Lecture 18 Cramer's Rule, Volume, & Linear Transformations (Section 3.3)

Today's lecture
- Cramer's rule → tool to solve $A\vec{x} = \vec{b}$
- classical adjoint → connection to A^{-1}
- linear transformations and volume

I. Cramer's Rule

theoretical (not practical) tool to solve $A\vec{x} = \vec{b}$.

Notation Let $A = [\vec{a}_1 \cdots \vec{a}_n]$ be any $n \times n$ matrix and $\vec{b} \in \mathbb{R}^n$. Let

$$A_i(\vec{b}) = [\vec{a}_1 \cdots \vec{b} \cdots \vec{a}_n]$$

← replace column i with $\vec{b}$

Ex: If $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$, then

$$A_1(\vec{b}) = \begin{bmatrix} 5 & 2 \\ 3 & 4 \end{bmatrix} \qquad A_2(\vec{b}) = \begin{bmatrix} 3 & 5 \\ 1 & 3 \end{bmatrix}$$

(Cramer's Rule) Suppose A is an $n \times n$ invertible matrix. For any $\vec{b} \in \mathbb{R}^n$, $A\vec{x} = \vec{b}$ has a unique soln $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ where

$$x_i = \frac{\det A_i(\vec{b})}{\det A} \quad \text{for } i = 1, \ldots, n.$$

Ex (as above)

$\det(A) = 10$
$\det(A_1(\vec{b})) = 14$
$\det(A_2(\vec{b})) = 4$

$$\begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

has soln

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 14/10 \\ 4/10 \end{bmatrix}$$

Formula for A^{-1} via Cramer's Rule

Observation If A has an inverse A^{-1}, then the j^{th} column of A^{-1} satisfies

$$A\vec{x} = \vec{e}_j = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \leftarrow j^{th} \text{ row}$$

Ex $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$ $A^{-1} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ -3/2 & -2 & -1/2 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix} \begin{bmatrix} 7 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \vec{e}_2$$

Consequence: To find j^{th} column of A^{-1}, need to solve $A\vec{x} = \vec{e}_j$

CRAMER'S RULE

$\Rightarrow$ i^{th} entry of column j of A^{-1} = $\dfrac{\det A_i(\vec{e}_j)}{\det A}$

(= a sol^n to $A\vec{x} = \vec{e}_j$)

Note $A_i(\vec{e}_j) = \begin{bmatrix} * & & 0 & \cdots & * \\ & & \vdots & & \vdots \\ & & 1 & & \\ * & & 0 & & * \end{bmatrix}$ ← j^{th} row

(j^{th} column ↑)

Do cofactor expansion down i^{th}-column

$$\det(A_i(\vec{e}_j)) = (-1)^{j+i} \cdot 1 \det A_{ji} = C_{ji}$$

(A_{ji}: row j, column removed)

also called the cofactor

So, $(i,j)^{th}$ entry of $A^{-1} = \frac{C_{ji}}{\det(A)}$ (note: subscripts are reversed)

Hence

$$A^{-1} = \frac{1}{\det(A)} \underbrace{\begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & & & \\ \vdots & & & \\ C_{1n} & \cdots & & C_{nn} \end{bmatrix}}_{\text{adjoint}} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Octave: adjoint (A)

Ex (of adjoint and inverse)

Find A^{-1} of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$ $\qquad C_{ij} = (-1)^{i+j} \det A_{ij}$

$C_{11} = (-1)^{1+1} \begin{vmatrix} 2 & 4 \\ 3 & -3 \end{vmatrix} = 1 \cdot (-18) = -18$

$C_{12} = (-1)^{1+2} \begin{vmatrix} 2 & 4 \\ 1 & -3 \end{vmatrix} = 10$

$C_{13} = (-1)^{1+3} \begin{vmatrix} 2 & 2 \\ 1 & 3 \end{vmatrix} = 4$

$C_{21} = (-1)^{2+1} \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix} = 3$

$C_{22} = (-1)^{2+2} = \begin{vmatrix} 1 & -1 \\ 1 & -3 \end{vmatrix} = -2$

$C_{23} = (-1)^{2+3} \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = -1$

$C_{31} = (-1)^{3+1} \begin{vmatrix} 2 & -1 \\ 2 & 4 \end{vmatrix} = 10$

$C_{32} = (-1)^{3+2} \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix} = -6$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 1 & 2 \\ 2 & 2 \end{vmatrix} = -2$

