Lecture 17 3.2 Properties of Determinants

Today's lecture * how row operations change $\det(A)$
* relate $\det(A)$ to inverse

From last lecture: If A is square triangular matrix,
$\det(A)$ = product of diagonal entries

Ex $\begin{vmatrix} 2 & 4 & 6 \\ 0 & 5 & 3 \\ 0 & 0 & -3 \end{vmatrix} = 2 \cdot 5 \cdot (-3) = -30$

This suggests a way to compute $\det(A)$. In particular put A into echelon form (upper triangular)

Problem row operations can change the determinant!

Ex $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = \boxed{-2}$ $\quad \begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = 3\cdot 2 - 1\cdot 4 = \boxed{2}$

Thm (row operations and $\det(A)$) Let A be a square matrix

(a) If a multiple of one row is added to another row to form B, then $\det B = \det A$

(b) If two rows of A swapped to form B, then $\det B = -\det(A)$

(c) If one row of A multiplied by k to form B, then $\det B = k \det(A)$

Ex Apply rules to compute $\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix}$

$$\begin{vmatrix} 1 & 5 & -6 \\ -1 & -4 & 4 \\ -2 & -7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 3 & -3 \end{vmatrix} \begin{matrix} \\ \leftarrow \text{row } 1 + \text{row } 2 \\ \leftarrow \text{row } 1 (\times 2) + \text{row } 3 \end{matrix}$$

:)

$$= 3 \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix} = 3 \begin{vmatrix} 1 & 5 & -6 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{vmatrix} = 3 \cdot 1 \cdot 1 \cdot 1 = 3$$

Proof of (a) 2×2 case only

If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $B=\begin{bmatrix} a+ck & b+dk \\ c & d \end{bmatrix}$ ← row 2 × k + row 1

Then $\det(B)=(a+ck)(d)-c(b+dk)$

$= ad+cdk-cb-cdk$

$= ad-cb = \det(A)$ ∎

Consequence If A has two proportional rows (one row is a multiple of other), then $\det(A)=0$

Ex $\begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 5 & 10 & 15 \end{bmatrix}$ ← proportional

$\begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 5 & 10 & 15 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 0 & 0 & 0 \end{vmatrix}$ ← row 1 × (−5) + row 3

$= (-1)^{3+1}\cdot 0\cdot \det A_{31} + (-1)^{3+2}\cdot 0\cdot \det A_{32} + (-1)^{3+3}\cdot 0\cdot \det A_{33}$

$= 0$

Note Any matrix with a row/column all zeroes has det = 0
(do a row/column expansion on the row/column of zeroes)

Ex (hybrid approach) Do row operations to make a row/column with more zeroes. Then do cofactor expansion

Compute $\begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 4 & 10 & -4 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 5 & -3 & -1 \\ 3 & 0 & 1 & -3 \\ -6 & 0 & -4 & 9 \\ 0 & 0 & 2 & 1 \end{vmatrix}$ row 1 $\times(-2)$ + row 4

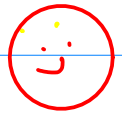
Cofactor expansion down 2nd column

$$= (-1)^{1+2} \cdot 5 \cdot \begin{vmatrix} 3 & 1 & -3 \\ -6 & -4 & 9 \\ 0 & 2 & 1 \end{vmatrix} + 0$$

$$= (-1)^{3} \cdot 5 \begin{vmatrix} 3 & 1 & -3 \\ 0 & -2 & 3 \\ 0 & 2 & 1 \end{vmatrix}$$ ← row 1 $(\times 2)$ + row 2

$$= (-1)^{3} \cdot 5 \cdot 3 \begin{vmatrix} -2 & 3 \\ 2 & 1 \end{vmatrix} + 0$$

$$= (-1) \cdot 5 \cdot 3 (-8) = \boxed{120}$$



Remark Can apply column operations just like row operations. Behaviour of determinants under column operations similar to row operations

Determinants and inverses

$\det(A)$ "measures" whether A is invertible

Thm A is invertible if and only if $\det A \neq 0$

Proof Let A be a square matrix. There are row operations that change A into its reduced row echelon form R

Each row operation changes sign of $\det(A)$ or multiplies it by a non zero constant. So

$$\det(A) = (-1)^a \cdot k \cdot \det R \quad k \neq 0 \text{ a constant}, \quad a = 0, 1$$

The matrix R is I_n or it has a row of zeroes

($\Rightarrow$) If A is invertible, $R = I_n$. So

$$\det(A) = (-1)^a \cdot k \cdot 1 \neq 0$$

($\Leftarrow$) If $\det A \neq 0$, then $\det R \neq 0$. So $R = I_n$ (if R has a row of zeroes, $\det R = 0$). But then A is invertible ∎

Other properties

Thm Let A, B be square matrices, $0 \neq k \in \mathbb{R}$

1. $\det(A^T) = \det(A)$
2. $\det(kA) = k^n \det(A)$ if A is $n \times n$
3. $\det(AB) = \det(A)\det(B)$
4. If A invertible, $\det(A^{-1}) = \frac{1}{\det(A)}$

Proof of (4)

A invertible means that $A \cdot A^{-1} = I_n$. So by (3)

$$1 = \det(I_n) = \det(A \cdot A^{-1}) = \det(A)\det(A^{-1})$$

$$\Rightarrow \det(A^{-1}) = 1/\det(A)$$ ∎

WARNING! $\det(A+B) \neq \det(A) + \det(B)$

Ex $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$ $A+B = \begin{bmatrix} 0 & 2 \\ 5 & 5 \end{bmatrix}$

$\det(A) = -5$ $\det B = -2$ $\det(A+B) = -10$

Key ideas
* determinants & inverses
* determinants & row operations/prop of determinants