Lecture 6 — Solution sets of linear equations (Section 1.5)

Today's lecture * homogeneous systems of linear equations
* describe all sol'ns to a SLE as a span of vectors

Homogeneous linear systems

Def^n A s.l.e. is homogeneous if it has the form $A\vec{x} = \vec{0}$ (zero vector)

A homogeneous s.l.e. always has at least one sol^n:

namely, $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ← normally called trivial sol^n

Ex homog s.l.e

$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= 0 \\ -2x_1 - 5x_2 + 4x_3 &= 0 \\ -x_1 + 2x_2 + 2x_3 &= 0 \end{aligned} \iff \begin{bmatrix} 1 & 3 & -2 \\ -2 & -5 & 4 \\ -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Fundamental Q for homog s.l.e.:

does a homog s.l.e have only the trivial sol^n (one sol^n) or an infinite # of sol^ns?

Thm $A\vec{x}=\vec{0}$ has a non-trivial sol^n if and only if the system has at least one free variable

Ex (Con't)

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ -2 & -5 & 4 & 0 \\ -1 & 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \textcircled{1} & 3 & -2 & 0 \\ 0 & \textcircled{1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

pivots; x_3 is free variable

The homog s.l.e. has a nontrivial sol^n:

x_1, x_2 basic variables

x_3 free variable

Describe basic variables in terms of free variables:

$$x_1 + 3x_2 - 2x_3 = 0 \implies x_1 - 2x_3 = 0 \implies x_1 = 2x_3$$
$$x_2 = 0 \qquad\qquad x_2 = 0 \qquad\qquad x_2 = 0$$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is a sol^n to $A\vec{x} = \vec{0}$ if and only if

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \leftarrow x_3 \in \mathbb{R}$$

Every sol^n of $A\vec{x} = \vec{0}$ "lives in"

$$\text{span}\left\{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}\right\} = \left\{c\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \mid c \in \mathbb{R}\right\}$$

Nonhomogeneous SLE

Defn A s.l.e. is **non-homogeneous** if $A\vec{x}=\vec{b}$ with $\vec{b}\neq\vec{0}$

Ex Find all sol'ns to $A\vec{x}=\vec{b}$ when

$$A=\begin{bmatrix}1 & 3 & -2\\ -2 & -5 & 4\\ 1 & 2 & 2\end{bmatrix} \quad \text{and} \quad \vec{b}=\begin{bmatrix}1\\ 2\\ 19\end{bmatrix}$$

Sol'n

r.r.e.f

$$\left[\begin{array}{ccc|c}1 & 3 & -2 & 1\\ -2 & -5 & 4 & 2\\ -1 & 2 & 2 & 19\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 3 & -2 & 1\\ 0 & 1 & 0 & 4\\ 0 & 0 & 0 & 0\end{array}\right] \sim \left[\begin{array}{ccc|c}1 & 0 & -2 & -11\\ 0 & 1 & 0 & 4\\ 0 & 0 & 0 & 0\end{array}\right]$$

x_1, x_2 are basic variables and x_3 is free variable

Describe basic var. in terms of free var.

$$x_1 = -11 + 2x_3$$
$$x_2 = 4$$
$$x_3 = x_3$$

As a vector, soln to $A\vec{x}=\vec{b}$ has form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -11+2x_3 \\ 4 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_3 \\ 0 \\ x_3 \end{bmatrix} + \begin{bmatrix} -11 \\ 4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -11 \\ 4 \\ 0 \end{bmatrix}$$

$x_3 \in \mathbb{R}$

Let $\vec{p} = \begin{bmatrix} -11 \\ 4 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Soln has form

$\vec{x} = \vec{p} + c\vec{v}$ with $c \in \mathbb{R}$

parametric form

Note $c\vec{v}$ is a soln to homog s.le. $A\vec{x}=\vec{0}$!!

