

## Lecture 34 5.5 Complex Eigenvalues/vectors

Today: Geometry of  $2 \times 2$  matrices with complex eigenvalues

### Complex eigenvalues and eigenvectors

Def<sup>n</sup> If  $A$  is an  $n \times n$  complex matrix, a complex scalar  $\lambda$  is a complex eigenvalue if there is a nonzero vector  $\vec{v} \in \mathbb{C}^n$  such that

$$A\vec{v} = \lambda\vec{v}.$$

Call  $\vec{v}$  the complex eigenvector



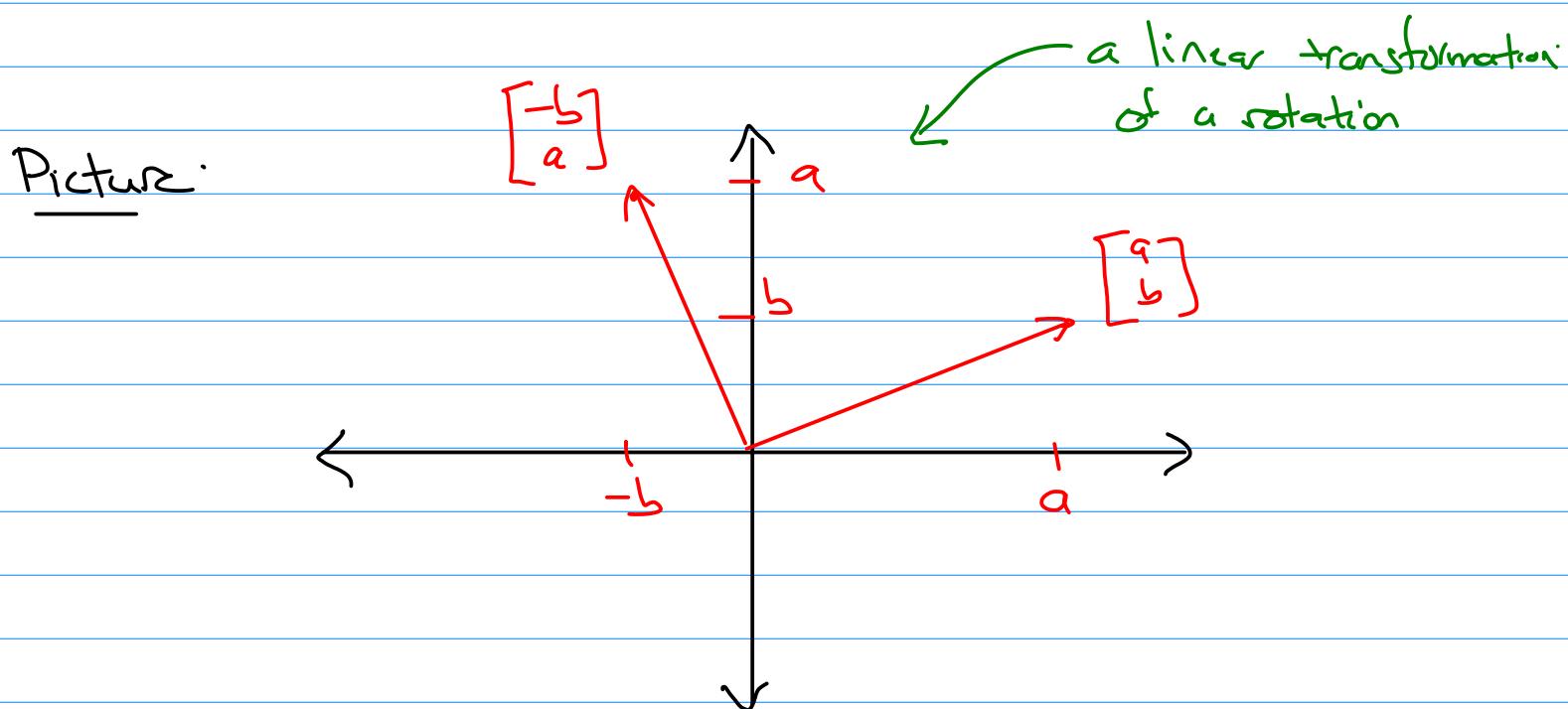
Recall over  $\mathbb{R}^n$ , eigenvectors correspond to "stretching"

Ex Let  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ , and consider the

linear transformation  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by,  $T(\vec{x}) = A\vec{x}$ .

