SOLUTIONS

MATH 1B03 C01 and C02: Midterm 2 (Version 1)

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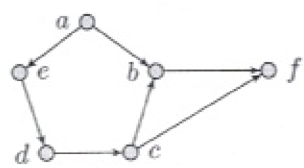
Duration: 75 min.

Instructions:

(YOUR RESULTS DEPEND UPON PROPER ATTENTION TO THESE INSTRUCTIONS.)

- Fill in your name and student ID **neatly** in the box above.
- This test paper contains **16** multiple choice questions and **2** short answer questions printed on both sides of the page. The questions are on pages 2 through 10. Page 11 is a blank page for calculation, and Page 12 is the bubble sheet for the multiple choice questions. Scrap paper is also available for rough work.
- For Questions 1 through 16, select the one correct answer to each question and enter that answer on the bubble sheet.
- For Questions 17 and 18, write your answer directly in this test booklet.
- The midterm is graded out of 20. Questions 1 through 16 are worth 1 point each, Question 17 and 18 are worth 2 points.
- **NO CALCULATORS ALLOWED.**
- YOU ARE RESPONSIBLE FOR ENSURING THAT YOUR COPY OF THE TEST IS COMPLETE. LET THE INVIGILATOR KNOW OF ANY DISCREPANCIES.

1. Which matrix is the vertex matrix of the directed graph given below?



← a connected to e and b. But no directed edge to a.

So matrix D is correct matrix

~~a)~~ $\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}$ ~~b)~~ $\begin{bmatrix}0&1&1&1&1&1\\0&0&1&1&1&1\\0&0&0&1&1&1\\0&0&0&0&1&1\\0&0&0&0&0&1\\0&0&0&0&0&0\end{bmatrix}$ ~~c)~~ $\begin{bmatrix}1&1&0&0&1&0\\0&1&0&0&0&1\\0&1&1&0&0&1\\0&0&1&1&0&0\\0&0&0&1&1&0\\0&0&0&0&0&1\end{bmatrix}$

$\boxed{d)}$ $\begin{array}{c|c} & a\ \ b\ \ c\ \ d\ \ e\ \ f \\ \hline \begin{matrix}a\\b\\c\\d\\e\\f\end{matrix} & \begin{bmatrix}0&1&0&0&1&0\\0&0&0&0&0&1\\0&1&0&0&0&1\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&0&0\end{bmatrix}\end{array}$ ~~e)~~ $\begin{bmatrix}0&1&0&0&1&0\\1&0&1&0&0&1\\0&1&0&1&0&1\\0&0&1&0&1&0\\1&0&0&1&0&0\\0&1&1&0&0&0\end{bmatrix}$

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2. Let $T : \mathbb{R}^{2019} \to \mathbb{R}^2$ be a linear transformation. If

$$T(e_1) = \begin{bmatrix}4\\0\end{bmatrix} \quad \text{and} \quad T(e_1 + 2e_2) = \begin{bmatrix}10\\6\end{bmatrix}$$

what is $T(e_2)$? (Here e_i denotes the i-th standard vector of $\mathbb{R}^{2019}$).

$\boxed{\text{a) } \begin{bmatrix}3\\3\end{bmatrix}}$ b) $\begin{bmatrix}6\\6\end{bmatrix}$ c) $\begin{bmatrix}10\\3\end{bmatrix}$ d) $\begin{bmatrix}6\\3\end{bmatrix}$ e) Not enough information given.

$\begin{bmatrix}10\\6\end{bmatrix} = T(\vec{e}_1 + 2\vec{e}_2) = T(\vec{e}_1) + 2T(\vec{e}_2) = \begin{bmatrix}4\\0\end{bmatrix} + 2\begin{bmatrix}a\\b\end{bmatrix}$ So b=3 and a=3

property of linear transf

3. Compute the determinant of the matrix below:

$$A = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 2 & 2 & 0 & -3 \\ 1 & 0 & 3 & 4 \\ 1 & 0 & 4 & 0 \end{bmatrix}$$

← do column expansion down 2^{nd} column

a) -48 b) -24 c) 0 d) 24 $\boxed{\text{e) 48}}$

$\det(A) = (-1)^{2+2} \cdot 2 \begin{vmatrix} -1 & 2 & 0 \\ 1 & 3 & 4 \\ 1 & 4 & 0 \end{vmatrix}$.