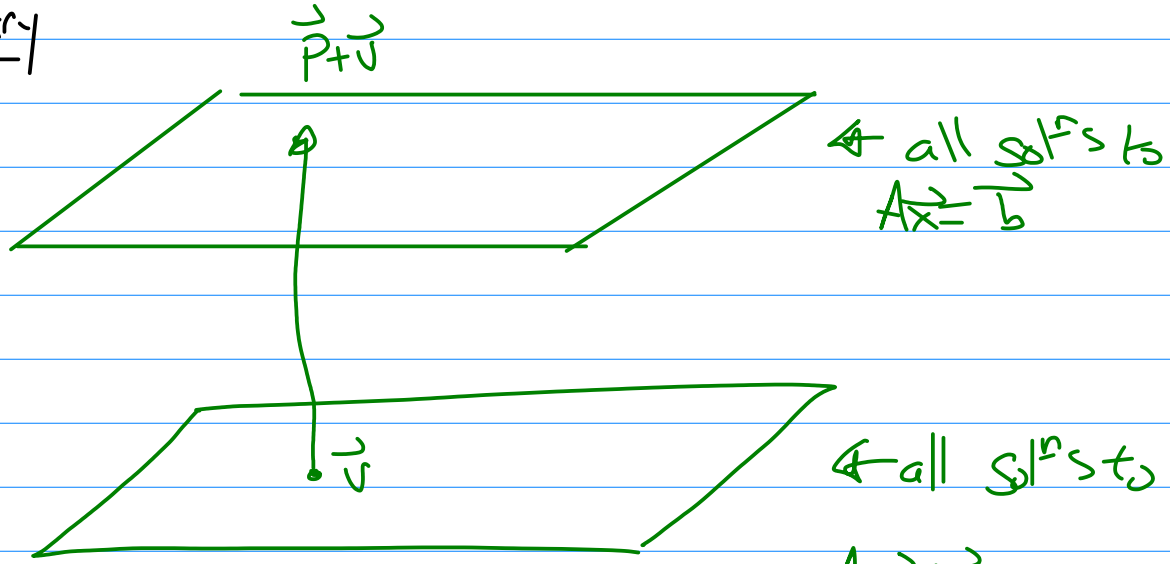
Q How are the solution sets of $A\vec{x}=\vec{0}$ and $A\vec{x}=\vec{b}$ related?

Thm Let $\vec{p}$ be any soln to $A\vec{x}=\vec{b}$. Then

$$B=\{\vec{w} \mid A\vec{w}=\vec{b}\} = \{\vec{p}+\vec{v} \mid \vec{v} \text{ any soln to } A\vec{x}=\vec{0}\} = C$$

$\vec{w} \mid A\vec{w}=\vec{b}$: all solns to $A\vec{x}=\vec{b}$

Geometry



← all solns to $A\vec{x}=\vec{b}$

← all solns to $A\vec{x}=\vec{0}$

Solution sets are "translations"

Consequence: to find all sol'n's to $A\vec{x}=\vec{b}$, find one sol'n to $A\vec{x}=\vec{b}$ and add it to all sol'n's of $A\vec{x}=\vec{0}$.

Why? Want to show $B=C$

Need to show $B\subseteq C$ and $C\subseteq B$

$\subseteq$ B is a subset of C

Let $\vec{p}+\vec{v}\in C$. Then

$$A(\vec{p}+\vec{v}) = A\vec{p}+A\vec{v} = A\vec{p}+\vec{0} = \vec{b}.$$

So $\vec{p}+\vec{v}\in B$ So $C\subseteq B$

Let $\vec{w}\in B$. This means $A\vec{w}=\vec{b}$

So

$$A(\vec{w}+\vec{p}-\vec{p})=\vec{b}.$$

Thus

$$A\vec{p}+A(\vec{w}-\vec{p})=\vec{b}$$

So

$$\vec{b}+A(\vec{w}-\vec{p})=\vec{b} \Rightarrow A(\vec{w}-\vec{p})=\vec{0}.$$

Thus $\vec{w}=\vec{p}+(\vec{w}-\vec{p})$

$\vec{w}-\vec{p}$ a sol'n to $A\vec{x}=\vec{0}$.

So $B\subseteq C$. So $B=C$.

Writing sol^n's in parametric form

1. Row reduce augmented matrix to reduced row echelon form.
2. Express each basic variable in terms of free variables
3. Express typical sol^n $\vec{x}$ as a vector whose entries depend upon free variables (if any)
4. Decompose $\vec{x}$ into linear combinations of vectors with numeric entries using free variables as parameters.

Problem Write sol^n to $A\vec{x}=\vec{b}$ in parametric form

$$A=\begin{bmatrix}1 & -1 & -1 & 2\\ 2 & -2 & -1 & 3\\ -1 & 1 & -1 & 0\end{bmatrix} \qquad \vec{b}=\begin{bmatrix}1\\3\\-3\end{bmatrix}$$

Step 1

$$\left[\begin{array}{cccc:c}1 & -1 & -1 & 2 & 1\\ 2 & -2 & -1 & 3 & 3\\ -1 & 1 & -1 & 0 & -3\end{array}\right] \sim \left[\begin{array}{cccc:c}\textcircled{1} & -1 & 0 & 1 & 2\\ 0 & 0 & \textcircled{1} & -1 & 1\\ 0 & 0 & 0 & 0 & 0\end{array}\right]$$

Step 2 basic variables $= x_1, x_3$
free variables $= x_2, x_4$

$$x_3 - x_4 = 1 \Rightarrow x_3 = 1 + x_4$$
$$x_1 - x_2 + x_4 = 2 \Rightarrow x_1 = 2 + x_2 - x_4$$

Step 3 Typical soln

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 + x_2 - x_4 \\ x_2 \\ 1 + x_4 \\ x_4 \end{bmatrix}$$

Step 4 $\vec{x} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ $x_2, x_4 \in \mathbb{R}$

parametric form

Key ideas: * homogeneous vs. non-homogeneous
* parametric soln sets