Lecture 33 — Intro to Complex Numbers (Appendix B)

Today's Goal: introduce properties of complex numbers

Complex numbers & linear algebra

Recall $x^2 = -1$ has no real root (i.e. no sol^n in $\mathbb{R}$)

Let $i = \sqrt{-1}$ ← imaginary number. i is a sol^n

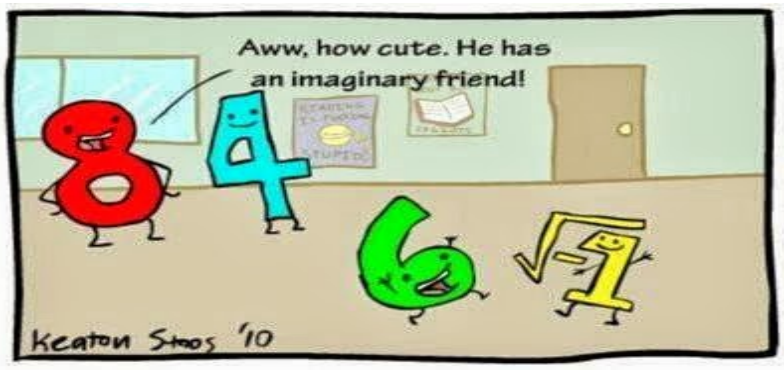
$(i)^2 = (\sqrt{-1})^2 = -1$

A complex number is a number of the form

$a + bi$ with $a, b \in \mathbb{R}$

e.g. $3+2i$, $3-2i$, $4i$, $17 = 17 + 0i$

Complex numbers: $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$



← ha! ha!

Motivation $\Rightarrow$ Complex eigenvalues & eigenvectors

Ex Find eigenvalues of $A=\begin{bmatrix} -1 & -5 \\ 4 & 7 \end{bmatrix}$

$$\det(A-\lambda I_2)=\begin{vmatrix} -1-\lambda & -5 \\ 4 & 7-\lambda \end{vmatrix}$$

$$=(-1-\lambda)(7-\lambda)-(-20)=\lambda^2-6\lambda+13=0$$

Via quadratic formula:

$$\lambda=\frac{6\pm\sqrt{(-6)^2-4(13)}}{2}=\frac{6\pm\sqrt{36-52}}{2}=\frac{6\pm\sqrt{-16}}{2}$$

$$=\frac{6\pm\sqrt{16}\sqrt{-1}}{2}=\frac{6\pm 4i}{2}=3\pm 2i$$

So eigenvalues are $\lambda_1=3+2i$ and $\lambda_2=3-2i$

Complex numbers show up naturally since $\det(A-\lambda I_n)=0$ has exactly n roots, counting multiplicity, if we allow complex numbers

Arithmetic of Complex numbers

Write complex numbers as $z = a + bi$

real part of z $= Re(z) = a$
imaginary part of z $= Im(z) = b$
Ex $z = 3 - 2i \Rightarrow Re(z) = 3$ and $Im(z) = -2$

1. addition $(a+bi)+(c+di) = (a+c) + (b+d)i$
2. Subtraction $(a+bi)-(c+di) = (a-c) + (b-d)i$

3. multiplication $(a+bi)(c+di) = ac + adi + bci + bdi^2$

$= ac + adi + bci - bd$ $\qquad i^2 = -1$
$= (ac - bd) + (ad + bc)i$

Ex $(2+3i)(4+6i) = 8 + 12i + 12i + 18(i^2)$
$= 8 + 24i - 18$
$= -10 + 24i$

For division, need

a. conjugate: if $z = a+bi$, then $\bar{z} = a-bi$
b. modulus: if $z = a+bi$, then $|z| = \sqrt{a^2+b^2}$

Ex If $z = 2+3i$, conjugate $\bar{z} = 2-3i$
modulus $|z| = \sqrt{2^2+3^2} = \sqrt{13}$

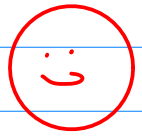
4. division. If $z_1 = a+bi$ and $z_2 = c+di$,

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \bar{z}_2}{z_2 \cdot \bar{z}_2} = \frac{z_1 \bar{z}_2}{|z_2|^2} \Rightarrow \frac{a+bi}{c+di} = \frac{(a+bi)(c-di)}{c^2+d^2}$$

