

Lecture 24 Gram-Schmidt I

- 6.1 Inner products, length, & orthogonality
- 6.2 Orthogonal Sets

Last time: bases & coordinate systems

Today "good" bases in \mathbb{R}^n

Properties of vectors in \mathbb{R}^n

Defⁿ Given $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in \mathbb{R}^n , the

dot product of \vec{u} and \vec{v} is

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

Ex $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, then

$$\vec{u} \cdot \vec{v} = 1 \cdot (3) + 2(-1) = 1$$

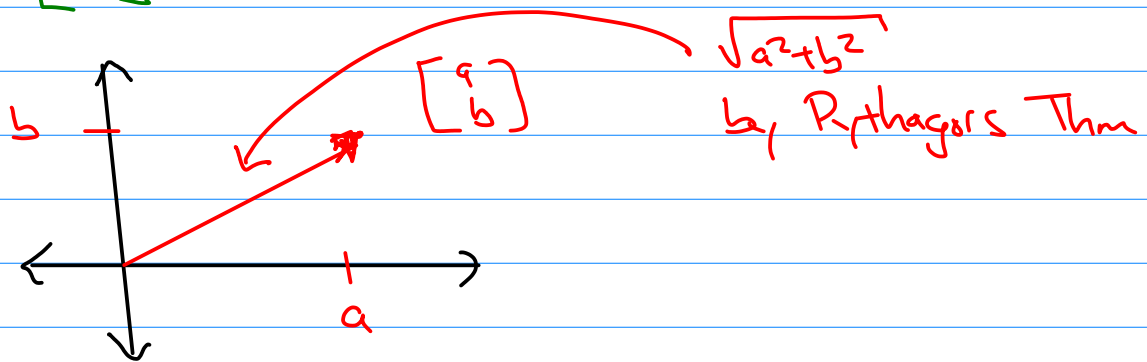
(Properties of dot product)

1. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
2. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$
3. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
4. $\vec{u} \cdot \vec{u} = 0 \Leftrightarrow \vec{u} = \vec{0}$. Otherwise $\vec{u} \cdot \vec{u} > 0$

Defⁿ length (or norm) of $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \in \mathbb{R}^n$ is

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$$

Ex If $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$, then $\|\vec{v}\| = \sqrt{a^2 + b^2}$



Note $\cdot \|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

$\cdot \vec{u}$ is a unit vector if $\|\vec{u}\| = 1$

Defⁿ $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$

Big idea: \vec{u} and \vec{v} orthogonal \Leftrightarrow \vec{u} and \vec{v} are "perpendicular" to each other

Orthogonal & Orthonormal Sets

Defⁿ A set $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ in \mathbb{R}^n is

- an orthogonal set if $\vec{u}_i \cdot \vec{u}_j = 0$ for all $i \neq j$
- an orthonormal set if S is orthogonal and $\|\vec{u}_i\| = 1$ for all i

Ex $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$ is an orthonormal set in \mathbb{R}^n

Ex $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ is an orthogonal set, but not orthonormal

(orthogonal check) $1 \cdot 0 + 0 \cdot 1 + 0 \cdot 1 = 0 \quad \checkmark$

(orthonormal check) $\sqrt{1^2 + 0^2 + 0^2} = 1 \leftarrow \text{correct length} = \|\vec{u}_1\|$
 $\sqrt{0^2 + 1^2 + 1^2} = \sqrt{2} \leftarrow \text{incorrect length} = \|\vec{u}_2\|$

Fact: Given any orthogonal set $S = \{\vec{u}_1, \dots, \vec{u}_p\}$,
can make an orthonormal set

Precisely, if $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ is orthogonal, then

$$S^* = \left\{ \frac{1}{\|\vec{u}_1\|} \vec{u}_1, \frac{1}{\|\vec{u}_2\|} \vec{u}_2, \dots, \frac{1}{\|\vec{u}_p\|} \vec{u}_p \right\} \text{ is orthonormal}$$

vectors are rescaled so have length 1
(called normalizing)

Ex $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ orthogonal, so

$$S^* = \left\{ \frac{1}{1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \right\}$$

is orthonormal

Orthogonal sets and linear independence

Thm If $S = \{\vec{u}_1, \dots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in \mathbb{R}^n , then S is also linearly independent.

