

Lecture 4 Vector Equations (Section 1.3)

Today's lecture:

- introduce vectors in \mathbb{R}^n
- introduce linear combinations
- introduce span

Vectors in \mathbb{R}^n

Defⁿ A matrix with one column is a column vector or simply a vector

Ex $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

$$\mathbb{R}^n = \left\{ \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \mid u_i \in \mathbb{R} \right\} \leftarrow \text{n-space (all vectors with n-entries)}$$

two vectors $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ are equal if and only if $u_i = v_i$ for all $i = 1, \dots, n$

Two operations on \mathbb{R}^n

Vector addition $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix}$

Scalar multiplication $c \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} \quad c \in \mathbb{R}$

Ex $\vec{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \vec{u} + \vec{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ 7 \end{bmatrix}$

$$6\vec{u} = 6 \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 18 \\ 24 \end{bmatrix}$$

Defⁿ The zero-vector in \mathbb{R}^n is $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

(Algebraic Properties of \mathbb{R}^n) For all $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$, and $c, d \in \mathbb{R}$

$$(i) \vec{u} + \vec{v} = \vec{v} + \vec{u} \quad (v) c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$$

$$(ii) \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w} \quad (vi) (c+d)\vec{u} = c\vec{u} + d\vec{u}$$

$$(iii) \vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u} \quad (vii) c(d\vec{u}) = (cd)\vec{u}$$

$$(iv) \vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0} \quad (viii) 1\vec{u} = \vec{u}$$

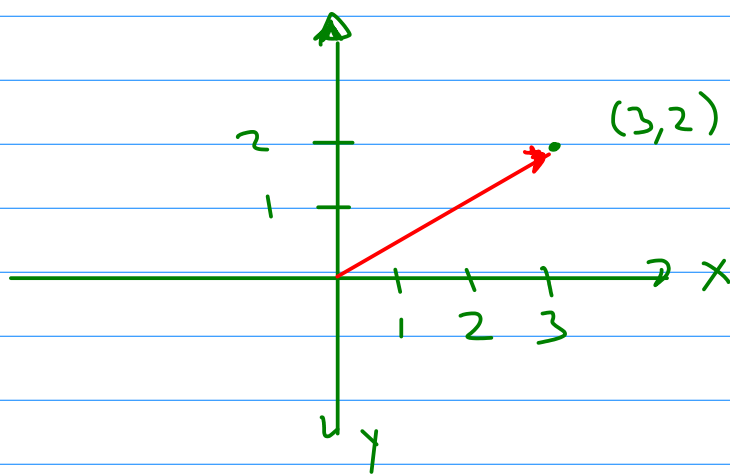
$$\text{where } -\vec{u} = (-1)\vec{u}$$

Looking ahead: these properties imply that \mathbb{R}^n is "vector space"

Vectors and Geometry

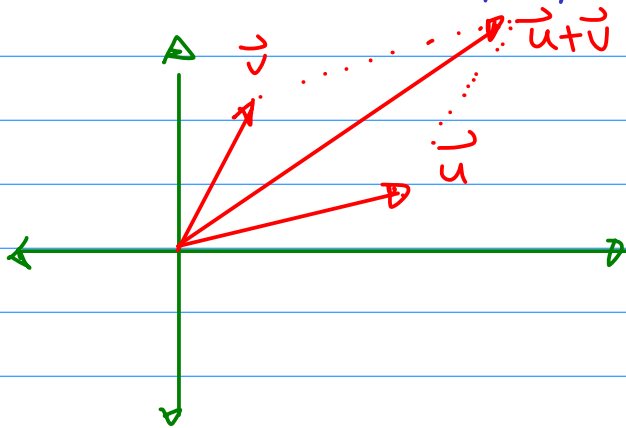
When $n=2$ or $n=3$, can visualize \mathbb{R}^2 or \mathbb{R}^3

View \mathbb{R}^2 as the plane. Identify $\begin{bmatrix} a \\ b \end{bmatrix}$ with the pt (a, b) and draw directed arrow from $(0, 0)$ to (a, b)

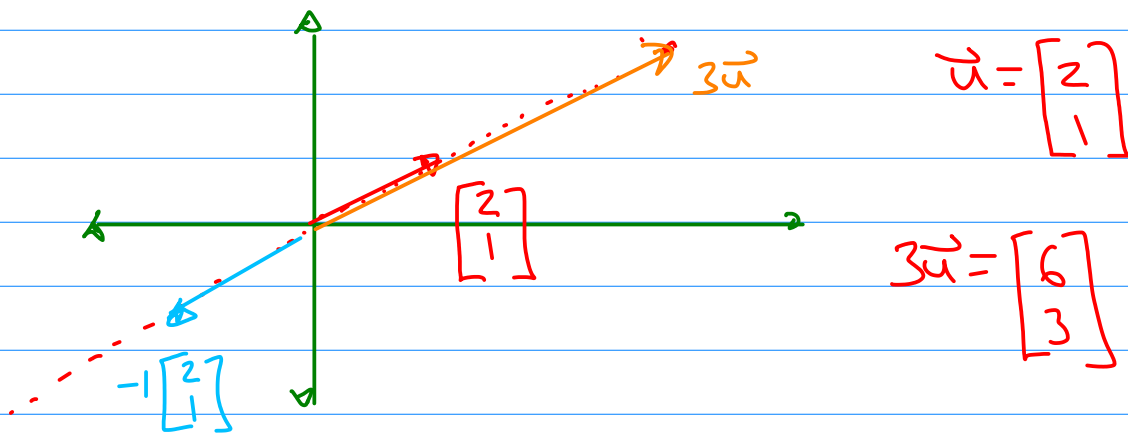


$$\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The sum $\vec{u} + \vec{v}$ is the fourth vertex of the parallelogram whose other vertices are $\vec{0}, \vec{u}, \vec{v}$



Scalar multiplication "stretches" the line through $\vec{0}$ and \vec{u}



Geometric idea extends to \mathbb{R}^n (although hard to draw!)

