Lecture 14 Characterization of invertible matrices II (Section 2.3) Partition of Matrices (Section 2.4)

Today: • finish discussion of invertible matrices
• introduce partitions of matrices.

(Classification Thm) Let A be an $n \times n$ matrix. The following are equivalent:

(a) A invertible
(b) A is row equivalent to I_n
(d) $A\vec{x} = \vec{0}$ has only the trivial sol^n
(f) linear transformation $T(\vec{x}) = A\vec{x}$ is one-to-one
(i) linear transformation $T(\vec{x}) = A\vec{x}$ is onto
(l) A^T is invertible

} a subset from the previous lecture

Ex For what k is the matrix below invertible?

$$\begin{bmatrix} k & 0 & 0 \\ 1 & k & 0 \\ 0 & 1 & k \end{bmatrix}$$

$$\left[\begin{array}{ccc:ccc} k & 0 & 0 & 1 & 0 & 0 \\ 1 & k & 0 & 0 & 1 & 0 \\ 0 & 1 & k & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/k & 0 & 0 \\ 1 & k & 0 & 0 & 1 & 0 \\ 0 & 1 & k & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/k & 0 & 0 \\ 0 & k & 0 & -1/k & 1 & 0 \\ 0 & 1 & k & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/k & 0 & 0 \\ 0 & 1 & 0 & -1/k^2 & 1/k & 0 \\ 0 & 1 & k & 0 & 0 & 1 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/k & 0 & 0 \\ 0 & 1 & 0 & -1/k^2 & 1/k & 0 \\ 0 & 0 & k & 1/k^2 & -1/k & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/k & 0 & 0 \\ 0 & 1 & 0 & -1/k^2 & 1/k & 0 \\ 0 & 0 & 1 & 1/k^3 & -1/k^2 & 1/k \end{array}\right]$$

matrix is invertible $\Leftrightarrow k \neq 0$.

Defn An $n \times n$ matrix is

1. upper triangular if all nonzero entries are on or above diagonal
2. lower triangular if all nonzero entries are on or below diagonal

Ex

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

upper triangular

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 0 \\ 6 & 0 & 8 \end{bmatrix}$$

lower triangular

Fact Triangular matrices invertible if and only if diagonal entries not zero.

Why? If A upper triangular

diagonal entries are not zero $\iff$ pivots on the diagonal
$\iff$ A can be reduced to I_n
$\iff$ A is invertible

If A lower triangular, then A^T upper triangular

A invertible $\iff$ A^T is invertible $\iff$ diagonal entries are not zero

Ex This explains the first example, i.e. $k \neq 0$

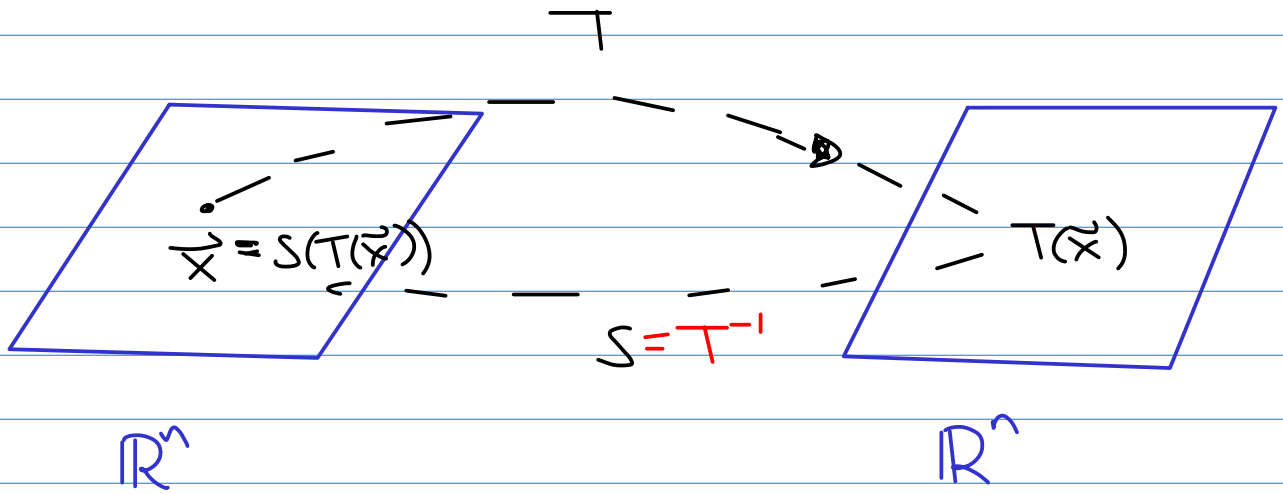
Linear transformations + invertible matrices

