Lecture 9 Matrix of a linear transformation (Sec 1.9)

Today's lecture * show every linear transformation is a matrix transf.
* geometry of linear transformation
* onto and one-to-one

I. Standard Matrix

Recall A linear transformation is a function $T:\mathbb{R}^n \to \mathbb{R}^m$ such that
- $T(\vec{u}+\vec{v}) = T(\vec{u})+T(\vec{v})$ for all $\vec{u},\vec{v}\in\mathbb{R}^n$
- $T(c\vec{u}) = cT(\vec{u})$ for all $c\in\mathbb{R}, \vec{u}\in\mathbb{R}^n$

Def^n $n\times n$ identity matrix $I_n = \underbrace{\begin{bmatrix} 1 & & & 0 \\ 0 & 1 & & \\ 0 & & \ddots & \\ & & & 1 \end{bmatrix}}_{n}$ n

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Notation $\vec{e}_j = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$ ← j^{th} spot $\Leftrightarrow$ j^{th} column of I_n

Def^n $\{\vec{e}_1, \ldots, \vec{e}_n\}$ is the standard basis of $\mathbb{R}^n$

Fact If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then T is completely determined by what T does to $\vec{e}_1, \ldots \vec{e}_n$.

Ex Suppose $T: \mathbb{R}^2 \to \mathbb{R}^4$ is a linear transformation

$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ ← standard basis of $\mathbb{R}^2$

Suppose $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

Find a formula for $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$ for <u>any</u> $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

Observation $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= T\left(x_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + T\left(x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \quad \text{(by prop of lin. transf.)}$$

$$= x_1 T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + x_2 T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \quad \text{(by prop of lin. transf.)}$$

matrix transformation

$$= x_1\begin{bmatrix}1\\2\\3\\4\end{bmatrix} + x_2\begin{bmatrix}1\\1\\0\\0\end{bmatrix} = \begin{bmatrix}x_1+x_2\\2x_1+x_2\\3x_1+0\\4x_1+0\end{bmatrix} = \begin{bmatrix}1&1\\2&1\\3&0\\4&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

output of $T(\vec{e}_1)$ — output of $T(\vec{e}_2)$

Thm Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation. Then there exists a (unique) matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$

(i.e. every linear transf. is a matrix transf.)

Procedure to find A

① Compute $T(\vec{e}_1), T(\vec{e}_2), \ldots, T(\vec{e}_n)$ for standard basis elements of $\mathbb{R}^n$

② $A = [\,T(\vec{e}_1)\ T(\vec{e}_2) \cdots T(\vec{e}_n)\,]$

Defn A is called the standard matrix for linear transformation T

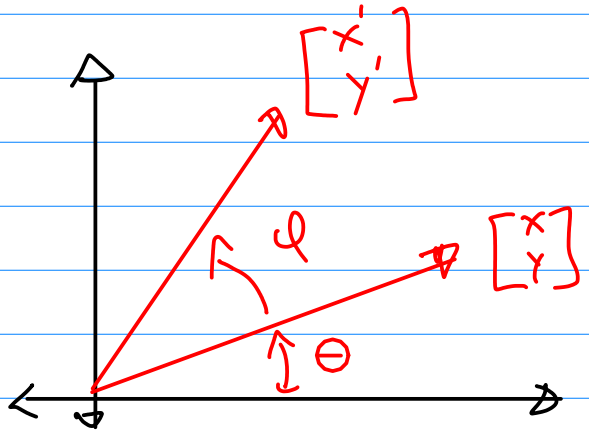
Geometry of linear transformation in $\mathbb{R}^2$

Many geometric transformations (e.g. reflection, dilation, rotation, shear) in $\mathbb{R}^2$ are examples of linear transformations

Ex Fix angle φ. Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that rotates each point in $\mathbb{R}^2$ about origin through angle φ

(1) Show T is a linear transf.

(2) Find standard matrix of T



Given $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x' \\ y' \end{bmatrix}$

Need to express x', y' in terms of x, y, and φ

$r = \sqrt{x^2+y^2}$ = length of $\begin{bmatrix} x \\ y \end{bmatrix}$ = length of $\begin{bmatrix} x' \\ y' \end{bmatrix}$

So $x = r\cos\theta$ and $y = r\sin\theta$

$x' = r\cos(\theta+\varphi)$ and $y' = r\sin(\theta+\varphi)$

Apply trig identities

$$x' = r(\cos\theta\cos\varphi - \sin\theta\sin\varphi)$$
$$= r\cos\theta\cos\varphi - r\sin\theta\sin\varphi$$
$$= x\cos\varphi - y\sin\varphi$$
$$y' = r(\sin\theta\cos\varphi + \cos\theta\sin\varphi)$$
$$= y\cos\varphi + x\sin\varphi$$

So

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x\cos\varphi - y\sin\varphi \\ y\cos\varphi + x\sin\varphi \end{bmatrix}$$

can check this is a linear transformation

For standard matrix

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \qquad T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$$

So

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

See the text for other matrices

Onto and one-to-one

Def^n A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>onto</u> (or surjective) if for each $\vec{b} \in \mathbb{R}^m$, there exists some $\vec{x} \in \mathbb{R}^n$ such that $T(\vec{x}) = \vec{b}$

A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is <u>one-to-one</u> if for each $\vec{b} \in \mathbb{R}^m$, the equation $T(\vec{x}) = \vec{b}$ has either a unique sol^n or no <u>sol^n</u>

$\iff$ for all $\vec{x} \neq \vec{y}$ in $\mathbb{R}^n$, then $T(\vec{x}) \neq T(\vec{y})$

Note

- Onto is asking about existence, i.e. does there exist an $\vec{x}$ such that $T(\vec{x}) = \vec{b}$
- One-to-one is asking about uniqueness, i.e., is there only one $\vec{x}$ such that $T(\vec{x}) = \vec{b}$

When T is a linear transf, onto and one-to-one encoded in standard matrix

Thm Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transf with standard matrix A. Then

(a) T is onto $\mathbb{R}^m$ if and only if $T(\vec{x}) = A\vec{x} = \vec{b}$ has a sol^n for all $\vec{b} \in \mathbb{R}^m$

if and only if columns of A span $\mathbb{R}^m$

if and only if A has a pivot in each row

(b) T is one-to-one if and only if columns of A are linearly independent

if and only if $A\vec{x} = \vec{0}$ has only trivial sol^n

if and only if A has no free variables!

Ex The linear transf $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + x_2 + 4x_3 \\ x_2 - x_3 \end{bmatrix}$$

Has standard matrix $A = \begin{bmatrix} \textcircled{2} & 1 & 4 \\ 0 & \textcircled{1} & -1 \end{bmatrix}$

↑ pivots

Since A has a pivot in each row, T is onto
Since A has a free variable, T is not one-to-one

Key ideas
* Standard matrix
* one-to-one, onto