# Lecture 35 5.6 Discrete Dynamical Systems

Today: Introduce dynamical systems and connections to eigenvalues/eigenvectors

## Setup

Let $A$ be an $n \times n$ matrix and $\vec{x}_0$ a vector. Consider sequence:

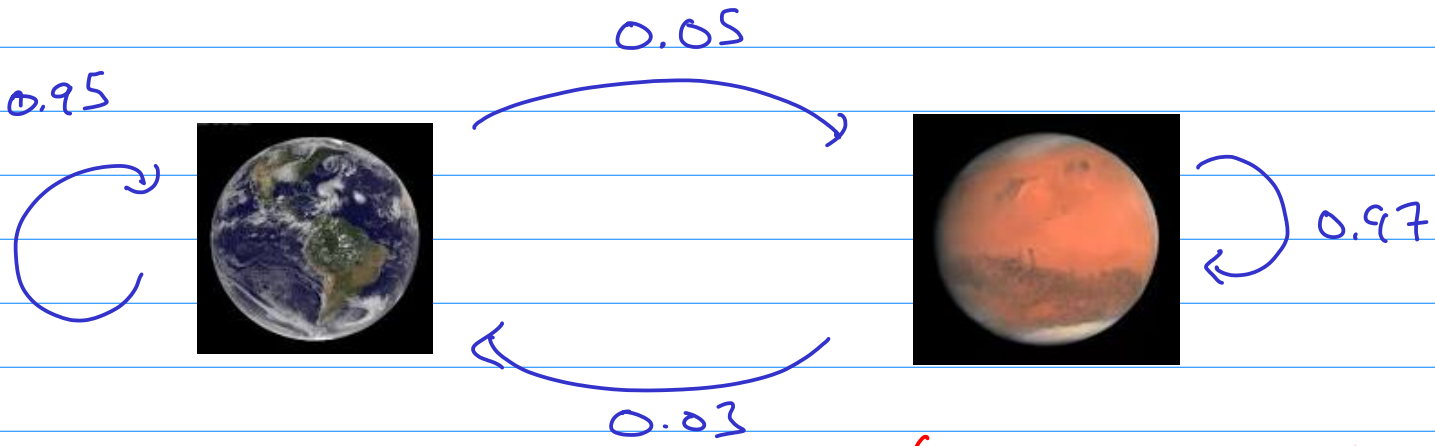
$$\vec{x}_1 = A\vec{x}_0 \quad \vec{x}_2 = A\vec{x}_1, \; \vec{x}_3 = A\vec{x}_2, \ldots, \; \vec{x}_{k+1} = A\vec{x}_k$$

Def$^n$. The equation $\vec{x}_{k+1} = A\vec{x}_k$ is called a difference equation.

- a dynamical system is a finite set of variables whose values change with time.

(in the above, if $\vec{x}_0 = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$, then $y_i$'s are changing)

**Ex (Migration)** In a given year, migration between Earth and Mars




(for any given year)

$$A = \begin{array}{c|cc} \text{To} \backslash \text{From} & E & M \\ \hline E & 0.95 & 0.03 \\ M & 0.05 & 0.97 \end{array} \Rightarrow A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$$

Suppose $\vec{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ ← 60% on Earth, ← 40% on Mars

After 1 year $\vec{x}_1 = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = \begin{bmatrix} 0.582 \\ 0.418 \end{bmatrix}$

Q What does $\vec{x}_k$ look like as $k \to \infty$?

$\vec{x}_2 = A\vec{x}_1$, $\vec{x}_3 = A\vec{x}_2$, ...

[A] information captured in eigenvalues and eigenvectors

Ex (con't) The matrix $A = \begin{bmatrix} 0.95 & 0.03 \\ 0.05 & 0.97 \end{bmatrix}$ is diagonalizable

$$A = \begin{bmatrix} 0.6 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.92 \end{bmatrix} \begin{bmatrix} 0.6 & -1 \\ 1 & 1 \end{bmatrix}^{-1} = PDP^{-1}$$

Note: Columns of $P$ form basis of $\mathbb{R}^2$, i.e.

$$\mathbb{R}^2 = \text{span}\left\{ \begin{bmatrix} 0.6 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

↑ eigenvectors

Write $\vec{x}_0 = \begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix}$ in terms of basis:

$$\begin{bmatrix} 0.6 \\ 0.4 \end{bmatrix} = 0.625 \begin{bmatrix} 0.6 \\ 1 \end{bmatrix} - 0.225 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

So $A\vec{x}_0 = A\left(0.625\begin{bmatrix}0.6\\1\end{bmatrix} - 0.225\begin{bmatrix}-1\\1\end{bmatrix}\right)$