Show geometrically that  $A$  has no real eigenvalues

For any point  $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ ,  $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ a \end{bmatrix}$



The matrix  $A$  has no real eigenvalues, since every point is sent to a new direction

$$\text{Indeed } \det(A - \lambda I_2) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1 = 0$$

$$\Leftrightarrow \lambda = \pm i$$

BIG IDEA If  $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is a linear transformation given by  $T(\vec{x}) = A\vec{x}$  and  $A$  has complex eigenvalues, then  $T$  corresponds to a rotation.

### Computing eigenvectors

conjugate of  $z = a+bi \Rightarrow \bar{z} = a-bi$

Useful fact: Let  $A$  be an  $n \times n$  real matrix. If  $\lambda$  is an eigenvalue of  $A$  with eigenvector  $\vec{x}$ , then  $\bar{\lambda}$  is an eigenvalue of  $A$  with eigenvector  $\bar{\vec{x}}$ .

Proof Given  $A\vec{x} = \lambda\vec{x}$ . So

$$\overline{A\vec{x}} = \overline{A\vec{x}} = \overline{\lambda\vec{x}} = \overline{\lambda}\overline{\vec{x}}$$

Since  $A$  is real,  $\overline{A} = A$ . So  $\overline{A\vec{x}} = \overline{\lambda}\overline{\vec{x}}$ , i.e  $\bar{\lambda}$  is an eigenvalue with eigenvector  $\bar{\vec{x}}$ .  $\blacksquare$

Problem Find eigenvectors of  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

Recall, eigenvalues  $\lambda = i, -i$

$$\boxed{\lambda = i}$$

$$A - iI_2 = \begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \sim \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix} \text{ so } x_2 \text{ free}$$

$$x_1 - x_2 i = 0 \Leftrightarrow x_1 = i x_2$$

eigenvalue is  $\vec{v} = \begin{bmatrix} ix_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} i \\ 1 \end{bmatrix}$  an eigenvector of  $\lambda = i$  is  $\vec{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$$\boxed{\lambda = -i} \text{ by thm } \overline{\vec{v}} = \begin{bmatrix} \bar{i} \\ 1 \end{bmatrix} = \begin{bmatrix} -i \\ 1 \end{bmatrix} \text{ is the}$$

eigenvector of  $\lambda = -i$

Ex  $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$  eigenvalue  $\lambda = 2+i$  with eigen vector  $\vec{x} = \begin{bmatrix} 2 \\ -1-i \end{bmatrix}$

$\Rightarrow \bar{\lambda} = 2-i$  is an eigenvalue with eigenvector  $\begin{bmatrix} 2 \\ -1+i \end{bmatrix}$

## Special case of $2 \times 2$ matrices

Thm If  $C = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$  with  $a, b$  real,

then eigenvalues are  $a \pm bi$  polar form

If  $\lambda = a + bi = |\lambda|(\cos \theta + i \sin \theta)$ , then

$$C = \begin{bmatrix} |\lambda| & 0 \\ 0 & |\lambda| \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Stretching

Proof (first part only)

$$\det(C - \lambda I_2) = \begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = (a-\lambda)^2 - b^2 = 0$$

$$\text{So } (a-\lambda)^2 - b^2 = 0 \Rightarrow a-\lambda = \pm bi \Rightarrow a \pm ib = \lambda$$

Ex  $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$  has form  $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$  with  $a=0$ ,  $b=1$

Eigenvalues are  $\lambda = 0 \pm i = \pm i$  ← as before

$$\lambda = 0 + i \Rightarrow \lambda = |\lambda|(\cos(\pi/2) + i\sin(\pi/2)) \\ = 1(\cos(\pi/2) + i\sin(\pi/2))$$

so  $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix}$

"stretch" by  $\times 1$       ↑      "rotate by  $\pi/2 = 90^\circ$ "

### General case of $2 \times 2$ matrices

If  $A$  is a real  $2 \times 2$  matrix with complex eigenvalues, has a nice decomposition.

