# Lecture 11 Matrix Operations (Section 2.1)

Today's lecture * basic operations that can be applied to matrices

## Terminology

**Def^** A matrix is a rectangular array of numbers. Numbers are called entries

Size of a matrix (# of rows) x (# of columns)

e.g. A is an $m \times n$ matrix $\Rightarrow$ $m$ rows and $n$ columns

Denote entry in row $i$ column $j$ by $a_{ij}$

General $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & \cdots & \cdots & a_{mn} \end{bmatrix} = [\vec{a}_1 \cdots \vec{a}_n] = [a_{ij}]$$

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \end{bmatrix}$ ← 2x3 matrix

Octave: `A=[1 2 3; 3 4 5]`

Square matrix: an $n \times n$ matrix

diagonal matrix: a square matrix of form $\begin{bmatrix} a_{11} & & \bigcirc \\ & a_{22} & \\ \bigcirc & & a_{nn} \end{bmatrix}$

identity matrix $I_n$: a $n \times n$ diagonal matrix with 1's on diagonal

Zero matrix: a matrix with all zeroes

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Octave:

- `eye(3)`
- $V = [a_1\ a_2 \cdots a_n]$
  $D = diag(v)$

# Operations on Matrices

A. Transpose ← swap rows and columns

Ex $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

B. Sums and Scalars

Let A and B be two matrices of the Same size.
If $A = [a_{ij}]$ and $B = [b_{ij}]$, then

$A + B = [a_{ij} + b_{ij}]$ ← add $(i,j)^{th}$ entries

$rA = [ra_{ij}]$ ← multiply all entries by r

↑ $r \in \mathbb{R}$

Ex $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$

$A + B = \begin{bmatrix} 0 & 1 \\ 3 & 4 \end{bmatrix}$ $3A = \begin{bmatrix} 3 & 3 \\ 6 & 6 \end{bmatrix}$

Octave: transpose(A) } find transpose
A'

