Lecture 12 The inverse of a matrix I (Section 2.2)

Today's lecture * introduce invertible matrices
* procedure to find inverses
* connection to SLE

Inverses

The real numbers have the property: if $a \neq 0$, then there exists a multiplicative inverse a^{-1} such that $a \cdot a^{-1} = 1$

Ex $a = 17$, then $a^{-1} = 1/17$ since $17 \cdot 1/17 = 1$

Want a similar property for matrices.

Def^{n} A $n \times n$ matrix A is invertible if there is an $n \times n$ matrix

C such that $CA = I_n = AC$

- C is the inverse of A and denoted by A^{-1}
- If no C exists, A is called singular

Ex $A=\begin{bmatrix}1&1\\0&1\end{bmatrix}$ $C=\begin{bmatrix}1&-1\\0&1\end{bmatrix}$

$$AC=\begin{bmatrix}1&0\\0&1\end{bmatrix}=I_2=AC \quad \text{So } C=A^{-1}$$

determinant of A

Thm (2x2 case) Let $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$. If $ad-cb\neq 0$,

then A^{-1} exists and $A^{-1}=\frac{1}{ad-cb}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

Ex Find A^{-1} of $A=\begin{bmatrix}1&2\\4&7\end{bmatrix}$

$$A^{-1}=\frac{1}{(1\cdot 7-4\cdot 2)}\begin{bmatrix}7&-2\\-4&1\end{bmatrix}=\frac{1}{-1}\begin{bmatrix}7&-2\\-4&1\end{bmatrix}=\begin{bmatrix}-7&2\\4&-1\end{bmatrix}$$

CHECK $\begin{bmatrix}1&2\\4&7\end{bmatrix}\begin{bmatrix}-7&2\\4&-1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ ☺

Properties of inverses: If A and B invertible, then

1. A^{-1} is invertible with $(A^{-1})^{-1} = A$
2. A^T is invertible with $(A^T)^{-1} = (A^{-1})^T$
3. cA is invertible with $(cA)^{-1} = c^{-1}A^{-1}$ $(c \in \mathbb{R})$
4. AB invertible with $(AB)^{-1} = B^{-1}A^{-1}$
5. A^{-1} is unique

Proof (4) To show AB is invertible, need a matrix C such that $(AB)C = I_n = C(AB)$

Claim $C = B^{-1}A^{-1}$ works!

- $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = AA^{-1} = I_n$
- $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}I_nB = B^{-1}B = I_n$

(5) Suppose B, B' inverses of A. So $AB = I_n$ and $B'A = I_n$

But then $B'(AB) = B'I_n \Rightarrow (B'A)B = B'$

$\Rightarrow (I_nB) = B' \Rightarrow B = B'$

∎

Procedure to find A^{-1}

Joke! Ha! Ha!

Theorem: Every matrix is invertible.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} \frac{d}{ad-cb} & \frac{-b}{ad-cb} \\ \frac{-c}{ad-cb} & \frac{a}{ad-cb} \end{bmatrix}$$

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- Have a formula for 2x2 case
- WARNING A^{-1} may not exist. E.g., in 2x2 case, no A^{-1} if $ad-cb=0$

Find A^{-1} for $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$

1. make the matrix $[A \vdots I_n]$
2. apply row operations to A to make I_n. Do operations to entire matrix $[A : I_n]$
3. When matrix has form $[I_n : B]$, B is the inverse

$$\left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

inverse of A!

$$\begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}\begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7\frac{1}{2} & 3\frac{1}{2} & 1\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Octave inv(A)

Try $A = \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & 7 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & -1 & -4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & -1 & -4 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & -7 & 2 \\ 0 & 1 & 4 & -1 \end{bmatrix}$$

A^{-1}

Ex If you get a row of zeros on LHS, no inverse!

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 4 & 2 & 6 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & -4 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & -2 & 1 \end{array}\right]$$

no inverse exists!

Linking ideas: SLE & inverses

Thm If A is an $n \times n$ invertible matrix, then $A\vec{x} = \vec{b}$ has a unique sol^n for all $\vec{b} \in \mathbb{R}^n$, namely $\vec{x} = A^{-1}\vec{b}$

Why? Let $\vec{x} = A^{-1}\vec{b}$, then

$$A(A^{-1}\vec{b}) = (AA^{-1})\vec{b} = I_n\vec{b} = \vec{b}$$

Ex Solve: $x_1 + 2x_2 = 5$
$4x_1 + 7x_2 = 18$

old approach (Gaussian Elim)

$$\left[\begin{array}{cc:c} 1 & 2 & 5 \\ 4 & 7 & 18 \end{array}\right] \sim \left[\begin{array}{cc:c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right] \qquad \begin{array}{l} x_1 = 1 \\ x_2 = 2 \end{array}$$

new approach (inverses)

same answer!

$$\text{SLE} \iff \begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}\vec{x} = \begin{bmatrix} 5 \\ 18 \end{bmatrix} \leftarrow \text{form } A\vec{x} = \vec{b}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 7 \end{bmatrix}^{-1} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix} \qquad \text{So } \vec{x} = \begin{bmatrix} -7 & 2 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Recall Can have 3 types of sol^{n}s:

- 0 sol^{n}
- 1 sol^{n}
- ∞ sol^{n}s

BIG IDEA # of solutions related to the invertibility of the coefficient matrix

SLE with n equations and n unknowns has exactly one sol^{n}

coefficient matrix is invertible

coefficient matrix row reduces to I_n

Key points: · inverse of matrix

procedure to find A^{-1}