So $\operatorname{adj}(A) = \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}$

$\det(A) = \boxed{-2}$

So $A^{-1} = \frac{1}{-2}\begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}$

II Determinants: Area and Volume

Thm • If A is a 2×2 matrix, and P a parallelogram defined by columns of P, then

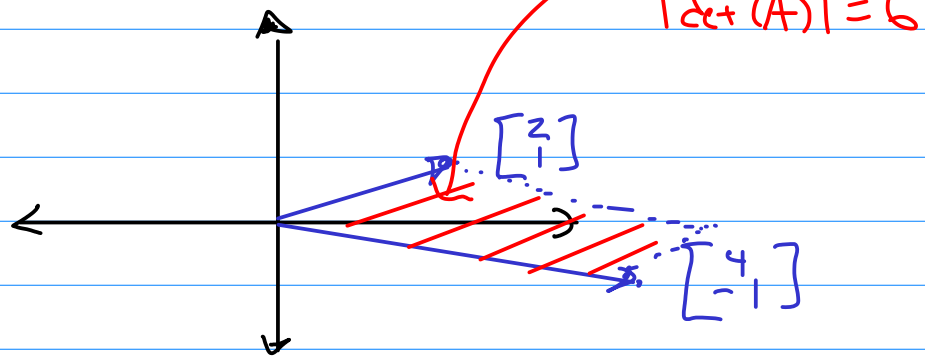
area of $P = |\det(A)|$

• If A is 3×3 matrix, and V parallelepiped defined by columns of A, then

volume of $V = |\det(A)|$

Ex $A = \begin{bmatrix} 2 & 4 \\ 1 & -1 \end{bmatrix}$

$\det(A) = -6$

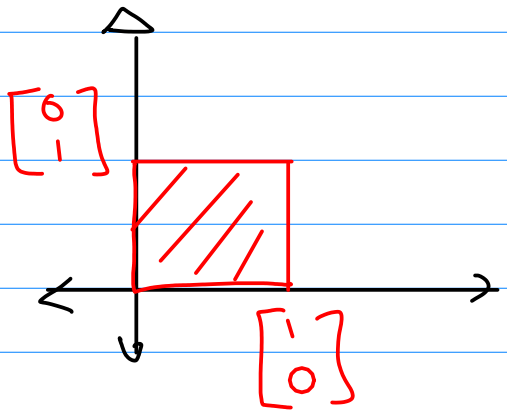


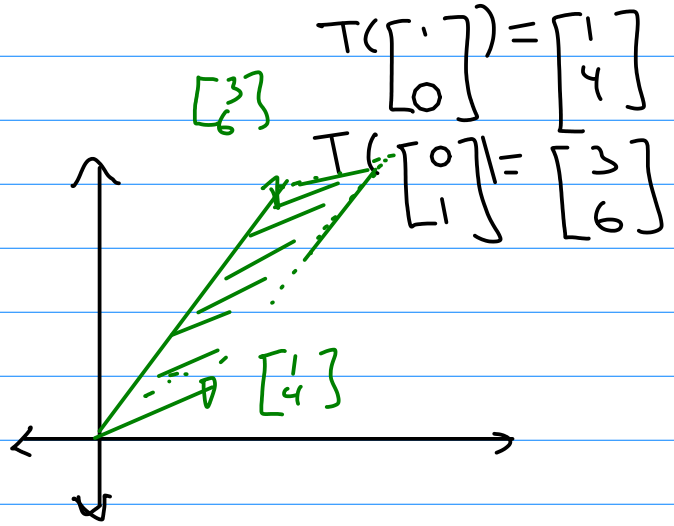
Thm (area & linear transformations)

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation w/ standard matrix A. If S is a parallelogram in $\mathbb{R}^2$, then

$$\{\text{area of } T(S)\} = |\det(A)| \{\text{area of } S\}$$

Ex Let S be unit square and suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $\vec{x} \mapsto \begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} \vec{x}$





Area of $T(S)$ = $|\det(A)|$ Area of S

(6 under Area of $T(S)$; 1 under Area of S)

green box = red box

Note Similar statements hold for $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ (see text)

key ideas: * applications of determinants:
cramer's rule, inverse formula,
area/volume