If $z = a+bi$, then $\frac{1}{z} = \frac{a-bi}{a^2+b^2}$

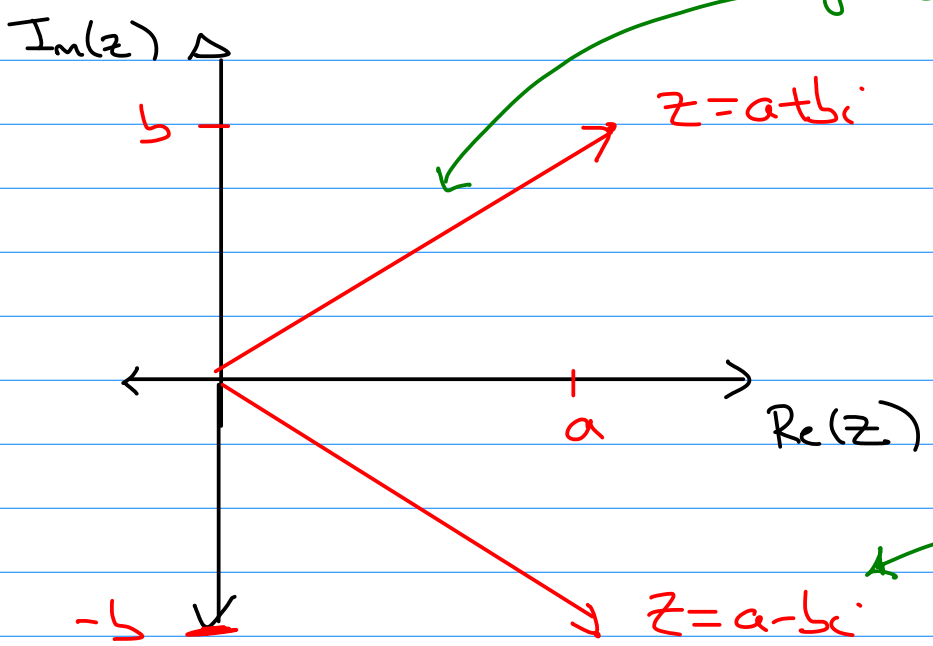
Ex $\frac{1+i}{2+3i} = \frac{(1+i)(2-3i)}{2^2+3^2} = \frac{2-3i+2i+3}{13} = \frac{5-i}{13}$