A vector $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ corresponds to directed arrow from $(0, \dots, 0)$ to (u_1, \dots, u_n)

Linear combinations

Given vectors $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$ and scalars $c_1, c_2, \dots, c_p \in \mathbb{R}$, the vector

$$\vec{y} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p$$

is a linear combination of $\vec{v}_1, \dots, \vec{v}_p$. The c_i 's are weights

Ex $\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$, then $2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 8 \\ 16 \\ 0 \end{bmatrix}$
↑ a linear combination of \vec{a}_1, \vec{a}_2

Ex Let $\vec{a}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$

Write \vec{b} as a linear combination of \vec{a}_1 and \vec{a}_2

Want to find c_1, c_2 such that $c_1 \begin{bmatrix} 6 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$

Via vector operations, LHS reduces to

$$\begin{bmatrix} 6c_1 - 3c_2 \\ -c_1 + 4c_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$$

Get a SLE!!

$$\begin{array}{l} \textcircled{1} \quad 6c_1 - 3c_2 = -3 \\ \textcircled{2} \quad -c_1 + 4c_2 = 11 \end{array} \Rightarrow \left[\begin{array}{cc|c} 6 & -3 & -3 \\ -1 & 4 & 11 \end{array} \right] \sim \left[\begin{array}{cc|c} 1 & -4 & -11 \\ 0 & 21 & 63 \end{array} \right]$$

So $c_2 = 3$ and $c_1 = 1$

Thus $1 \cdot \begin{bmatrix} 6 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -3 \\ 4 \end{bmatrix} = \begin{bmatrix} -3 \\ 11 \end{bmatrix}$.

Fact A vector equation

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n = \vec{b}$$

has the same solⁿ set as the linear system
with augmented matrix

$$[\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n \ \vec{b}]$$

Spanning Sets

Defⁿ If $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$, the subset of \mathbb{R}^n spanned by $\vec{v}_1, \dots, \vec{v}_p$ is the set

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_p\} = \underbrace{\{c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_p \vec{v}_p \mid c_1, \dots, c_p \in \mathbb{R}\}}_{\text{a linear combination}} \quad \leftarrow \begin{array}{l} \text{all} \\ \text{linear} \\ \text{combinations} \\ \text{of } \vec{v}_1, \dots, \vec{v}_p \end{array}$$

Ex 1 $\begin{bmatrix} -3 \\ 11 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 6 \\ -1 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ since $\begin{bmatrix} -3 \\ 11 \end{bmatrix} = 1 \cdot \begin{bmatrix} 6 \\ -1 \end{bmatrix} + 3 \begin{bmatrix} -3 \\ 4 \end{bmatrix}$

Ex 2 For any $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n$, $\vec{0} \in \text{span} \{ \vec{v}_1, \dots, \vec{v}_p \}$ since

$$\vec{0} = 0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \dots + 0 \cdot \vec{v}_p$$

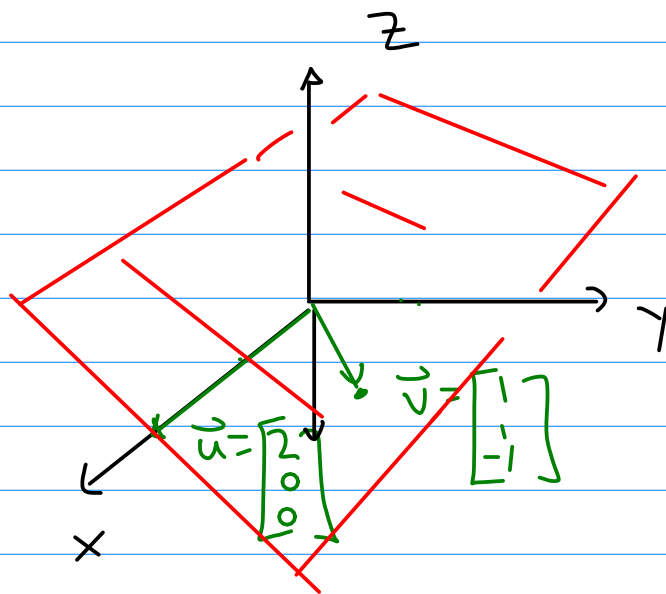
Ex 3 $\text{span} \{ \vec{v}_1, \dots, \vec{v}_p \}$ contains all scalar multiples of \vec{v}_i
Since

$$c \vec{v}_i = 0 \vec{v}_1 + \dots + c \vec{v}_i + \dots + 0 \vec{v}_p$$

Geometrically: If $\vec{u}, \vec{v} \in \mathbb{R}^3$

- $\text{span}\{\vec{u}\}$ is the line in \mathbb{R}^3 through the origin and $\vec{u} \in \mathbb{R}^3$
- If \vec{v} is not a multiple of \vec{u} , then $\text{span}\{\vec{u}, \vec{v}\}$ is the plane in \mathbb{R}^3 through \vec{u}, \vec{v} .

Ex $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$



Fact Determining if $\vec{b} \in \text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$

equivalent to finding a solution to
$$x_1 \vec{v}_1 + x_2 \vec{v}_2 + \dots + x_p \vec{v}_p = \vec{b}$$

which is equivalent to finding a solⁿ to
$$[\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_p \ \vec{b}]$$

Key ideas *

- * vector, vector sum, scalar multiplication
- * linear combinations of vectors
- * span