$= 0.625\left(A\begin{bmatrix}0.6\\1\end{bmatrix}\right) - 0.225\left(A\begin{bmatrix}-1\\1\end{bmatrix}\right)$

$A\begin{bmatrix}0.6\\1\end{bmatrix} = 1\begin{bmatrix}0.6\\1\end{bmatrix}$

$= 0.625\left(1\cdot\begin{bmatrix}0.6\\1\end{bmatrix}\right) - 0.225\left(0.92\begin{bmatrix}-1\\1\end{bmatrix}\right)$

$= \vec{x}_1$

Then $A\vec{x}_1 = A\left(0.625\cdot 1\begin{bmatrix}0.6\\1\end{bmatrix} - 0.225(0.92)\begin{bmatrix}-1\\1\end{bmatrix}\right)$

$= 0.625\cdot 1\left(A\begin{bmatrix}0.6\\1\end{bmatrix}\right) - 0.225(0.92)\left(A\begin{bmatrix}-1\\1\end{bmatrix}\right)$

$= 0.625\cdot 1^2\begin{bmatrix}0.6\\1\end{bmatrix} - 0.225(0.92)^2\begin{bmatrix}-1\\1\end{bmatrix}$

$= \vec{x}_2$

For any $k$

$\vec{x}_k = 0.625\cdot 1^k\begin{bmatrix}0.6\\1\end{bmatrix} - 0.225(0.92)^k\begin{bmatrix}-1\\1\end{bmatrix}$

As $k \to \infty$, $(0.92)^k \to 0$

So $\vec{x}_k \longrightarrow 0.625 \cdot \begin{bmatrix} 0.6 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.375 \\ 0.625 \end{bmatrix}$

So eventually Earth contains 37.5% of the population
and Mars contains 62.5% of the population

## General Case

Assumptions
- A is diagonalizable
- A has n linearly independent eigenvectors $\vec{v}_1, \ldots, \vec{v}_n$ corresponding to $\lambda_1, \ldots, \lambda_n$
- order eigenvalues $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$

Since $\{\vec{v}_1, \ldots, \vec{v}_n\}$ linearly independent, form a basis for $\mathbb{R}^n$

$\mathbb{R}^n = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}$

Write initial vector as

$\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$ ← using eigenvector basis

So $\vec{x}_1 = A\vec{x}_0 = c_1(A\vec{v}_1) + \cdots + c_n(A\vec{v}_n)$

$= c_1\lambda_1\vec{v}_1 + \cdots + c_n\lambda_n\vec{v}_n$

In general

$$\boxed{\vec{x}_k = c_1\lambda_1^k\vec{v}_1 + c_2\lambda_2^k\vec{v}_2 + \cdots + c_n\lambda_n^k\vec{v}_n}$$

Note eigenvectors/values determine $\vec{x}_k$

Observation Suppose $\lambda_1 \geq 1$ and
$1 > |\lambda_2| \geq \cdots \geq |\lambda_n| \geq 0$

For large $k$,

$\vec{x}_k \approx c_1\lambda_1^k\vec{v}_1 + 0$ since $\lambda_i^k \to 0$ for $i = 2, \ldots, n$

## Graphical Sol^n's (2x2 case)

Suppose also assume $A$ is $2\times 2$ So

$$\vec{x}_k = c_1 \lambda_1^k \vec{v}_1 + c_2 \lambda_2^k \vec{v}_2$$

The graph of $\vec{x}_0, \vec{x}_1, \ldots$ is <u>trajectory</u> of the dynamical system

<u>Ex</u> $A = \begin{bmatrix} .7 & 0 \\ 0 & .42 \end{bmatrix}$ eigenvalue $\lambda_1 = 0.7$ eigenvector $= \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

eigenvalue $\lambda_2 = 0.42$ eigenvector $= \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

