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$

So $\left\{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}\right\}$ is an orthogonal basis of W

why this works ⇒ projection

Fix a vector $\vec{a} \in \mathbb{R}^n$. Given any vector $\vec{u}$, want to decompose $\vec{u}$ into a scalar multiple of $\vec{a}$ and a vector orthogonal to $\vec{a}$

Picture



← line = span$\{\vec{a}\}$ = W

$\vec{u} = \vec{w}_1 + \vec{w}_2$

↑ scalar multiple of $\vec{a}$

↑ orthogonal to $\vec{a}$

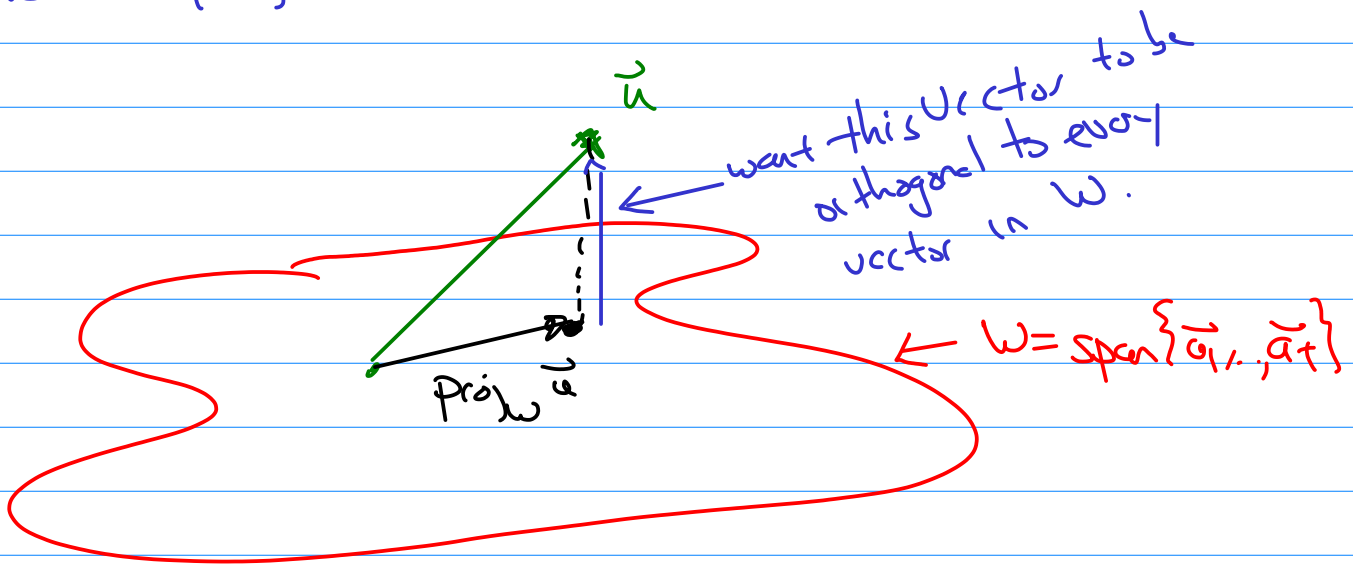
$\vec{w}_1$ = orthogonal projection of $\vec{u}$ onto $W = \text{span}\{\vec{a}\}$
$= \text{proj}_W \vec{u}$

$\vec{w}_2$ = complement of $\vec{u}$ orthogonal to W

Fact If $W = \text{span}\{\vec{a}\}$, $\text{proj}_W \vec{u} = \vec{w}_1 = \left(\frac{\vec{u} \cdot \vec{a}}{\vec{a} \cdot \vec{a}}\right)\vec{a}$

Idea generalizes: If $W = \text{span}\{\vec{a}_1, \ldots, \vec{a}_t\}$ and $\vec{u} \in \mathbb{R}^n$

$\text{proj}_W \vec{u}$ = projection of $\vec{u}$ onto W



Fact: If $W = \text{span}\{\vec{a}_1, \ldots, \vec{a}_p\}$ and $\vec{a}_1, \ldots, \vec{a}_p$ orthogonal

$$\text{proj}_W \vec{u} = \left(\frac{\vec{u}\cdot\vec{a}_1}{\vec{a}_1\cdot\vec{a}_1}\right)\vec{a}_1 + \left(\frac{\vec{u}\cdot\vec{a}_2}{\vec{a}_2\cdot\vec{a}_2}\right)\vec{a}_2 + \cdots + \left(\frac{\vec{u}\cdot\vec{a}_p}{\vec{a}_p\cdot\vec{a}_p}\right)\vec{a}_p$$

Connection to G-S: At step p, we compute

$$\vec{v}_p = \vec{x}_p - \left[\left(\frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 + \left(\frac{\vec{x}_p \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2} \right) \vec{v}_2 + \cdots + \left(\frac{\vec{x}_p \cdot \vec{v}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}} \right) \vec{v}_{p-1} \right]$$

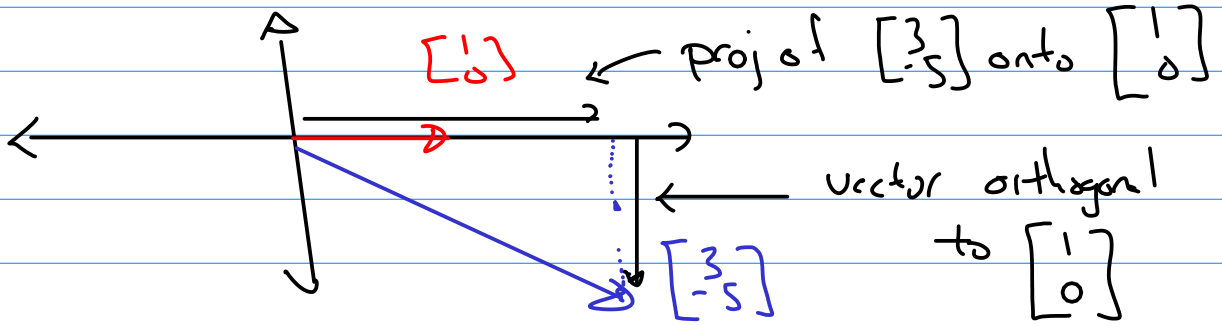
projection of $\vec{x}_p$ onto $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{p-1}\}$

Vector orthogonal to every vector in $\operatorname{Span}\{\vec{v}_1, \ldots, \vec{v}_{p-1}\}$

Ex Find an orthogonal basis for $\mathbb{R}^2$ if $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \end{bmatrix} \right\}$

Soln $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\vec{v}_2 = x_2 - \left(\frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \right) \vec{v}_1 = \begin{bmatrix} 3 \\ -5 \end{bmatrix} - \frac{3}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$



Key ideas: · Gram-Schmidt
· projections