Lecture 20 4.1 Vector spaces and subspaces
4.2 Null and column spaces

Last time: introduced vector spaces and subspaces
Today: more on subspaces
subspaces from SLE

Recall A subspace of a vector space V is a subset $H \subseteq V$ such that

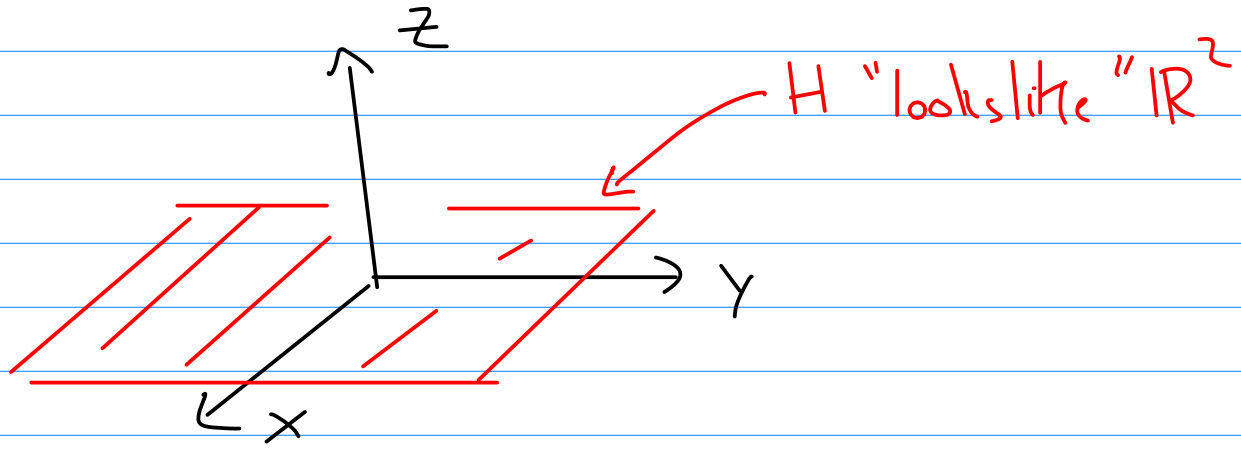
1. $\vec{0} \in H$
2. if $\vec{u}, \vec{v} \in H$, then $\vec{u}+\vec{v} \in H$
3. if $\vec{u} \in H$ and $c \in \mathbb{R}$, then $c\vec{u} \in H$

Ex $\mathbb{R}^2$ is not a subspace of $\mathbb{R}^3$

Note: $\mathbb{R}^2$ is not even a subset of $\mathbb{R}^3$!
Elements of $\mathbb{R}^2$ have the form $\begin{bmatrix} a \\ b \end{bmatrix}$
Elements of $\mathbb{R}^3$ have the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Ex $H = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \mid a, b \in \mathbb{R} \right\} \subseteq \mathbb{R}^3$

Picture:



His a subspace of $\mathbb{R}^3$

1. $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H$ since we can take $a = b = 0$

2 Let $\vec{u}, \vec{v} \in H$. So $\vec{u} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} c \\ d \\ 0 \end{bmatrix}$ with $a, b, c, d \in \mathbb{R}$

Then $\vec{u} + \vec{v} = \begin{bmatrix} a+c \\ b+d \\ 0 \end{bmatrix} \in H$

3. Let $\vec{u} = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} \in H$ and $c \in \mathbb{R}$. Then $c\vec{u} = \begin{bmatrix} ca \\ cb \\ 0 \end{bmatrix} \in H$

Ex any plane in $\mathbb{R}^3$ through origin is a subspace of $\mathbb{R}^3$
any line in $\mathbb{R}^2$ through origin is a subspace of $\mathbb{R}^2$

Spanning set (special case)

Let $\vec{v}_1, \vec{v}_2 \in V$ be any two vectors

$$\text{span}\{\vec{v}_1, \vec{v}_2\} = \{\underbrace{t_1\vec{v}_1 + t_2\vec{v}_2}_{\text{all linear combinations}} \mid t_1, t_2 \in \mathbb{R}\}$$

Then $\text{span}\{\vec{v}_1, \vec{v}_2\}$ is a subspace of V

Proof: 1. $\vec{0} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$ since $\vec{0} = 0\cdot\vec{v}_1 + 0\cdot\vec{v}_2$

2. Let $\vec{u}, \vec{w} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$. So

$$\vec{u} = a\vec{v}_1 + b\vec{v}_2 \quad \text{and} \quad \vec{w} = c\vec{v}_1 + d\vec{v}_2, \; a,b,c,d \in \mathbb{R}$$

Then $\vec{u} + \vec{w} = (a\vec{v}_1 + b\vec{v}_2) + (c\vec{v}_1 + d\vec{v}_2)$

$$= (a+c)\vec{v}_1 + (b+d)\vec{v}_2 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$$