A+B
r*A

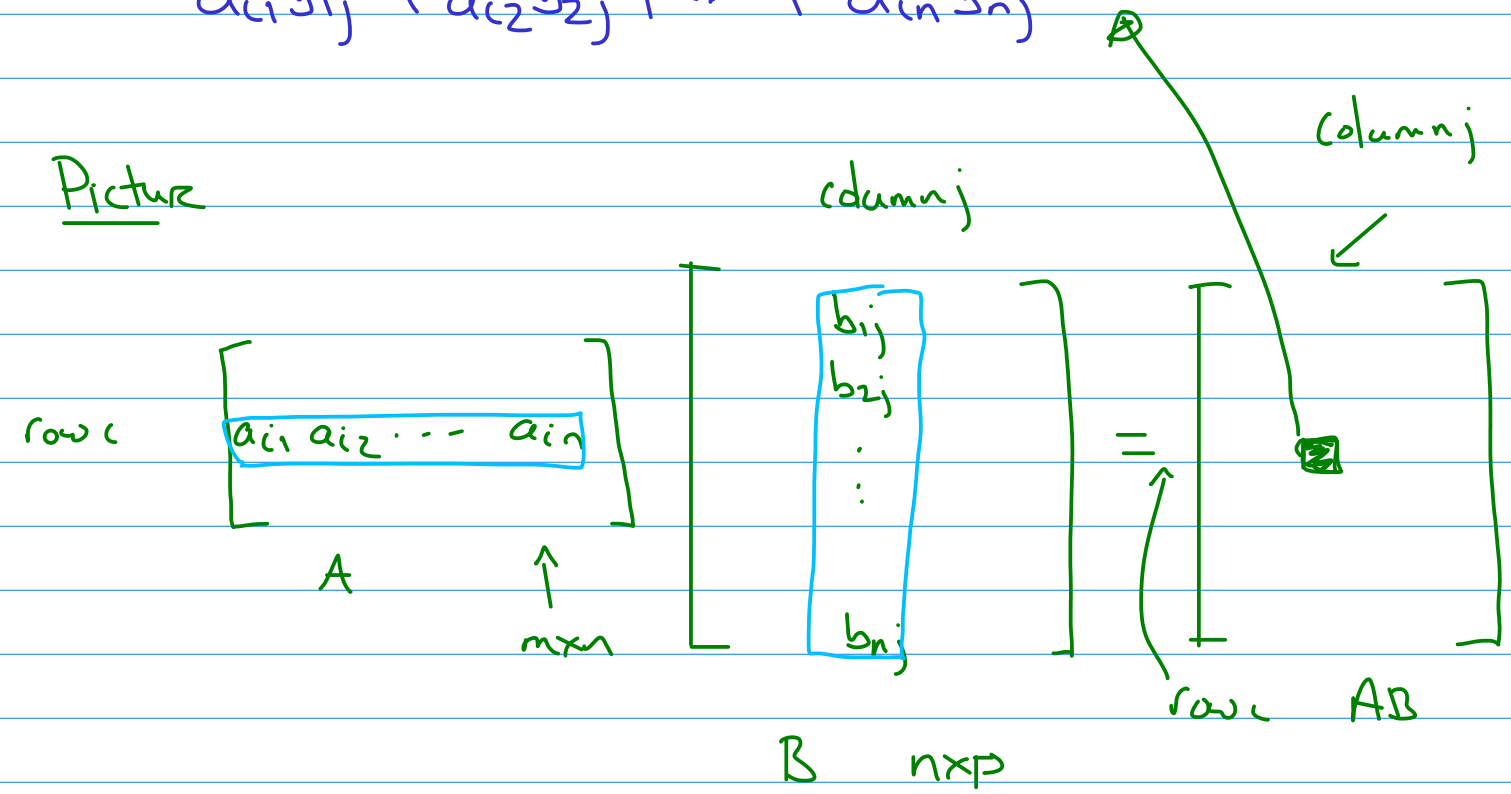
## C. Multiplication

same

Let A be an $m \times n$ matrix and B an $n \times p$ matrix.
The product AB is the $m \times p$ matrix with $(i,j)^{th}$ entry

$$a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$$

**Picture**




**Ex** $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ Compute AB

$2 \times 3$ $3 \times 3$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1\cdot1+2\cdot0+3\cdot1 & 1\cdot0+2\cdot1+3\cdot0 & 1\cdot1+2\cdot0+3(-1) \\ 4\cdot1+5\cdot0+6\cdot1 & 4\cdot0+5\cdot1+6\cdot0 & 4\cdot1+5\cdot0+6\cdot(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & -2 \\ 10 & 5 & -2 \end{bmatrix}$$

Octave A*B

Alt. P.O.V. : If $B = [\vec{b}_1 \cdots \vec{b}_p]$, then

$$AB = [A\vec{b}_1 \quad A\vec{b}_2 \cdots A\vec{b}_p]$$

(Matrices & linear transformations) Both A and B define linear transformations, i.e

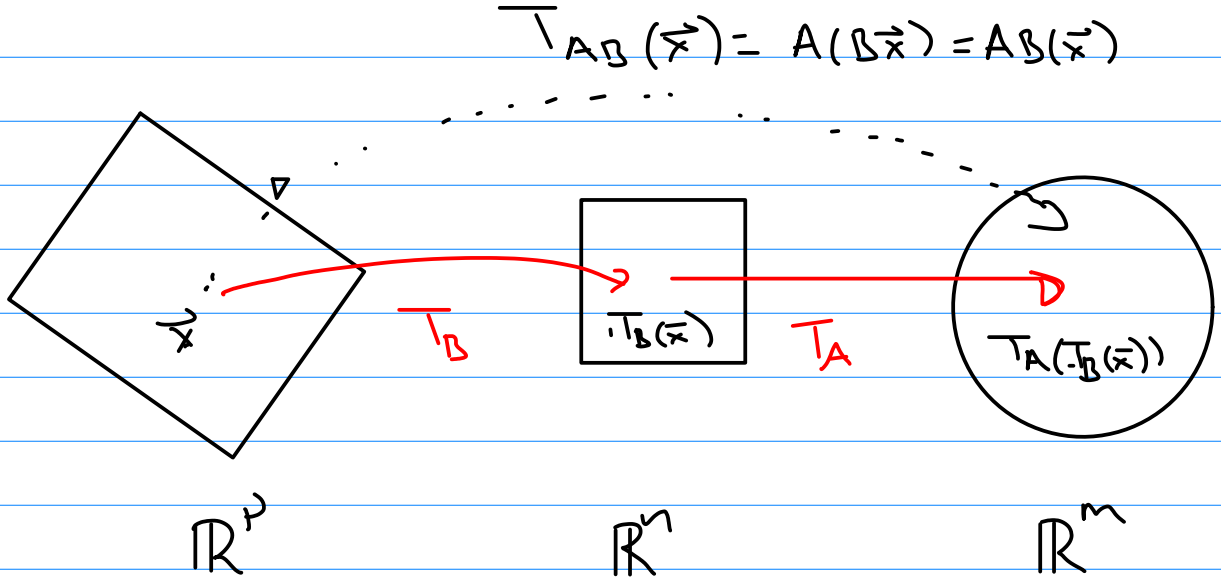
$T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^p \to \mathbb{R}^n$

where

$T_A(\vec{x}) = A\vec{x}$ and $T_B(\vec{x}) = B\vec{x}$

Then AB defines the linear transf $T_{AB} : \mathbb{R}^p \to \mathbb{R}^m$ which is defined by composing the two functions, i.e

