Lecture 3 Gaussian Elimination II (Section 1.2)

Last lecture: introduced Gaussian Elimination

This lecture: use Gaussian Elimination to solve SLE

introduce linear algebra software

Ex 1 Solve $x_1 - 2x_2 + 3x_3 = 9$

$-x_1 + 3x_2 = -4$

$2x_1 - 5x_2 + 5x_3 = 17$

Make augmented matrix and row reduce into echelon form

$$\left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ -1 & 3 & 0 & -4 \\ 2 & -5 & 5 & 17 \end{array}\right] \sim \left[\begin{array}{ccc:c} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & -1 & -1 & -1 \end{array}\right] \begin{array}{l} \\ \leftarrow \text{row}1 + \text{row}2 \\ \leftarrow \text{row}1 \times (-2) + \text{row}3 \end{array}$$

$$\begin{bmatrix} 1 & -2 & 3 & 9 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 4 \end{bmatrix} \leftarrow \text{row2} + \text{row3}$$

Turn back into equations:

$$x_1 - 2x_2 + 3x_3 = 9 \quad (1)$$
$$x_2 + 3x_3 = 5 \quad (2)$$
$$2x_3 = 4 \quad (3)$$

Use "back substitution" to solve for x_1, x_2, x_3

$(3) \Rightarrow x_3 = 2$

$(2)+(3) \Rightarrow x_2 + 3(2) = 5$
$\Rightarrow x_2 = 5 - 6 = -1$

$(1)+(2)+(3) \Rightarrow x_1 - 2(-1) + 3(2) = 9$
$\Rightarrow x_1 = 9 - 8 = 1$

Only one soln $(x_1, x_2, x_3) = (1, -1, 2)$

Ex 2 Solve $x_1 - 3x_2 + x_3 = 1$

$2x_1 - x_2 - 2x_3 = 2$

$x_1 + 2x_2 - 3x_3 = -1$

Repeat process of last example:

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ 2 & -1 & -2 & 2 \\ 1 & 2 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 5 & -4 & 0 \\ 0 & 5 & -4 & -2 \end{bmatrix} \begin{matrix} \\ \leftarrow \text{row 1} \times (-2) + \text{row 2} \\ \leftarrow \text{row 1} \times (-1) + \text{row 3} \end{matrix}$$

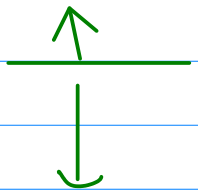
$$\sim \begin{bmatrix} \textcircled{1} & -3 & 1 & 1 \\ 0 & \textcircled{5} & -4 & 0 \\ 0 & 0 & 0 & \textcircled{-2} \end{bmatrix} \leftarrow \text{row 2} \times (-1) + \text{row 3}$$

Turn back into equations

$$x_1 - 3x_2 + x_3 = 1$$
$$5x_2 - 4x_3 = 0$$
$$0 = -2 \quad !! \ \text{☹}$$

Recall: A SLE has either

1. no sol^{n}
2. exactly one sol^{n}
3. infinite # of sol^{n}s



inconsistent

consistent

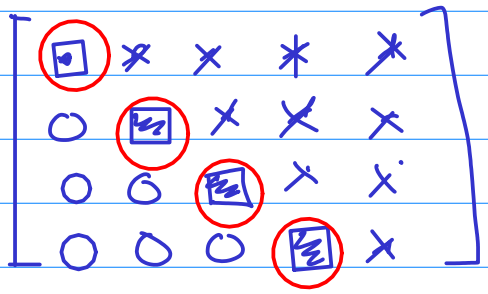
The "shape" of the echelon form of the augmented matrix determines which case.

NO SOLUTION: echelon form has a row of the form

$$[0\ 0\ 0 \cdots 0 : b] \text{ with } b \neq 0$$

$$\Leftrightarrow 0x_1 + 0x_2 + \cdots + 0x_n = 0 = b \leftarrow \text{no sol}^{n}$$

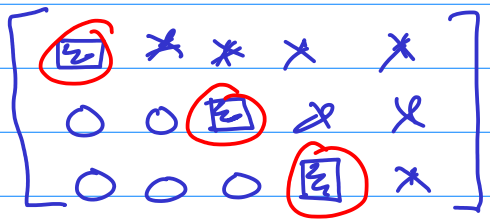
EXACTLY ONE SOLUTION: echelon form has a pivot (nonzero leading entry) in each row and column except the last column



□ = nonzero entry
* = any value

← exactly one soln

INFINITE NUMBER OF SOLNS. echelon form has no pivot in last column, and # of pivots < (# of columns − 1)



← will have an infinite # of soln's

Ex 3 (infinite # of sol'ns)

Solve $x_1 - 2x_2 - x_3 + 3x_4 = 1$

$2x_1 - 4x_2 + x_3 = 5$

$x_1 - 2x_2 + 2x_3 - 3x_4 = 4$

Augmented

$$\begin{bmatrix} 1 & -2 & -1 & 3 & 1 \\ 2 & -4 & 1 & 0 & 5 \\ 1 & -2 & 2 & -3 & 4 \end{bmatrix}$$

$\sim$ ↑ steps stripped!

reduced row echelon form

$$\begin{bmatrix} \textcircled{1} & -2 & 0 & 1 & 2 \\ 0 & 0 & \textcircled{1} & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots = 2

columns − 1 = 5 − 1 = 4

Since $2 < 4$, there are infinitely many sols!

How to express all solns?

Variables corresponding to pivot columns = basic variables or leading variables

Variables corresponding to non-pivot columns = free variables

Our example: basic variables x_1, x_3
free variables x_2, x_4

Express each basic variable in terms of free variables

$$\begin{aligned} x_1 - 2x_2 + x_4 &= 2 \\ x_3 - 2x_4 &= 1 \end{aligned} \implies \begin{aligned} x_1 &= 2x_2 - x_4 + 2 \\ x_3 &= 1 + 2x_4 \end{aligned}$$

x_2, x_4 can be arbitary. Let $x_2 = r$ and $x_4 = t$. All sol^{n}s have form.

$$\begin{aligned} x_1 &= 2r - t + 2 \\ x_2 &= r \\ x_3 &= 1 + 2t \\ x_4 &= t \end{aligned} \quad \text{with } r, t \in \mathbb{R}$$

$\in$: an element of
$\mathbb{R}$: all real numbers

} parametric sol^n

Ex $r = 0, t = 1$
then $(x_1, x_2, x_3, x_4) = (1, 0, 3, 1)$ is a sol^n to the SLE

FACT: If # free variables > 0, then infinite # of sol^{n}s.

Introduction to Octave/Matlab

Octave and Matlab are two computer algebra programs that can do calculations with matrices

* non-covid years: use Matlab on campus labs
* can purchase, but not needed

Octave has the required functionality for Math 1B03

https://octave-online.net/

Can make an account but not needed

Key ideas:
- "shape" of the echelon matrix tells us the # of soln's
- Gaussian elimination can find these soln's
- we have computer help :)