MATH 1B03 C01 and C02: Midterm 1 (Version 1)

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Duration: 75 min.

First name (please write as legibly as possible within the boxes)

SOLUTIONS

Last name

Student ID number

Instructions:

(YOUR RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.)

- Fill in your name and student ID **neatly** in the box above.
- This test paper contains **16** multiple choice questions and **2** short answer questions printed on both sides of the page. The questions are on pages 2 through 10. Page 11 is a blank page for calculation, and Page 12 is the bubble sheet for the multiple choice questions. Scrap paper is also available for rough work.
- For Questions 1 through 16, select the one correct answer to each question and enter that answer on the bubble sheet.
- For Questions 17 and 18, write your answer directly in this test booklet.
- The midterm is graded out of 21. Questions 1 through 16 are worth 1 point each, Question 17 is worth 2 points, and Question 18 is worth 3 points.
- **NO CALCULATORS ALLOWED.**
- YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE TEST IS COMPLETE. LET THE INVIGILATOR KNOW OF ANY DISCREPANCIES.

1. Which equation is **NOT** a linear equation in x_1, x_2 and x_3.

a) $x_1^2 + x_2^3 + x_3^4 = 2019.$

b) $(\sin(2019))x_1 + (\cos(2019))x_3 = 0.$

c) $\sqrt{2019}x_1 + \pi^{2019}x_2 + e^{2019}x_3 = 42.$

d) $2^{2019}x_1 + 6x_2 + (\log_{10} 11)x_3 = 2019.$

e) $-x_1 - 2x_2 - 3x_3 - 4 = 0.$

2. Which of the following matrices are in *reduced row echelon form?*

not reduced

i) $\begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 1 & 0 & 2 & -2 \\ 0 & 0 & 1 & -9 & 8 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$

not echelon

ii) $\begin{bmatrix} 0 & 1 & -5 & 0 & 0 \\ 0 & 0 & 0 & \pi & 6 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

not row echelon

iii) $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$

not row echelon

iv) $\begin{bmatrix} 42 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

v) $\begin{bmatrix} 0 & 1 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$ reduced row echelon

a) i), ii) and v) only

b) All of them

c) None of them

d) i) and v) only

e) v) only

3. How many solutions does the following system of linear equations have?

$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ 5x_1 + x_2 + x_3 &= 10 \\ 4x_1 + 6x_2 + 2x_3 &= 22 \end{aligned}$$

a) None

b) One

c) Two

d) 2019

e) Infinitely many

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 5 & 1 & 1 & 10 \\ 4 & 6 & 2 & 22 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -4 & -4 & -20 \\ 0 & 2 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & 5 \\ 0 & 0 & -2 & -6 \end{bmatrix}$$

← from the echelon form we see the SLE has exactly 1 sol^n.

4. Suppose that the augmented matrix of a system of linear equations has been placed into the following reduced row echelon form:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 5 & 0 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

← from the matrix, x_2 and x_4 are free and $x_5 = 2$

If q, r are arbitrary elements of $\mathbb{R}$, then the set of solutions for this systems is described by

(a)
$$\begin{aligned} x_1 &= -3 \\ x_2 &= 0 \\ x_3 &= 1 \\ x_4 &= 0 \\ x_5 &= 2 \end{aligned}$$

no free variables →

(b)
$$\begin{aligned} x_1 &= -3 - 2q - 5r \\ x_2 &= q \\ x_3 &= 1 + r \\ x_4 &= r \\ x_5 &= 2 \end{aligned}$$

(c)
$$\begin{aligned} x_1 &= 3 + 2q + 5r \\ x_2 &= q \\ x_3 &= 1 - r \\ x_4 &= r \\ x_5 &= 2 \end{aligned}$$

← wrong

(d)
$$\begin{aligned} x_1 &= 2q + 5r \\ x_2 &= q \\ x_3 &= -1 + r \\ x_4 &= r \\ x_5 &= -2 \end{aligned}$$

wrong →

The second row implies $x_3 - x_4 = 1 \iff x_3 = 1 + x_4$

So sol^n is (b)

5. The **rank** of a matrix A is the number of leading 1's in the reduced row echelon form of A. What is the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1-a & a^2+1 \\ 1 & 3 & -a & a^2+1 \end{bmatrix}$$

a) 0

b) 1

$\boxed{\text{c) 2}}$

d) 3

e) Not enough information; answer will depend upon the value of a.

