Lecture 23 4.4 Coordinate Systems

Last time: A set $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for V if

1. $\mathcal{B}$ is linearly independent
2. $\mathcal{B}$ spans V, i.e. $V = \text{span}\{\vec{b}_1, \ldots, \vec{b}_n\}$

Today: coordinate systems with respect to $\mathcal{B}$

Thm (Unique Representation Thm) Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a vector space V. For each $\vec{x} \in V$ there exists unique scalars $c_1, c_2, \ldots, c_n$ such that

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n$$

Proof Since $\mathcal{B}$ is a basis, there exists $c_1, \ldots, c_n$ such that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$$

Suppose $d_1, \ldots, d_n$ are also scalar vectors such that

$$\vec{x} = d_1\vec{b}_1 + \cdots + d_n\vec{b}_n$$

Then $\vec{0} = \vec{x} - \vec{x} = (c_1\vec{b}_1 + \cdots + c_n\vec{b}_n) - (d_1\vec{b}_1 + \cdots + d_n\vec{b}_n)$

$$= (c_1 - d_1)\vec{b}_1 + \cdots + (c_n - d_n)\vec{b}_n$$

Since the $\vec{b}_i$'s are linearly independent, $c_i - d_i = 0$ for $i = 1, \ldots, n$. So $c_i = d_i$ for all i. ∎

Defn Suppose $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for V. If $\vec{x} \in V$, the $\mathcal{B}$-coordinates of $\vec{x}$ are the weights

$c_1, c_2, \ldots, c_n$ such that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$$

If $c_1, \ldots, c_n$ are the $\mathcal{B}$-coordinates of $\vec{x}$, write

$$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \leftarrow \mathcal{B}\text{-coordinate of } \vec{x}$$

Examples

Ex 1 $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$

If $\vec{x} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$, find $[\vec{x}]_{\mathcal{B}}$

Soln Want c_1, c_2 such that

via inspection

$$c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}_{\mathcal{B}}$$

Ex 2 Standard basis $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$

For any $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ $\begin{bmatrix} a \\ b \\ c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} a \\ b \\ c \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

For standard basis of $\mathbb{R}^n$ $[\vec{x}]_{\mathcal{E}} = \vec{x}$

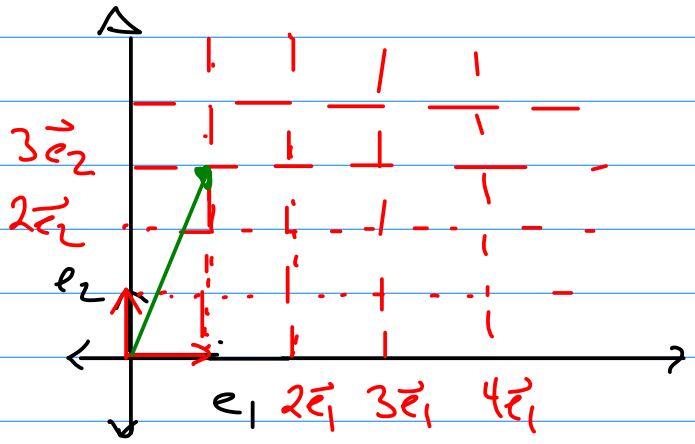
Procedure If $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for $\mathbb{R}^n$, here is how to find $[\vec{x}]_{\mathcal{B}}$

1. Set $P_{\mathcal{B}} = [\vec{b}_1 \cdots \vec{b}_n]$ ← change of coordinate matrix
2. Solve $P_{\mathcal{B}}\vec{c} = \vec{x}$ where $\vec{c} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Then $[\vec{x}]_{\mathcal{B}} = \vec{c}$

Graphical Interpretation of coordinates & bases

When plotting points in $\mathbb{R}^2$, do this relative to $\{\vec{e}_1, \vec{e}_2\}$