Def A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if there exists a linear transformation $S: \mathbb{R}^n \to \mathbb{R}^n$ such that

- $S(T(\vec{x})) = \vec{x}$ for all $\vec{x} \in \mathbb{R}^n$
- $T(S(\vec{x})) = \vec{x}$

S is called the inverse of T and denoted T^{-1}

Picture



Recall A linear transf T has a standard matrix A, i.e
$T(\vec{x}) = A\vec{x}$

Thm Let $T: \mathbb{R}^n \to \mathbb{R}^n$ be a lin. transf. with standard matrix A Then

T is invertible $\iff$ A is invertible

If T is invertible, the standard matrix of T^{-1} is A^{-1}
i.e $T^{-1}(\vec{x}) = A^{-1}\vec{x}$.

Rough idea:

T is invertible $\iff$ T is onto and one-to-one
$\iff$ $A\vec{x}$ is onto and one-to-one
$\iff$ A is invertible. ∎

Partitioned Matrices

Recall If A is an $m \times n$ matrix, can write A as

$$A = [\vec{a}_1 \; \vec{a}_2 \cdots \vec{a}_n] \quad \text{where each } \vec{a}_i \text{ a column vector}$$

↑ each $\vec{a}_i$ is also an $m \times 1$ matrix

Consider other partitions:

Ex $A = \left[\begin{array}{ccc:cc} 1 & 5 & -2 & 3 & 4 \\ 6 & 7 & 11 & 0 & 2 \\ \hdashline 4 & 9 & 12 & 11 & 8 \\ -1 & -2 & 0 & 4 & 1 \end{array}\right] \Rightarrow A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ ← each A_{ij} is a matrix or block

where $A_{11} = \begin{bmatrix} 1 & 5 & -2 \\ 6 & 7 & 11 \end{bmatrix}$, and so

In general, can partition matrix into blocks, e.g.

$$A = \begin{bmatrix} A_{11} & A_{12} \cdots & & A_{1n} \\ \vdots & & & \\ A_{m1} & A_{m2} \cdots & & A_{mn} \end{bmatrix}$$

← each matrix has the same # of rows

↑ each matrix has the same # of columns

Operations

(Addition + Scalar multiplication) If A and B are partitioned in the same way, $A+B$ is the matrix corresponding to sum of blocks

$$A+B = \begin{bmatrix} A_{11} & A_{12} \cdots & A_{1n} \\ \vdots & & \\ A_{m1} \cdots & & A_{mn} \end{bmatrix} + \begin{bmatrix} B_{11} \cdots & B_{1n} \\ & \\ B_{m1} \cdots & B_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} A_{11}+B_{11} & \cdots & A_{1n}+B_{1n} \\ \vdots & & \vdots \\ A_{m1}+B_{m1} & & A_{mn}+B_{mn} \end{bmatrix}$$

$$cA = \begin{bmatrix} cA_{11} \cdots cA_{1n} \\ \vdots \\ cA_{m1} \cdots cA_{mn} \end{bmatrix}$$

(multiplication) partitioned matrices can be multiplied by usual row-column rules, provided partitions are coformable i.e column partition of A's matrices line up with the row partitions of B.

Ex $A = \left[\begin{array}{ccc:cc} 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 0 & 1 \\ \hdashline 0 & 1 & 0 & 1 & 0 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$

$$B = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 2 & 0 \\ \hdashline 3 & 1 \\ 4 & 2 \end{array}\right] = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$

# of columns of A_{11} = # of rows of B_{11}

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix} = \left[\begin{array}{cc} 39 & 16 \\ 7 & 2 \\ \hdashline 3 & 2 \end{array}\right]$$

Octave Suppose A_1, A_2, A_3, A_4 are matrices that have been inputted. Can input

$$A = [A_1 \; A_2 \; ; \; A_3 \; A_4]$$

Key ideas
- properties of inverses
- invertible linear transformations & invertible matrices
- partition of matrices

Have a good fall break!