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1-a & a^2+1 \\ 1 & 3 & -a & a^2+1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1-a & a^2+1 \\ 0 & 1 & 1-a & a^2+1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 1-a & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Regardless of value of a, matrix always has two leading 1's

6. You are given the following three matrices

$$A = \begin{bmatrix} 5 & -6 & 4 & -4 \\ -8 & 9 & -2 & 3 \\ -4 & 7 & 3 & -1 \end{bmatrix}, B = \begin{bmatrix} -8 & 9 & -2 & 3 \\ 8 & -5 & 4 & -1 \\ -2 & 5 & -3 & 2 \end{bmatrix}, C = \begin{bmatrix} -4 & 9 \\ 6 & -5 \\ -9 & 3 \\ -1 & 5 \end{bmatrix}$$

What matrix multiplication will yield a 4×4 matrix?

~~a) $CABC$~~ ← 3 × Something

~~b) C^TA^TBC~~ ← 2 × Something

$\boxed{\text{c) } A^TBCC^T}$ ← 4×4

~~d) BC^TA^TB~~ ← 3 × Something

~~e) ABA^TC~~ ← 3 × Something

7. If A, B, C and D are invertible matrices of the same size and

$$C^{-1}BC^TA^2A^T = D$$

which of the following must be B?

$C^{-1}BC^TA^2A^T = D$

a) $(A^{-1})^T(A^2)^{-1}C(C^T)^{-1}D$

b) $\boxed{CD(A^{-1})^T(A^2)^{-1}(C^T)^{-1}}$

$\Rightarrow B = CD(A^T)^{-1}(A^2)^{-1}(C^T)^{-1}$

c) $CDA^TA^2C^T$

d) $CDA(C^T)^{-1}$

use the fact that $(A^T)^{-1} = (A^{-1})^T$

e) None of the above

8. Compute A if $(B+3A)^{-1} = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$.

a) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$B+3A = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{8-9}\begin{bmatrix} 4 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$

b) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

So $3A = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 3 \\ 0 & -3 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

So $A = \frac{1}{3}\begin{bmatrix} -6 & 3 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}$

d) $\begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$

e) $\boxed{\begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}}$

9. What are the diagonal entries of A^{-1} if

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

a) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

b) $0, 1, \frac{1}{2}$.

c) $-1, 0, 2$

d) $\frac{1}{2}, 0, 1$

e) $-\frac{1}{2}, 0, 1$

$$\left[\begin{array}{ccc|ccc} 0 & 2 & 2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -2 & 1 & 0 & -2 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & \frac{1}{2} & 1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} & 0 & 1 \end{array}\right]$$

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10. Which one of the following statements is not equivalent to the others?

a) A is invertible.

b) $\boxed{A\mathbf{x} = \mathbf{0} \text{ has a non-trivial solution.}}$ ← All other sol'ns equiv

c) The reduced row echelon form of A is I_n

d) A is a product of elementary matrices.

e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.

11. The system of 5 equations in 4 unknowns $A\mathbf{x} = B$ has solutions

$$\mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

If performing row operations on the augmented matrix $[A|B]$ can produce the following matrix

$$\left[\begin{array}{cccc|c} 2 & 2 & 3 & -1 & 1 \\ 0 & 1 & 5 & 0 & a \\ 12 & 0 & 7 & -6 & b \\ 8 & 3 & 4 & -4 & c \\ 22 & 0 & 11 & -11 & d \end{array}\right]$$

what is $b + c$? Since $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ is a sol^n, we have

a) -123

b) -8

$\boxed{\text{c) -7}}$

d) 7

e) 123

$$\begin{bmatrix} * \\ * \\ 12\ 0\ 7\ -6 \\ 8\ 3\ 4\ -4 \\ * \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ x \\ b \\ c \\ x \end{bmatrix} \Rightarrow b = -6 \text{ and } c = -1$$

So $b + c = -7$

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12. In Matlab, suppose that we have defined a $n \times n$ matrix M, and we want make a new matrix where each (i, j)-th entry is the square of the (i, j)-th entry of the matrix M. Which command could accomplish this?

a) `square(M)`

b) `M(1,1)^2,M(1,2)^2,...,M(n,n)^2`

c) `M^2`

$\boxed{\text{d) M.\^{}2}}$

e) `M*M'`

The following questions are all **TRUE-FALSE** questions.