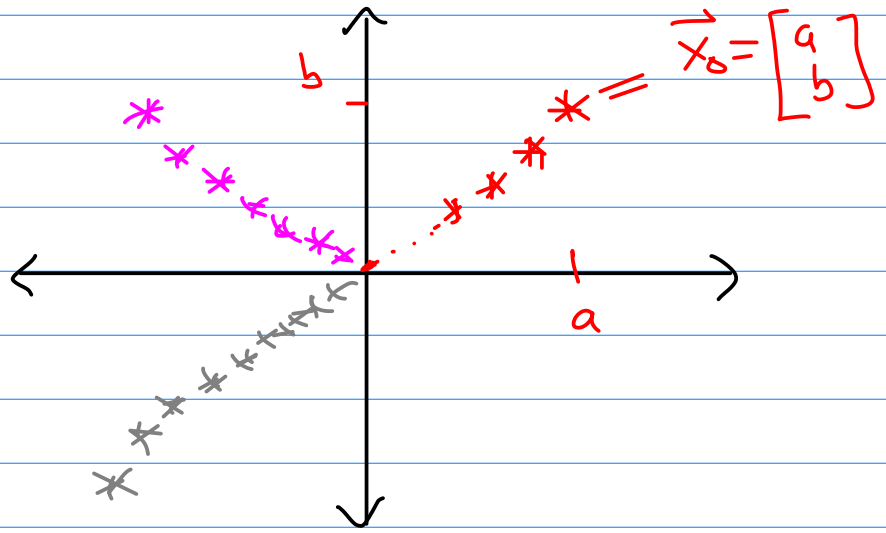
So, if $\vec{x}_0 = c_1\vec{v}_1 + c_2\vec{v}_2$

$$\Rightarrow \vec{x}_k = c_1 (0.7)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 (0.42)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

As $k \to \infty$, $(0.7)^k \to 0$ and $(0.42)^k \to 0$

So, for all $\vec{x}_0$, have $\vec{x}_k \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Picture




Def^n The origin is

- an attractor if eigenvalues $< 1$
- a repellor if eigenvalues $> 1$
- a saddle point if one eigenvalue $> 1$ and other $< 1$

Ex For $A = \begin{bmatrix} 1.2 & 0 \\ 0 & 1.4 \end{bmatrix}$

$$\vec{x}_k = c_1(1.2)^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2(1.4)^k \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

← a repellor

Key ideas: · dynamical system
· trajectory determined by eigenvalues/eigenvectors

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##################################################################
## Math 1B03
## Lecture 35
##################################################################

## Example from class
## migration

clear
A = [.95 0.03; 0.05 0.97]
b = [.6; .4] # initial vector
P1 = [b]
for i=1:25 b = A*b; P1 = [P1, b]; end;
x1 = P1(1,:);
y1 = P1(2,:);
plot(x1,y1,"*")

## Example from class
## attractor
clear
A = [.7 0; 0 0.42]
b = [1; 1] # initial vector
P1 = [b]
for i=1:25 b = A*b; P1 = [P1, b]; end;
c = [-1;1] # second initial vector
P2 = [c];
for i=1:25 c = A*c; P2 = [P2, c]; end;
d = [2;-3] # thider initial vector
P3 = [d];
for i=1:25 d = A*d; P3 = [P3, d]; end;
x1 = P1(1,:);
y1 = P1(2,:);
x2 = P2(1,:);
y2 = P2(2,:);
x3 = P3(1,:);
y3 = P3(2,:);
plot(x1,y1,"*",x2,y2,"*",x3,y3,"*")
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## make a loop to do lots of points at once
P = [];
for i=1:30
  # pick a random vecotr
  b = [100*(-1)^(ceil(10*rand()))*rand();100*(-1)^(ceil(10*rand()))*rand()];
  P = [P b];
  # for each vector, find trajector
for i=1:30 b = A*b; P = [P, b]; end;  # 4 iterations only
end;
x = P(1,:);
y = P(2,:);
# plot all trajectories
plot(x,y,"*")
```

.

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##############################################################################

## Example of repellor
##
## Example from class
clear
A = [1.2 0; 0 1.4]
b = [1; 1] # initial vector
P1 = [b]
for i=1:25 b = A*b; P1 = [P1, b]; end;
c = [-1;1] # second initial vector
P2 = [c];
for i=1:25 c = A*c; P2 = [P2, c]; end;
d = [2;-3] # third initial vector
P3 = [d];
for i=1:25 d = A*d; P3 = [P3, d]; end;
e = [1; -.01] # initial vector
P4 = [e]
for i=1:25 e = A*e; P4 = [P4, e]; end;
x1 = P1(1,:);
y1 = P1(2,:);
x2 = P2(1,:);
y2 = P2(2,:);
x3 = P3(1,:);
y3 = P3(2,:);
x4 = P4(1,:);
y4 = P4(2,:);
plot(x1,y1,"*",x2,y2,"*",x3,y3,"*",x4,y4,"*")
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### Example of Sadle Point
###
clear
A = [1.7 -.3; -1.2 .8]
eig(A)
P = [];
for i=1:400
  b = [100*(-1)^(ceil(10*rand()))*rand();100*(-1)^(ceil(10*rand()))*rand()]; # pick random vecto
  P = [P b];
  for i=1:3 b = A*b; P = [P, b]; end;  # 3 iterations only
end;
x = P(1,:);
y = P(2,:);
plot(x,y,"*")
```