LECTURE 26 — 4.5 Dimension of a Vector Space

Today's topic: dimension (the "size" of a vector space)

Dimension

Note that $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}3\\-5\end{bmatrix}\right\}$, $\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\-5\end{bmatrix}\right\}$

all bases of $\mathbb{R}^2$. Although different, all have the same number of elements.

Coincedence? NO

Thm If V is a vector space with a basis with n elements, then every basis of V has n elements.

Def^n If V has a basis β, then
V is finite dimensional if $|\beta| < \infty$

- The dimension of V is $\dim V = |\beta|$
- If $V = \{\vec{0}\}$, then $\dim V = 0$
- If V does not have a finite basis, then
V is infinite dimensional

Ex

1. basis for $\mathbb{R}^n = \{\vec{e}_1, \ldots, \vec{e}_n\} \Rightarrow \dim \mathbb{R}^n = n$

2. basis for $P_n = \{1, t, t^2, \ldots, t^n\} \Rightarrow \dim P_n = n+1$

3 basis for $M_{2\times 2} = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$

$\Rightarrow \dim M_{2\times 2} = 4$

4. $\dim M_{m\times n} = mn$

5. $P_\infty = \text{span}\{1, t, t^2, \dots\} \Rightarrow \dim P_\infty = \infty$

all polynomials of any degree

Since the basis has an infinite # of elements.

Properties of dimension

Thm Let V be an n-dimensional vector space with basis $\{\vec{v}_1, \dots, \vec{v}_n\}$.

1. If a set of vectors in V has $> n$ vectors, then the set of vectors is linearly dependent

2. If a set of vectors in V has $< n$ vectors, then the set of vectors cannot span V.

Ex Since $\dim \mathbb{R}^2 = 2$

- any three vectors in $\mathbb{R}^2$ are linearly dependent
- for any $\vec{v} \in \mathbb{R}^2$, $\text{span}\{\vec{v}\} \neq \mathbb{R}^2$

Thm (Bases Thm) Let V be any vector space with $\dim V = p \geq 1$

1. Any linearly independent set of p vectors is a basis for V.

2. Any set of p vectors that spans V is a basis for V

Proof 1. Suppose $\vec{v}_1, \ldots, \vec{v}_p$ is linearly independent.

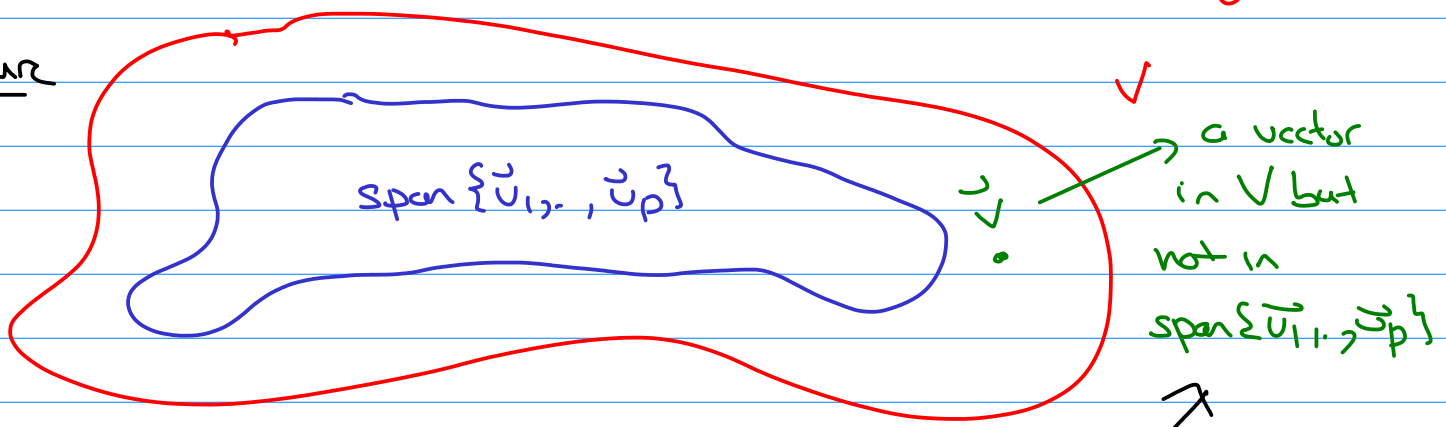
Claim

$$V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$$

Suppose $\text{span}\{\vec{v}_1, \ldots, \vec{v}_p\} \subsetneq V$

$\subsetneq$ contained in V but not equal

Picture



Take $\vec{v} \in V \setminus \text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$