13. Which of the following statements are true?

(1) Multiplying a row of an augmented matrix through by 0 is an acceptable elementary row operation. FALSE

(2) If a linear system has two equations in three unknowns, then the system is always inconsistent. FALSE

a) (1) is false and (2) is false.

b) (1) is true and (2) is false.

c) (1) is false and (2) is true.

d) (1) is true and (2) is true.

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14. Which of the following statements are true?

(1) If A and B are $n \times n$ matrices, then $\mathrm{tr}(AB) = \mathrm{tr}(A)\,\mathrm{tr}(B)$. FALSE

(2) If A and B are $n \times n$ matrices, then $\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B)$. TRUE

a) (1) is false and (2) is false.

b) (1) is true and (2) is false.

c) (1) is false and (2) is true.

d) (1) is true and (2) is true.

For (1)

$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

and $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

So $\mathrm{tr}(AB) = 0 \neq 1 \cdot 1 = \mathrm{trace}(A)\,\mathrm{trace}(B)$

15. Which of the following statements are true?

(1) If A is any $n \times n$ matrix and E is an $n \times n$ elementary matrix, then EA is invertible. FALSE

(2) If E_1 and E_2 are two $n \times n$ elementary matrices, then E_1E_2 is also an elementary matrix. FALSE

a) (1) is false and (2) is false.

b) (1) is true and (2) is false.

c) (1) is false and (2) is true.

d) (1) is true and (2) is true.

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16. Which of the following statements are true?

(1) If A and B are $n \times n$ matrix such that $A+B$ is symmetric, then A and B are symmetric. FALSE

(2) If A is an $n \times m$ matrix, then AA^T is an $n \times n$ symmetric matrix. TRUE

a) (1) is false and (2) is false.

b) (1) is true and (2) is false.

c) (1) is false and (2) is true.

d) (1) is true and (2) is true.

Counter-example for (1)

$$\underbrace{\underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}}_{A} + \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}}_{B}}_{\text{not symmetric}} = \underbrace{\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}}_{A+B}}_{\text{Symmetric}}$$

The following two questions are **SHORT ANSWER QUESTIONS**. Write your answer for each question directly into the space provided. You will be graded for both your answer and your explanation.

17. A square $n \times n$ matrix A is called **idempotent** if $A^2 = A$. Show that if A is an idempotent matrix, then the matrix $(I_n - A)$ is also an idempotent matrix.

We need to show $(I_n - A)^2 = (I_n - A)$. Now

$$(I_n - A)^2 = (I_n - A)(I_n - A) = I_n^2 - I_nA - AI_n + A^2$$

Since I_n is the identity $A = I_nA = AI_n$. Since A is idempotent, $A^2 = A$. By making a substitution, we get

$$(I_n - A)^2 = I_n^2 - I_nA - AI_n + A^2 = I_n - A - A + A = I_n - A,$$

as desired. $\boxed{3}$

18. Suppose that A is an $n \times n$ matrix such that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Prove that $A^{2019}\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Since $A\vec{x} = \vec{0}$ has only the trivial sol^n, A is invertible. But then A^{2019} is also invertible since the product of invertible matrices is invertible. But then $A^{2019}\vec{x} = \vec{0}$ has only the trivial sol^n $\boxed{4}$

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SCRAP

END OF TEST PAPER

(NEXT PAGE CONTAINS BUBBLE SHEET FOR MULTIPLE CHOICE)

1	A	B	C	D	E	21	A	B	C	D	E
2	A	B	C	D	E	22	A	B	C	D	E
3	A	B	C	D	E	23	A	B	C	D	E
4	A	B	C	D	E	24	A	B	C	D	E
5	A	B	C	D	E	25	A	B	C	D	E
6	A	B	C	D	E	26	A	B	C	D	E
7	A	B	C	D	E	27	A	B	C	D	E
8	A	B	C	D	E	28	A	B	C	D	E
9	A	B	C	D	E	29	A	B	C	D	E
10	A	B	C	D	E	30	A	B	C	D	E
11	A	B	C	D	E	31	A	B	C	D	E
12	A	B	C	D	E	32	A	B	C	D	E
13	A	B	C	D	E	33	A	B	C	D	E
14	A	B	C	D	E	34	A	B	C	D	E
15	A	B	C	D	E	35	A	B	C	D	E
16	A	B	C	D	E	36	A	B	C	D	E
17	A	B	C	D	E	37	A	B	C	D	E
18	A	B	C	D	E	38	A	B	C	D	E
19	A	B	C	D	E	39	A	B	C	D	E
20	A	B	C	D	E	40	A	B	C	D	E