Lecture 1 — 1.1 Systems of linear equations

Basic object of study: linear equations (l.e.)

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

$a_1, \ldots, a_n$ are coefficients
$x_1, \ldots, x_n$ are variables, and
$a_1, \ldots, a_n, b$ are real or complex numbers

Ex 1. $3x_1 + 2x_2 + 2020x_3 = \log_2 17$ ← l.e.

2. $3x_1^2 + 2x_1x_3 + \sin(x_4) = 8$ ← not l.e.

Def^n A system of linear equations (s.l.e) is a collection of one or more l.e.

Ex $\begin{aligned} 3x_1 + 2x_2 &= 3 \\ -x_1 + x_2 &= 4 \end{aligned}$ (*) sol^n $(s_1, s_2) = (-1, 3)$

Def^n A solution to a s.l.e. is an n-tuple $(s_1, s_2, \ldots, s_n)$ that makes each equation in the s.l.e true after you replace x_i with s_i

Ex $(s_1, s_2) = (-1, 3)$ is a sol^n to (*)

Def^n Solution set of a s.l.e. = set of all sol^{n}s to a s.l.e.

two s.l.e. are equivalent if they have same sol^n set.

FUNDAMENTAL QUESTIONS:
Given a s.l.e. can ask:

1. Does a sol^n to the s.l.e exist?
2. If a sol^n exists, is it unique?

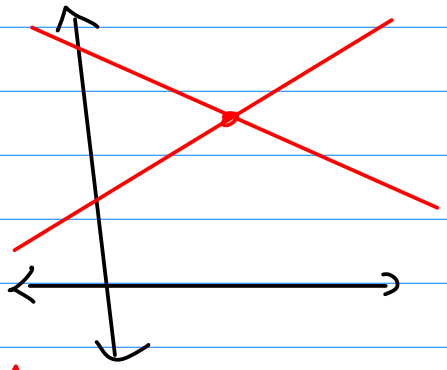
These two questions (in various disguises) appear throughout the course.

How big is a solution set to a S.l.e?

Intuition from 2-variable case

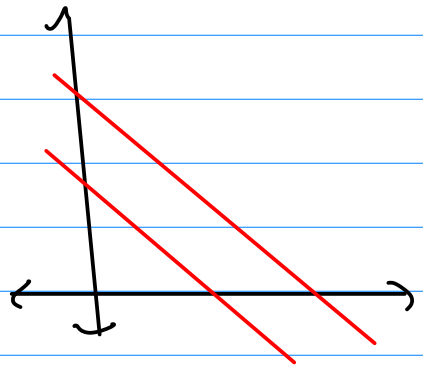
$$\left.\begin{aligned} a_1x_1 + a_2x_2 &= b \\ c_1x_1 + c_2x_2 &= d \end{aligned}\right\} \text{ equations of lines in the plane}$$

Two lines in the plane can intersect in 3 ways:



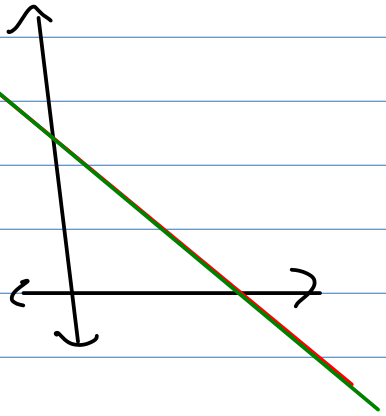
1 point

$x_1 + x_2 = 5$
$x_1 - x_2 = -1$



meet at no points (parallel lines)

$x_1 + x_2 = 3$
$x_1 + x_2 = 1$



everywhere (same line)

$x_1 + x_2 = 3$
$2x_1 + 2x_2 = 6$

GENERAL FACT (will prove)

A S.l.e. has either

1. no sol^n ↑ inconsistent
2. exactly one sol^n
3. infinitely many sol^{n}s. ↓ consistent

Def^n A s.l.e. is consistent if it has one or infinite number of sol^{n}s.

Otherwise, a s.l.e. is inconsistent

MATRIX NOTATION

Linear algebra also studies matrices, rectangular arrays of numbers

A s.l.e. can be stored compactly as a matrix

Ex $\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ 5x_1 \quad - 5x_3 &= 10 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$

augmented matrix of the s.l.e.

matrix

Solving S.l.e – a first look

To solve a S.l.e., our strategy is to replace one system with an equivalent system that is easier to solve. To solve, we manipulate the augmented matrix

Basic operations:

1. multiply one equation by a nonzero constant
2. interchange any two equations
3. add a constant times one equation to another equation

These three operations do not change the sol^n set

Ex Solve the Sle below (choices explained next lecture. Main idea: a l.e. in one variable easy to solve, so find equivalent S.l.e. with fewer variables)

$$\begin{aligned} 1.\ & x_1 - 2x_2 + x_3 = 0 \\ 2.\ & \quad 2x_2 - 8x_3 = 8 \\ 3.\ & 5x_1 \quad - 5x_3 = 10 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

Step 1 Replace (3) with (eq'n (1) x(-5)) + eq'n (3)

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$10x_2 - 10x_3 = 10$$

$\Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right]$$

$(-5)\,\text{Row}1 + \text{Row}3$

$$\begin{array}{cccc} -5 & 10 & -5 & 0 \\ 5 & 0 & -5 & 10 \\ \hline 0 & 10 & -10 & 10 \end{array}$$

$(-5)(x_1 - 2x_2 + x_3 = 0)$

$+ \quad (5x_1 \quad -5x_3 = 10)$

$0x_1 + 10x_2 - 10x_3 = 10$

Step 2 Replace new (3) with ((-5) x eqn (1)) + (3)

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_2 - 8x_3 = 8$$
$$30x_3 = -30$$

$\Rightarrow$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 0 & 30 & -30 \end{array}\right]$$

$\leftarrow$ row 2 x(-5) + row 3

Step 3 Last eqn $\Rightarrow 30x_3 = -30 \Rightarrow x_3 = -1$

eqn (2) $\Rightarrow 2x_2 - 8x_3 = 8 \Rightarrow 2x_2 - 8(-1) = 8 \Rightarrow x_2 = 0$

eqn (1) $\Rightarrow x_1 - 2x_2 + x_3 = 0 \Rightarrow x_1 - 2\cdot 0 + (-1) = 0 \Rightarrow x_1 = 1$

So, one soln is $(s_1, s_2, s_3) = (1, 0, -1)$ ← CHECK!

Next lecture: explain steps of this procedure

Key ideas:
- l.e. and s.l.e
- fundamental questions: does a s.l.e have a solution
- s.l.e. and matrix notation
- basic operations that preserves s.l.e