Proof: Suppose $c_1\vec{u}_1 + \dots + c_p\vec{u}_p = \vec{0}$. Want to show that $c_1 = \dots = c_p = 0$.

For each $i = 1, \dots, p$,

$$(c_1\vec{u}_1 + \dots + c_p\vec{u}_p) \cdot \vec{u}_i = \vec{0} \cdot \vec{u}_i = 0$$

Then

$$c_1(\vec{u}_1 \cdot \vec{u}_i) + \dots + c_i(\vec{u}_i \cdot \vec{u}_i) + \dots + c_p(\vec{u}_p \cdot \vec{u}_i) = 0 \quad (*)$$

But the set S is orthogonal, so $\vec{u}_j \cdot \vec{u}_i = 0$ for $j \neq i$.
So $(*)$ becomes

$$0 + \dots + 0 + c_i(\vec{u}_i \cdot \vec{u}_i) + 0 + \dots + 0 = 0$$

Since $\vec{u}_i \neq \vec{0}$, $\vec{u}_i \cdot \vec{u}_i \neq 0$. So $c_i = 0$. So
all $c_1 = c_2 = \dots = c_p = 0$, i.e., S is linearly independent. \square

Defⁿ A basis S for \mathbb{R}^n is

- an orthogonal basis if S is an orthogonal set
- an orthonormal basis if S is orthonormal

Ex $\{\vec{e}_1, \dots, \vec{e}_n\}$ is an orthonormal basis for \mathbb{R}^n

Orthogonal & orthonormal bases and coordinates

Recall If $B = \{\vec{b}_1, \dots, \vec{b}_n\}$ is a basis for \mathbb{R}^n , every vector $\vec{x} \in \mathbb{R}^n$ can be expressed uniquely as

$$\vec{x} = c_1 \vec{b}_1 + \dots + c_n \vec{b}_n$$

The B -coordinate of \vec{x} : $[\vec{x}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$

Difficulty: can be lots of work to find $[\vec{x}]_B$

However...

Thm Let $S = \{\vec{b}_1, \dots, \vec{b}_n\}$ be a basis for \mathbb{R}^n

1. If S is orthogonal, then for all $\vec{x} \in \mathbb{R}^n$

$$\vec{x} = \frac{(\vec{x} \cdot \vec{b}_1)}{\|\vec{b}_1\|^2} \vec{b}_1 + \frac{\vec{x} \cdot \vec{b}_2}{\|\vec{b}_2\|^2} \vec{b}_2 + \dots + \frac{\vec{x} \cdot \vec{b}_n}{\|\vec{b}_n\|^2} \vec{b}_n$$

$$\Rightarrow [\vec{x}]_S = \begin{bmatrix} \frac{\vec{x} \cdot \vec{b}_1}{\|\vec{b}_1\|^2} \\ \vdots \\ \frac{\vec{x} \cdot \vec{b}_n}{\|\vec{b}_n\|^2} \end{bmatrix}$$

2. If S is orthonormal, then for all $\vec{x} \in \mathbb{R}^n$

$$\vec{x} = (\vec{x} \cdot \vec{b}_1) \vec{b}_1 + (\vec{x} \cdot \vec{b}_2) \vec{b}_2 + \dots + (\vec{x} \cdot \vec{b}_n) \vec{b}_n$$

$$\Rightarrow [\vec{x}]_S = \begin{bmatrix} (\vec{x} \cdot \vec{b}_1) \\ \vdots \\ (\vec{x} \cdot \vec{b}_n) \end{bmatrix}$$

So, orthogonal/orthonormal bases "nice"
bases \rightarrow can quickly find $[\vec{x}]_B$

Ex $\beta = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for \mathbb{R}^2 (check!)

$$\text{Find } \begin{bmatrix} 2 \\ 3 \end{bmatrix}_\beta = \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}}{(\sqrt{1^2 + 1^2})^2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{\begin{bmatrix} 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix}}{(\sqrt{1^2 + (-1)^2})^2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$= \frac{5}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{(-1)}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix}_\beta = \begin{bmatrix} 5/2 \\ -1/2 \end{bmatrix}$$

Key ideas: dot product, norm, orthogonal
orthogonal & orthonormal sets & bases.

Next time: making orthogonal sets (bases)