Thm Let  $A$  be a real  $2 \times 2$  matrix with complex eigenvalue  $\lambda = a - bi$  and associated eigenvector  $\vec{v} \in \mathbb{C}^2$ . Then

$$A = P C P^{-1} \text{ where } P = [\operatorname{re}(\vec{v}) \operatorname{im}(\vec{v})]$$

and  $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

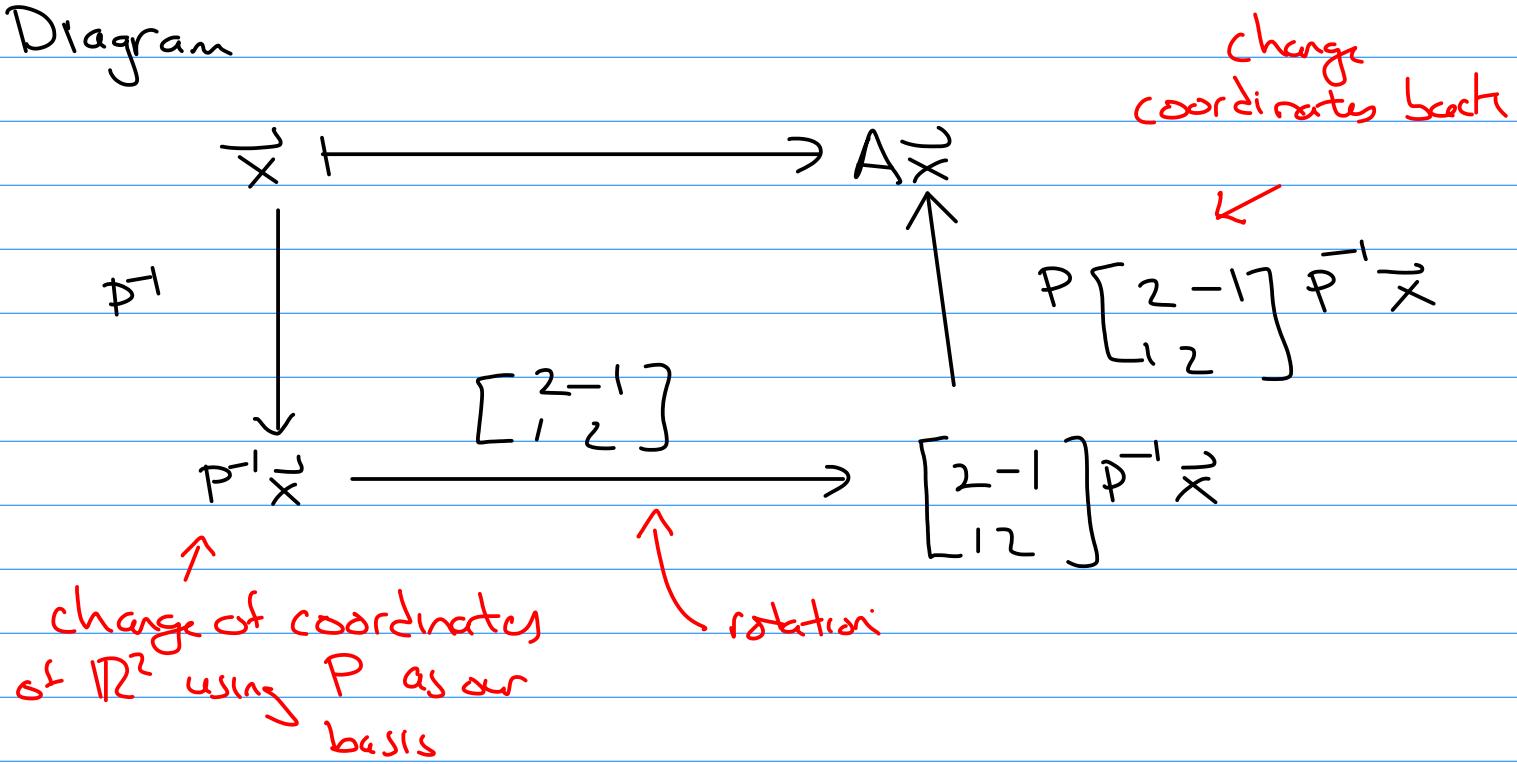
↑  
a factorization of  $A$

$$\text{Ex } A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix} \quad \lambda = 2-i \text{ and } \vec{v} = \begin{bmatrix} 2 \\ -1+i \end{bmatrix}$$

$$P = [r_{\mathbb{C}}(\vec{v}) \ im(\vec{v})] = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$$

$$\text{So } A = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}^{-1}$$

Diagram



Key idea:  $2 \times 2$  matrices with complex eigenvalues  $\Rightarrow$  rotations.