Lecture 8 Introduction to linear transformations (Sec 1.8)

Today's lecture: function $T: \mathbb{R}^n \to \mathbb{R}^m$
matrix transformations
linear transformations
SLE + linear transformations

Functions $T: \mathbb{R}^n \to \mathbb{R}^m$

Recall: $\mathbb{R}^n = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mid u_i \in \mathbb{R} \right\}$ ← all vectors in $\mathbb{R}^n$

Calculus studies functions $f: \mathbb{R}^1 \to \mathbb{R}^1$
e.g. $f(x) = x^2$, $f(x) = 3\cos x$, ...

Def^ A **transformation** is a function $T: \mathbb{R}^n \to \mathbb{R}^m$, i.e.

a rule that assigns to each $\vec{x} \in \mathbb{R}^n$ exactly one vector $T(\vec{x}) \in \mathbb{R}^m$.

Terminology:
- $\mathbb{R}^n$ the domain of T
- $\mathbb{R}^m$ is the codomain of T
- if $x \in \mathbb{R}^n$, $T(\vec{x})$ the image of $\vec{x}$
- range of $T = \{ T(\vec{x}) \mid \vec{x} \in \mathbb{R}^n \} \subseteq \mathbb{R}^m$

Ex Define $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1^2 + x_2 \\ x_2 \\ x_1^2 x_2 + x_2^3 \end{bmatrix}$

Each vector in $\mathbb{R}^2$ is mapped to a vector in $\mathbb{R}^3$

e.g. $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1^2 + 2 \\ 2 \\ 1^2 \cdot 2 + 2^3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 10 \end{bmatrix}$

Ex Every $m \times n$ matrix A defines a transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ by

$$T(\vec{x}) = A\vec{x}$$

Input is an $n \times 1$ matrix

Output is an $m \times 1$ matrix

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}$. Define a $T: \mathbb{R}^2 \to \mathbb{R}^3$ by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 + x_2 \end{bmatrix}$$

Matrix Transformations

Def^n If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a transformation that can be given by an $m \times n$ matrix A, i.e $T(\vec{x}) = A\vec{x}$, then T is a matrix transformation.

Ex Consider $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 + x_2 \end{bmatrix}$

matrix transf since $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_2 \\ x_2 \\ -x_1 + x_2 \end{bmatrix}$

Ex (not all transformations are matrix transformations)

Show $T: \mathbb{R}^2 \to \mathbb{R}$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = x_1 x_2$

is **not** a matrix transformation.

If it was a matrix transf, we could find a 1×2 matrix $A = [a \; b]$ such that $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = [a \; b]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 x_2$ for all $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$

So $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = [a \; b]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a = 1 \cdot 0 \Rightarrow a = 0$

$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = [a \; b]\begin{bmatrix} 0 \\ 1 \end{bmatrix} = b = 0 \cdot 1 \Rightarrow b = 0$

$\Big\} \; A = [0 \; 0]$

But $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = [0 \; 0]\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \neq 1 \cdot 1 = 1$ So **no** such A **exists**!

Fact: If T is a matrix transformation

range of $T = \{T(\vec{x}) = A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

= span of columns of A

Linear transformations

Def^n A transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear if

(i) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v} \in \mathbb{R}^n$

(ii) $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

Ex Show $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 \\ x_1 - 4x_2 \end{bmatrix}$ is linear

$$T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} 3(u_1+v_1) + 2(u_2+v_2) \\ (u_1+v_1) - 4(u_2+v_2) \end{bmatrix}$$

$$= \begin{bmatrix} 3u_1 + 2u_2 + 3v_1 + 2v_2 \\ u_1 - 4u_2 + v_1 - 4v_2 \end{bmatrix}$$

$$= \begin{bmatrix} 3u_1 + 2u_2 \\ u_1 - 4u_2 \end{bmatrix} + \begin{bmatrix} 3v_1 + 2v_2 \\ v_1 - 4v_2 \end{bmatrix} = T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

$$T\left(c\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} 3cu_1 + 2cu_2 \\ cu_1 - 4cu_2 \end{bmatrix} = c\begin{bmatrix} 3u_1 + 2u_2 \\ u_1 - 4u_2 \end{bmatrix}$$

$$= cT\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)$$

So T is a linear transformation!

Thm (Important!) $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformations if and only if $T: \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation

Ex $T: \mathbb{R}^2 \to \mathbb{R}^2$ above was linear.

Can check $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3 & 2 \\ 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Proof ($\Leftarrow$) Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a matrix transf.

So $T(\vec{x}) = A\vec{x}$ for some $m \times n$ matrix A.

In Section 1.4, we showed

$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v}$ and $A(c\vec{u}) = c(A\vec{u})$

So

$T(\vec{u}+\vec{v}) = A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v})$

$T(c\vec{u}) = A(c\vec{u}) = cA\vec{u} = c\,T(\vec{u})$.

Thus, T is a linear transformation.

($\Rightarrow$) next lecture! given a linear transf, we can explicitly find a matrix A such that $T(\vec{x}) = A\vec{x}$.

SLE and matrix transformations

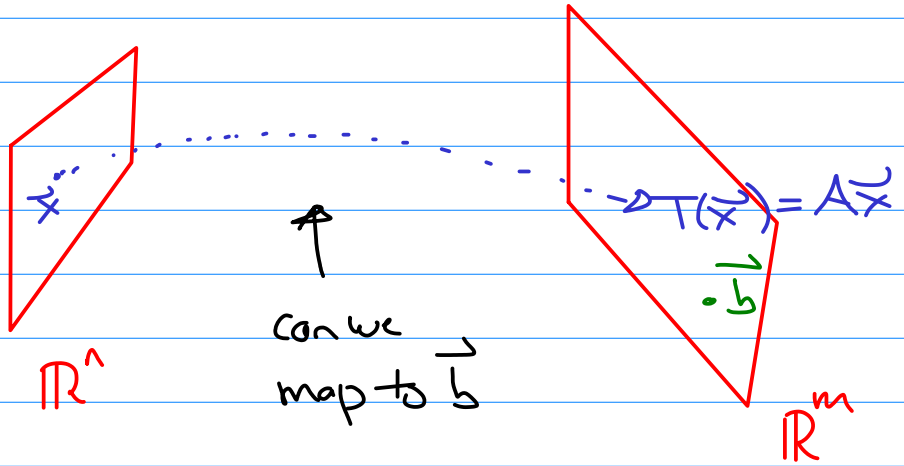
Matrix/linear transformations give a new point-of-view of SLE.

Solving $\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = b_m \end{matrix}$ $\Longleftrightarrow$ solving $A\vec{x} = \vec{b}$

(A: coefficient matrix)

In language of transformations:

Consider the matrix transformation $T: \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{x}) = A\vec{x}$. Given a vector $\vec{b} \in \mathbb{R}^m$, solving the SLE is asking if there is a vector $\vec{x} \in \mathbb{R}^n$ that maps to $\vec{b}$



SLE has a sol^n $\Leftrightarrow$ $\vec{b}$ in range of $T = \{T(\vec{x}) = A\vec{x} \mid \vec{x} \in \mathbb{R}^n\}$

SLE has a unique sol^n $\Leftrightarrow$ exactly one vector in $\mathbb{R}^n$ maps to $\vec{b}$

Key ideas
- transformation
- matrix transformation = linear transformation
- new P.O.V. of SLE