$$= \frac{5}{13} + \left(\frac{-1}{13}\right)i$$



Geometric Interpretation & Polar Coordinates

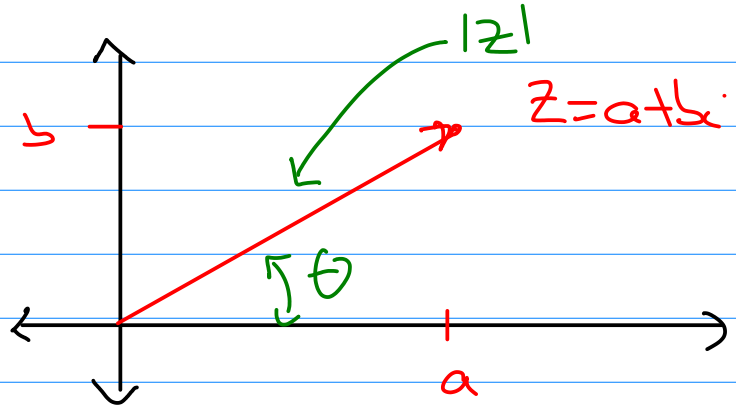
Plot $z = a + bi$ in a plane



length of this vector $= \sqrt{a^2+b^2} = |z|$ ← the modulus

conjugate is a reflection

Note



So $a = |z| \cos\theta$

$b = |z| \sin\theta$

Polar form of $z = a + bi$

$$z = |z|\cos\theta + (|z|\sin\theta)i = |z|(\cos\theta + i\sin\theta)$$

$$= |z|\,\mathrm{cis}(\theta)$$

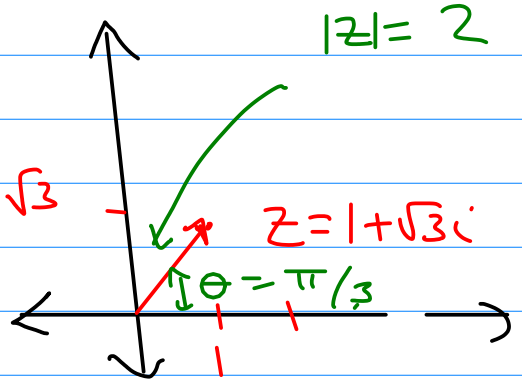
Note Call θ = argument of z =

$$\cos^{-1}\left(a/|z|\right) = \sin^{-1}\left(b/|z|\right)$$

Ex Write $z = 1+\sqrt{3}i$ in polar form

$$|z| = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{1+3} = \sqrt{4} = 2$$

$$\theta = \cos^{-1}\left(\tfrac{1}{2}\right) = \pi/3$$



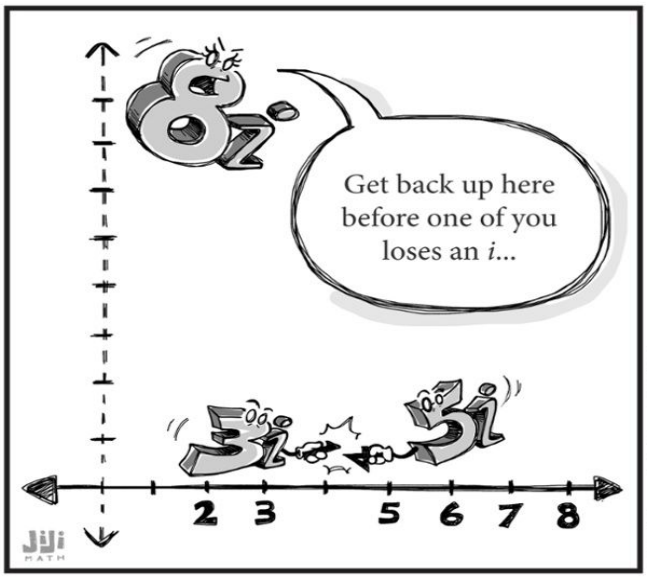
So

$$z = 1+\sqrt{3}i$$
$$= 2\left(\cos(\pi/3) + i\sin(\pi/3)\right)$$

Advantage: division & multiplication easier in polar form

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

- $z_1 z_2 = (r_1 r_2)\left(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right)$
- $z_1/z_2 = \left(r_1/r_2\right)\left(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right)$



← haha!!

Matrices of Complex Numbers

Let $\mathbb{C}^n = \left\{ \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \mid z_i \in \mathbb{C} \right\}$ ← n-tuples of complex numbers

A complex matrix is a matrix with real and complex numbers

Ex $A = \begin{bmatrix} 2+i & i & 4 \\ -1 & 2i & 4+6i \end{bmatrix}$ $\vec{u} = \begin{bmatrix} 3+2i \\ 1 \\ 0 \end{bmatrix}$

Conjugate of A or $\vec{u}$ ← take the conjugate of each entry.

Ex

$$\overline{A} = \begin{bmatrix} 2-i & -i & 4 \\ -1 & -2i & 4-6i \end{bmatrix} \quad \vec{u} = \begin{bmatrix} 3-2i \\ 1 \\ 0 \end{bmatrix}$$

$re(A)$ = real part of A
$im(A)$ = imaginary part of A

$$A = re(A) + im(A) = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 0 & 4 \end{bmatrix} + \begin{bmatrix} i & i & 0 \\ 0 & 2i & 6i \end{bmatrix}$$

Properties

1. $\overline{\overline{A}} = A$
2. $\overline{(A^T)} = (\overline{A})^T$
3. $\overline{(AB)} = \overline{A}\,\overline{B}$
4. $\overline{\overline{\vec{u}}} = \vec{u}$
5. $\overline{k\vec{u}} = \overline{k}(\overline{\vec{u}})$
6. $\overline{\vec{u}+\vec{v}} = \overline{\vec{u}} + \overline{\vec{v}}$

Next time: matrices with complex eigenvalues/vectors

Key ideas
* Complex numbers
* arithmetic ($+, -, \times, \div$)
* polar form
* geometry