Then $\vec{v}_1, \ldots, \vec{v}_p, \vec{v}$ must be linearly independent. If they weren't, will have

$$c_0\vec{v} + c_1\vec{v}_1 + \cdots + c_p\vec{v}_p = \vec{0} \text{ with } c_0 \neq 0 \iff$$

$$\vec{v} = -\left(\frac{c_1}{c_0}\right)\vec{v}_1 - \left(\frac{c_2}{c_0}\right)\vec{v}_2 - \cdots - \left(\frac{c_p}{c_0}\right)\vec{v}_p \in \text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$$

So V has at least $p+1$ linear vectors in a $\dim V = p$ vector space, contradicting first theorem. So no such $\vec{v}$ exists, i.e

$$V = \text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$$ ∎

Big idea: if you know two of the three (dimension, spanning, linear indep), you also get info about the third.

Additional facts:

1. If $\vec{v}_1, \ldots, \vec{v}_p$ linearly independent
   $\dim(\text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}) = p$

2. If $\vec{v}_1, \ldots, \vec{v}_p$ are linearly independent in V, then it can be extended to a basis for V (by adding additional linear indep. elements)

3. If $H \subseteq V$ is a subspace, $\dim H \leq \dim V$

Ex (classifying subspaces of $\mathbb{R}^3$ by dimension)

0-dimensional subspace: only one $= \{\vec{0}\}$ origin
1-dimensional subspace: infinite # = any line through the origin
2-dimensional subspace infinite # = any plane through the origin
3-dimensional subspace only one $= \mathbb{R}^3$

Ex Find the dimension of the subspace $H \subseteq \mathbb{R}^2$ spanned by

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \end{bmatrix}, \quad \text{i.e.} \quad H = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ -6 \end{bmatrix} \right\}$$

Soln Answer is not 3!

$H \subseteq \mathbb{R}^2$, so $\dim H = 0, 1, \text{or } 2$

Note $\begin{bmatrix} -2 \\ 4 \end{bmatrix} = -2\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ $\quad \begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

So $H = \text{span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\} \Rightarrow \dim H = 1$

(need to remove linear dependent vectors)

Given an $m \times n$ matrix A, we saw how to compute bases for

$\text{Nul}(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$ and $\text{Col}(A) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$

Facts

- $\dim \text{Nul}(A) = \#$ free variables in equation $A\vec{x} = \vec{0}$
- $\dim \text{Col}(A) = \#$ pivot columns of A

Ex $A = \begin{bmatrix} \textcircled{1} & -2 & 0 & -1 & 3 \\ 0 & 0 & \textcircled{1} & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ (pivots)

$\Rightarrow \dim \text{Nul}(A) = 3 \qquad \dim \text{Col}(A) = 2$

Ex First three Hermite polynomials:

$p_1(t) = 1, \quad p_2(t) = 2t, \quad p_3(t) = -2 + 4t^2$

Show $\{p_1(t), p_2(t), p_3(t)\}$ is a basis for $\mathbb{P}_2$

Soln $\dim \mathbb{P}_2 = 3$, so we only need to show $p_1(t), p_2(t), p_3(t)$ are linearly independent

$$c_1 \cdot 1 + c_2 \cdot (2t) + c_3(-2 + 4t^2)$$

$$= (c_1 \cdot 1 - 2c_3) + (2c_2)t + (4c_3)t^2 = 0 + 0t + 0t^2$$

$$\iff \quad 0 = c_3 = c_2 = c_1$$

By bases theorem, $p_1(t), p_2(t), p_3(t)$ is a basis since we have 3 linearly indep. elements in a 3-dimensional vector space

Key idea: dimension