Lecture 27 4.6 Rank and Nullity Theorem

Today's Lecture: rank

Row space

If A is an $m \times n$ matrix $A = [\vec{a}_1 \cdots \vec{a}_n]$,
column space $\text{Col}(A) = \text{span}\{\vec{a}_1, \ldots, \vec{a}_n\}$

Can do the same for rows:

Ex $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 3 & 8 & 9 & 10 \end{bmatrix}$ Set $\vec{r}_1 = [1\ 2\ 3\ 4]^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$\vec{r}_2 = [2\ 4\ 6\ 7]^T$

$\vec{r}_3 = [3\ 8\ 9\ 10]^T$

the row space of $A = \text{row}(A) = \text{span}\{\vec{r}_1, \vec{r}_2, \vec{r}_3\}$

Observation $A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 8 \\ 3 & 6 & 9 \\ 4 & 7 & 10 \end{bmatrix} \Rightarrow \text{col}(A^T) = \text{row}(A)$

Defn If A is an $m \times n$ matrix, row space of A, denoted $\text{row}(A)$, is the subspace of $\mathbb{R}^n$ spanned by the columns of A^T, i.e

$$\text{row}(A) = \text{col}(A^T)$$

Thm: If A and B row equivalent, then

$$\text{row}(A) = \text{row}(B)$$

If B is in echelon form, then the nonzero rows of B form a basis for $\text{row}(A)$ and $\text{row}(B)$

.

Why? rows of B are linear combinations of rows of A (rough idea)

Procedure to find basis/dim of Row A

Step 0: Given $A = \begin{bmatrix} 1 & 3 & -2 & 0 & 2 & 0 \\ 2 & 6 & -5 & -2 & 4 & -3 \\ 0 & 0 & 5 & 10 & 0 & 15 \\ 2 & 6 & 0 & 8 & 4 & 18 \end{bmatrix}$

Step 1 row echelon form

$B = \begin{bmatrix} \textcircled{1} & 3 & 0 & 4 & 2 & 0 \\ 0 & 0 & \textcircled{1} & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Step 2 Identify the rows with leading 1's. In our example, rows 1, 2, and 3

$\vec{r}_1 = [1\ 3\ 0\ 4\ 2\ 0]^T \quad \vec{r}_2 = [0\ 0\ 1\ 2\ 0\ 0]^T \quad \vec{r}_3 = [0\ 0\ 0\ 0\ 1\ 0]^T$

Step 3 $\text{row}(A) = \text{span}\{\vec{r}_1, \vec{r}_2, \vec{r}_3\}$ ← a basis

$\dim \text{row}(A) = 3$

$=$ # of leading entries (pivots) in A

Rank Theorem

Note $\dim \text{Col}(A) = \#$ of leading 1's in A
$= \dim \text{Row}(A)$

However, $\text{Col}(A) \neq \text{Row}(A)$. In fact, they don't even "live" in same space, i.e

$\text{Col}(A) \subseteq \mathbb{R}^m$ and $\text{Row}(A) \subseteq \mathbb{R}^n$

Defn The rank of A is
$\text{rank}(A) = \dim \text{Col}(A) = \dim \text{Row}(A)$

Sometimes called Rank-Nullity Thm

Thm (Rank Thm) Let A be an $m \times n$ matrix. Then

$\text{rank}(A) + \dim \text{Nul}(A) = n$

in echelon form

Proof
$n = \#$ columns of A
$= (\#$ of leading 1's$) + (\#$ of free variables$)$
$= \text{rank}(A) + \dim \text{Nul}(A)$

□

Ex 1. If A is 6×9, can $\dim \text{Nul}(A) = 2$?
2. If A is 7×5, what is largest possible rank? Smallest possible $\dim \text{Nul}(A)$?

Sol 1. Suppose $\dim \mathrm{Nul}(A) = 2$.

Then rank theorem implies

$$\mathrm{rank}(A) + 2 = 9 \Rightarrow \mathrm{rank}(A) = 7$$

But $\mathrm{Col}(A) \subseteq \mathbb{R}^6$, so $7 = \mathrm{rank}(A) = \dim \mathrm{Col}(A) \leq \dim \mathbb{R}^6 = 6$

This is a contradiction! So $\dim \mathrm{Nul}(A) \neq 2$.

Sol 2. Since $\mathrm{rank}(A) = \dim \mathrm{Col}(A)$ and $\mathrm{Col}(A) \subseteq \mathbb{R}^7$, we have $\dim \mathrm{Col}(A) \leq 7$

By rank theorem

$$\mathrm{rank}(A) \leq \mathrm{rank}(A) + \dim \mathrm{Nul}(A) = 5$$

So $\mathrm{rank}(A) \leq \min\{5, 7\} = 5$

By rank thm

$$\dim \mathrm{Nul}(A) = 5 - \mathrm{rank}(A) \geq 5 - \min\{5, 7\} = 0$$

□

In general, if A is $m \times n$,

$$\text{rank}(A) \leq \min\{m, n\}$$
$$\dim \text{Nul}(A) \geq n - \min\{m, n\}$$

Consequences for invertible matrices

Thm Let A be $n \times n$. Then A is invertible $\iff$ $\text{rank}(A) = n$

Proof

$$\begin{aligned}
\text{rank}(A) = n &\iff \dim \text{Nul}(A) = 0 \\
&\iff \text{Nul}(A) = \{\vec{0}\} \\
&\iff A\vec{x} = \vec{0} \text{ has only trivial sol}^{n} \\
&\iff A \text{ is invertible}
\end{aligned}$$

∎

New conditions for invertible matrix theorem

Thm Let A be an $n \times n$ matrix. The following are equivalent.

1. A is an invertible matrix
2. Columns of A form a basis for $\mathbb{R}^n$
3. $\text{Col}(A) = \mathbb{R}^n$
4. $\dim \text{Col}(A) = n$
5. $\text{rank}(A) = n$
6. $\dim \text{Nul}(A) = 0$
7. $\text{Nul}(A) = \{\vec{0}\}$

Key ideas:
* row space
* rank
* rank theorem