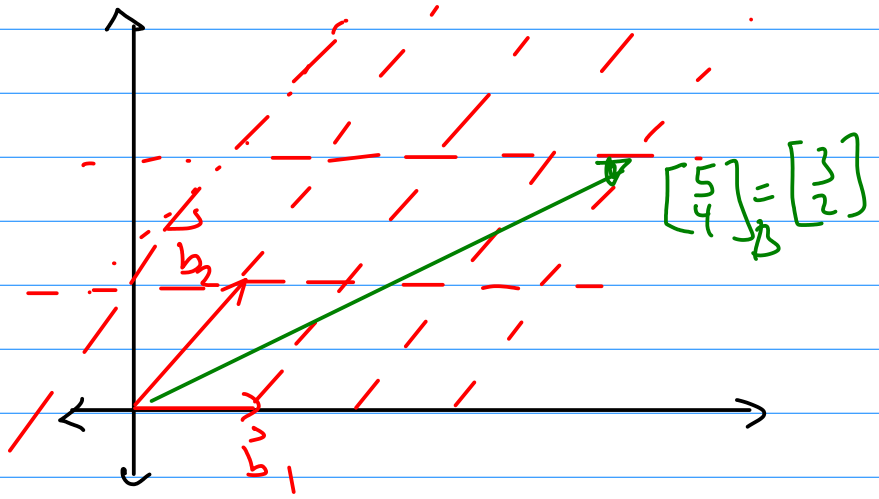


vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ is at the point $(1,3)$ in the grid described by $\vec{e}_1, \vec{e}_2$

$\begin{bmatrix} 1 \\ 3 \end{bmatrix}_{\mathcal{E}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ ← tells you where to plot vector with respect to this basis.

Suppose $\vec{b}_1, \vec{b}_2$ another basis for $\mathbb{R}^2$, e.g.

$\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{b}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ ← draw a grid with these vectors



$\begin{bmatrix} 5 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

vector $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ is drawn to $(3,2)$ in grid described by $\vec{b}_1$ and $\vec{b}_2$

Since $\begin{bmatrix} 5 \\ 4 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ ← this tells us where to plot the vector with respect to $\mathcal{B}$.

I.e. vector $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ in the normal grid has endpt at $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ in the grid defined by $\mathcal{B}$.

Roughly, a set of vectors is a basis if we can make a "grid" and write every element in terms of the grid.

Coordinate map

We are associating to each $\vec{v} \in V$ a vector in $\mathbb{R}^n$, i.e. a map (called the coordinate map)

$$\varphi: V \longrightarrow \mathbb{R}^n$$

$$\vec{x} \longmapsto [\vec{x}]_\beta \quad \text{once } \beta \text{ is fixed}$$

Extra properties of this map

Thm Fix a basis $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ of a vector space V. The coordinate map $\varphi: V \longrightarrow \mathbb{R}^n$ where $\varphi(\vec{x}) = [\vec{x}]_\beta$ is a linear transformation that is one-to-one and onto.

Def An isomorphism b/w vector spaces V and W is a linear transformation $\varphi: V \to W$ that is one-to-one and onto

Cor The coordinate map is an isomorphism

"Big idea" Isomorphism implies the vector spaces are the same, just have different labels/names

Ex Let $\mathcal{B} = \{1, t, t^2\}$ be standard basis for $\mathbb{P}_2$

$p(t) \in \mathbb{P}_2 \Rightarrow p(t) = a_0 + a_1 t + a_2 t^2 = a_0 \cdot \underline{1} + a_1 (\underline{t}) + a_2 (\underline{t^2})$

$$\Rightarrow [p(t)]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

So map $\varphi : \mathbb{P}_2 \longrightarrow \mathbb{R}^3$ given by

$$a_0 + a_1 t + a_2 t^2 \longmapsto \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

is an isomorphism. So $\mathbb{P}_2$ and $\mathbb{R}^3$ are the "same" vector space!

In fact, $\mathbb{P}_n$ and $\mathbb{R}^{n+1}$ are isomorphic

Key points
* $\mathcal{B}$-coordinates
* graphical meaning
* isomorphism