3 Let $\vec{u} \in \text{span}\{\vec{v}_1, \vec{v}_2\}$ and $c \in \mathbb{R}$. Then

$$\vec{u} = a\vec{v}_1 + b\vec{v}_2.$$

Then $c\vec{u} = (ca)\vec{v}_1 + (cb)\vec{v}_2 \in \text{span}\{\vec{v}_1, \vec{v}_2\}$ ∎

Spanning set (general case)

Let $\vec{v}_1, \ldots, \vec{v}_p$ be any set of vectors of V. Then

$$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\} = \{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \mid c_i \in \mathbb{R}\} \subseteq V$$

is a subspace of V

Proof generalize the above proof

Def^n • $\text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ is the subspace spanned (or generated) by $\vec{v}_1, \ldots, \vec{v}_p$

• given a subspace $H \subseteq V$, a spanning (or generating) set for H is a set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in H such that

$$H = \text{span}\{\vec{v}_1, \ldots, \vec{v}_p\}$$

SLE and Subspaces

When studying SLE, subspaces naturally arise

Def^n Let A be an $m \times n$ matrix. The null space of A, denoted $\text{Nul}(A)$, is

$$\text{Nul}(A) = \{ \vec{x} \in \mathbb{R}^n \mid A\vec{x} = \vec{0} \}$$

Equivalent Def^n

Let A be an $m \times n$ matrix that defines the linear transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ by $T(\vec{x}) = A\vec{x}$. Then

$$\text{Nul}(A) = \{ \vec{x} \in \mathbb{R}^n \mid T(\vec{x}) = \vec{0} \}$$

Thm $\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$

Proof 1. $\vec{0} \in \text{Nul}(A)$ since $A \cdot \vec{0} = \vec{0}$

2. Let $\vec{u}, \vec{v} \in \text{Nul}(A)$. Then $A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$
$= \vec{0} + \vec{0} = \vec{0}$

So $\vec{u}+\vec{v} \in \text{Nul}(A)$

3. Let $\vec{u} \in \text{Nul}(A)$ and $c \in \mathbb{R}$. Then

$A(c\vec{u}) = c(A\vec{u}) = c\vec{0} = \vec{0}$. So $c\vec{u} \in \text{Nul}(A)$

Ex $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ Describe $Nul(A) = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \;\middle|\; A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \vec{0} \right\}$

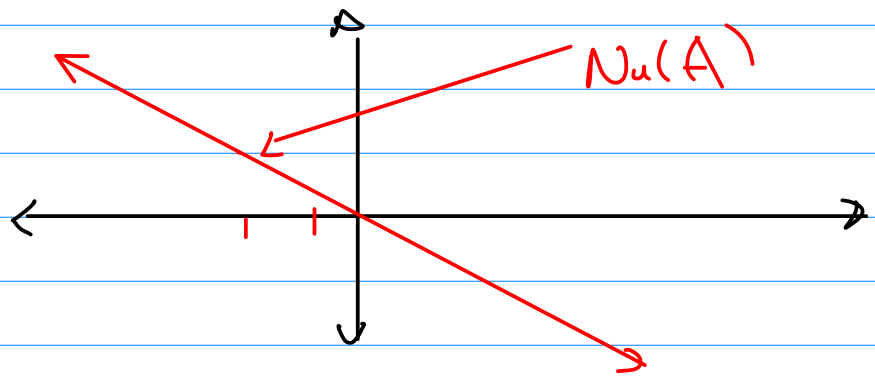
$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \leftarrow x_2 \text{ is free} \quad \text{Set } x_2 = t$$

$$x_1 = -2x_2$$

So

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \end{bmatrix} \quad \text{as } t \in \mathbb{R}$$

Thus $Nul(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\} = \left\{ t\begin{bmatrix} -2 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\} \subseteq \mathbb{R}^2$



Ex (Solution sets as spanning sets)

Let $A = \begin{bmatrix} 1 & -2 & 0 & 4 & 0 \\ 0 & 0 & 1 & -9 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ So $\mathrm{Nul}(A) \subseteq \mathbb{R}^5$ is a subspace

Find a spanning set of $\mathrm{Nul}(A)$

Sol^n A is already in row reduced echelon form

So x_2, x_4 free $\Rightarrow x_2 = r$ and $x_4 = s$

$$\left.\begin{array}{l} x_1 - 2x_2 + 4x_4 = 0 \\ x_3 - 9x_4 = 0 \\ x_5 = 0 \end{array}\right\} \quad \begin{array}{l} x_1 = 2x_2 - 4x_4 = 2r - 4s \\ x_3 = 9x_4 = 9s \\ x_5 = 0 = 0 \end{array}$$

$$\text{So } \vec{x} \in \mathrm{Nul}(A) \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2r - 4s \\ r \\ 9s \\ s \\ 0 \end{bmatrix} = r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix}$$

linear combination

$$\text{So } \mathrm{Nul}(A) = \mathrm{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 9 \\ 1 \\ 0 \end{bmatrix} \right\}$$

← nothing New!

Justnes terminology

Key ideas: • Spanning sets give subspaces
• Null spaces and their spanning sets