$T_{AB}(\vec{x}) = T_A \circ T_B = T_A(T_B(\vec{x}))$




Properties of operations:

Transpose

① $(A^T)^T = A$

② $(A+B)^T = A^T + B^T$

③ For scalar $c$, $(cA)^T = c(A^T)$

④ $(AB)^T = B^T A^T$ ← order reverses

**Thm** (Arithmetic of sums, scalars, and multiplication)

Let $A, B, C$ be appropriately sized matrices, and $a, b, c \in \mathbb{R}$

(a) $A+B = B+A$ (addition commutes)

(b) $A+(B+C) = (A+B)+C$ (addition associative)

(c) $A(BC) = (AB)C$ (multiplication associative)

(d) $A(B+C) = AB+AC$ (multiplication is left distributive)

(e) $(A+B)C = AC+BC$ ( " " right " )

(f) $a(A+B) = aA+aB$

(g) $(a+b)C = aC+bC$

(h) $a(bC) = (ab)C$

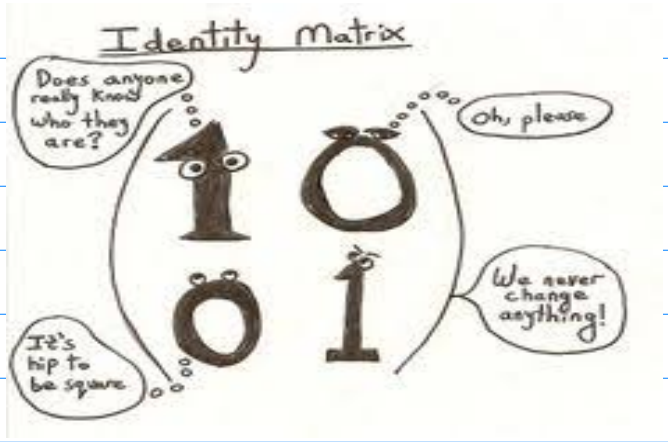
(i) $a(BC) = (aB)C$

**FACT** ① If $A$ is $m \times n$, then $I_m A = A = A I_n$

② $A + O = A$ ← zero matrix

$I_n$ acts like 1

zero matrix behaves like 0




Matrices have many (but not all) properties of integers $\mathbb{Z}$ and real numbers $\mathbb{R}$.

WARNING!! Three properties of integers/reals that do not hold for matrices

1. multiplication does not always commute
In $\mathbb{Z}$ and $\mathbb{R}$, $ab=ba$. Fails for matrices

$A=\begin{bmatrix}1&0\\2&1\end{bmatrix}$ $B=\begin{bmatrix}1&2\\0&3\end{bmatrix}$ $AB=\begin{bmatrix}1&2\\2&7\end{bmatrix}$ $\neq$ $BA=\begin{bmatrix}5&2\\6&3\end{bmatrix}$

order matters!

2. Cancellation law fails
In $\mathbb{Z}$ and $\mathbb{R}$, if $ab=ac$ and $a\neq 0$, then $b=c$. May fail for matrices!

$A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ $B=\begin{bmatrix}1&1\\2&2\end{bmatrix}$ $C=\begin{bmatrix}1&1\\-2&-2\end{bmatrix}$ $AB=\begin{bmatrix}1&1\\0&0\end{bmatrix}=AC$

but $B\neq C$ and $A\neq 0$

3. Product of two nonzero elements can be zero
In $\mathbb{Z}$ and $\mathbb{R}$, if $ab=0$, then $a=0$ or $b=0$. May fail for matrices!

$A=\begin{bmatrix}1&0\\0&0\end{bmatrix}$ $B=\begin{bmatrix}0&0\\0&1\end{bmatrix}$ $\Rightarrow AB=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

but $A\neq 0$ and $B\neq 0$

Key ideas: matrix operations: transpose, sum, scalar, multiplication