

Lecture 5

Matrix Equation $A\vec{x} = \vec{b}$ (Section 1.4)

Last lecture linear combinations $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^n, c_1, \dots, c_p \in \mathbb{R}$

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_p \vec{v}_p$$

Today's lecture The matrix equation $A\vec{x} = \vec{b}$
(a new framework to view SLE)

Defⁿ Let A be an $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n$.
If $\vec{x} \in \mathbb{R}^n$, then the product of A and \vec{x} , denoted $A\vec{x}$,
is the linear combination of the columns of A using
the corresponding entries of \vec{x} as weights, i.e.

$$A\vec{x} = [\vec{a}_1 \dots \vec{a}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_n \vec{a}_n$$

Ex $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix}$ $\vec{a}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ $\vec{a}_3 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

If $\vec{x} = \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$, then $A\vec{x} = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ -1 \end{bmatrix}$

$$= 1 \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 6 \begin{bmatrix} -2 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -12 \\ 22 \end{bmatrix}$$

Ex Write the following S.L.E. in form $A\vec{x}=\vec{b}$

$$\begin{array}{l} x_1 + 2x_2 + 3x_3 = 0 \\ 4x_1 + 5x_2 + 6x_3 = -3 \\ 7x_1 + 8x_2 + 9x_3 = 6 \end{array} \quad \text{lec 2+3} \quad \Leftrightarrow \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & -3 \\ 7 & 8 & 9 & 6 \end{array} \right] \quad \text{augmented matrix}$$

$$\Leftrightarrow \quad \text{last lecture} \quad x_1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix}$$

$$\Leftrightarrow \quad \text{first def'n} \quad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 6 \end{bmatrix}$$

Three ways to view SLE:

Thm Let A be an $m \times n$ matrix with columns $\vec{a}_1, \dots, \vec{a}_n \in \mathbb{R}^m$ and $\vec{b} \in \mathbb{R}^m$. Then

the equation $A\vec{x}=\vec{b}$ with $\vec{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

has the same solⁿ set as the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{b}$$

which has the same solⁿ set as the SLE with augmented matrix

$$[\vec{a}_1 \dots \vec{a}_n | \vec{b}]$$

Existence of Solⁿs

Recall existence question:

when does a SLE have a solⁿ \Leftrightarrow when does $A\vec{x}=\vec{b}$ have a solⁿ?

Fact $A\vec{x}=\vec{b}$ has a solⁿ if and only if exists x_1, \dots, x_n such that $\vec{b} = x_1\vec{a}_1 + \dots + x_n\vec{a}_n$, i.e. \vec{b} is a linear combination of the columns

Harder question: when does $A\vec{x}=\vec{b}$ have a solⁿ for all $\vec{b} \in \mathbb{R}^m$?

Ex Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 14 \\ 1 & 3 & 5 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

Does $A\vec{x} = \vec{b}$ have a solⁿ for all possible b_1, b_2, b_3 ?

Solⁿ Row reduce the augmented matrix for $A\vec{x} = \vec{b}$

$$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 8 & 14 & b_2 \\ 1 & 3 & 5 & b_3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 2 & 8 & 14 & b_2 \\ 0 & 1 & 2 & b_3 - b_1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & 4 & 8 & b_2 - 2b_1 \\ 0 & 1 & 2 & b_3 - b_1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 : & b_1 \\ 0 & 4 & 8 : & b_2 - 2b_1 \\ 0 & 0 & 0 : & b_3 - b_1 - \frac{1}{4}(b_2 - 2b_1) \end{bmatrix}$$

Matrix is consistent if and only if $b_3 - b_1 - \frac{1}{4}(b_2 - 2b_1) = 0$

$$\Leftrightarrow -\frac{1}{2}b_1 - \frac{1}{4}b_2 + b_3 = 0$$

Note if $\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $-\frac{1}{2} - \frac{1}{4} + 1 \neq 0 \Rightarrow A\vec{x} = \vec{b}$ has no solⁿ

Recall $\text{span}\{\vec{v}_1, \dots, \vec{v}_p\} = \underbrace{\{c_1\vec{v}_1 + \dots + c_p\vec{v}_p \mid c_i \in \mathbb{R}\}}_{\text{all linear combinations}}$

Defⁿ The columns of an $m \times n$ matrix $A = [\vec{a}_1 \dots \vec{a}_n]$ span \mathbb{R}^m is $\text{span}\{\vec{a}_1, \dots, \vec{a}_n\} = \mathbb{R}^m$
(This means every $\vec{b} \in \mathbb{R}^m$ is in $\text{span}\{\vec{a}_1, \dots, \vec{a}_n\}$)

Thm Let A be an $m \times n$ matrix. The following are equivalent: → all statements are all true or all false

- (a) For each $\vec{b} \in \mathbb{R}^m$, $A\vec{x} = \vec{b}$ has a solⁿ
- (b) Each $\vec{b} \in \mathbb{R}^m$ is a linear combination of the columns of A
- (c) columns of A span \mathbb{R}^m
- (d) A has a pivot in each row

Note: This statement is about the coefficient matrix
extremely useful

Ex. (Revisit previous example)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 8 & 14 \\ 1 & 3 & 5 \end{bmatrix}$$

row reduce A

$$A \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

← a row with no pivot

So $A\vec{x} = \vec{b}$ does not

have a solⁿ for all $\vec{b} \in \mathbb{R}^3$

Ex Show $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ has a solⁿ for all b_1, b_2

Solⁿ $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \sim$

Algebraic Properties of product $A\vec{x}$

Ex (computing $A\vec{x}$)

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \cdot x_1 + 2 \cdot x_2 \\ 3x_1 + 4x_2 \\ 5x_1 + 6x_2 \end{bmatrix}$$

← notice 3rd
entry comes from
3rd row

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1(-1) + 2(2) \\ 3(-1) + 4(2) \\ 5(-1) + 6(2) \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 7 \end{bmatrix}$$

Thm If A is an $m \times n$ matrix, $\vec{u}, \vec{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$

$$(a) A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$$

$$(b) A(c\vec{u}) = c(A\vec{u})$$

Proof (special case $n=2$) So $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\text{and } A = [\vec{a}_1 \ \vec{a}_2]$$

$$(a) A(\vec{u} + \vec{v}) = A\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = A\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right)$$

$$= (u_1 + v_1)\vec{a}_1 + (u_2 + v_2)\vec{a}_2$$

$$= u_1\vec{a}_1 + v_1\vec{a}_1 + u_2\vec{a}_2 + v_2\vec{a}_2$$

$$= (u_1\vec{a}_1 + u_2\vec{a}_2) + (v_1\vec{a}_1 + v_2\vec{a}_2)$$

$$= A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + A\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = A\vec{u} + A\vec{v}.$$

$$(b) A(c\vec{u}) = A\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix} = (cu_1)\vec{a}_1 + (cu_2)\vec{a}_2 \\ = c[u_1\vec{a}_1 + u_2\vec{a}_2] = c(A\vec{u})$$



Key ideas: * new viewpoint of SLE $\Leftrightarrow A\vec{x} = \vec{b}$
* when $A\vec{x} = \vec{b}$ has a solⁿ for all \vec{b} .
* properties of $A\vec{x}$