Now $\begin{vmatrix} -1 & 2 & 0 \\ 1 & 3 & 4 \\ 1 & 4 & 0 \end{vmatrix} = 8 - (-16) = 24$

So $\det(A) = 1 \cdot 2(24) = \boxed{48}$

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4. Let A be the matrix as in Question 3 above, and let $B = \text{adj}(A)$. What is b_{42}?

a) -3 b) -31 $\boxed{\text{c) 0}}$ d) 31 e) 3

$b_{42} = C_{24} = (-1)^{2+4} \begin{vmatrix} -1 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & 4 \end{vmatrix} = 0$ since the submatrix A_{24} has a column of zeros

5. Suppose that A is an invertible matrix with $\det(A) = 5$, B is an invertible matrix with $\det(B) = 4$, and C is an invertible matrix with $\det(C) = 2$. Furthermore, each matrix has size 2×2. Find the determinant of the matrix:

$$5A^{-1}(B^T)\,\mathrm{adj}(C)$$

a) $\frac{5}{4}$ b) 4 c) 8 d) 20 $\boxed{\text{e) 40}}$

Since the matrix is 2×2, $\det(5A^{-1}(B^T)\,\mathrm{adj}(C)) = 5^2 \det(A^{-1}) \det(B^T) \det(\mathrm{adj}(C))$

But $\det(A^{-1}) = 1/\det(A) = 1/5$, $\det(B^T) = \det(B)$, and since $\det(C)\, C^{-1} = \mathrm{adj}(C)$,

$\det(\mathrm{adj}(C)) = \det(C)^2 \cdot 1/\det(C) = \det(C)$. So

$\det(5A^{-1}(B^T)\,\mathrm{adj}(C)) = 5^2 \cdot 1/5 \cdot 4 \cdot 2 = 5 \cdot 4 \cdot 2 = \boxed{40}$

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6. Given that

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7,$$

Keep track of row operations

what is the determinant of the matrix below?

$$\begin{bmatrix} 2d & 2e & 2f \\ a & b & c \\ g-d & h-e & i-f \end{bmatrix}$$

$\boxed{\text{a) -14}}$ b) -7 c) $\frac{7}{2}$ d) 7 e) 14

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \sim \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix} \sim \begin{bmatrix} d & e & f \\ a & b & c \\ g-d & h-e & f-c \end{bmatrix} \sim \begin{bmatrix} 2d & 2e & 2f \\ a & b & c \\ g-d & h-e & f-c \end{bmatrix}$$

swap

add one row to another

scale a row

det = 7

det = -7

det = -7

$\det = 2(-7) = \boxed{-14}$

7. What 2×2 matrix given below has characteristic polynomial

$$p(\lambda) = \lambda^2 - 5\lambda - 2$$

a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ b) $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$ d) $\begin{bmatrix} -1 & -2 \\ 3 & 4 \end{bmatrix}$ $\boxed{\text{e) } \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}$

a) char poly $= (\lambda-1)^2$ c) $p(\lambda) = (\lambda-1)^2$

b) $p(\lambda) = (\lambda-2)^2 - 1$ d) $p(\lambda) = (\lambda+1)(\lambda-4) - (-6)$

e) $p(\lambda) = (\lambda-1)(\lambda-4) - 6$

$= \lambda^2 - 5\lambda + 4 - 6$

$= \lambda^2 - 5\lambda - 2$

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8. Let A be the matrix

$$A = \begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix}.$$

If $\mathbf{x} = \begin{bmatrix} 2 \\ -3 \\ k \end{bmatrix}$ is an eigenvector of A, what is the value of k?

a) -2019 $\boxed{\text{b) -1}}$ c) 1 d) 3 e) Not enough information given.

We know $\begin{bmatrix} -2 & -4 & 2 \\ -2 & 1 & 2 \\ 4 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ k \end{bmatrix} = \lambda \begin{bmatrix} 2 \\ -3 \\ k \end{bmatrix}$

So ① $-4 + 12 + 2k = 2\lambda$
② $-4 - 3 + 2k = -3\lambda$ $\Rightarrow$ ① $2k = 2\lambda - 8$. Sub into ②
$-7 + 2\lambda - 8 = -3\lambda$

So $-15 = -5\lambda \Rightarrow \lambda = 3$

But then $2k = 6 - 8$, so $\boxed{k = -1}$

9. For a given 3×3 matrix A, the following facts are known

$$A\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2019\\0\\0\end{bmatrix} \quad A\begin{bmatrix}2\\2\\0\end{bmatrix} = \begin{bmatrix}-2\\-2\\0\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

If D is the matrix that diagonalizes A, what are the diagonal entries of D?

a) $2, -1, 0$
b) $-2, -1, 0$
c) 0 (with multiplicity 3)
d) $1, 2, 3$
e) $2019, -1, 0$

NOT GRADED ! This question is poorly worded. It should have said "If A is similar to the diagonal matrix D, what are the diagonal entries of D?" The answer we were looking for was (e). However, as worded, the question appears to be asking for the invertible matrix that makes A a diagonal matrix. Since this was not the intention of the question, it was dropped

10. Compute $(-\frac{1}{i})^{2019}$.

a) $-i$ b) -1 c) 0 d) 1 e) i

First note that $\frac{-1}{i} = \frac{-1}{i}\left(\frac{-i}{-i}\right) = \frac{i}{-(i^2)} = \frac{i}{-(-1)} = i$

Also $i^4 = 1$. Now $2019 = 4(504) + 3$ So

$\left(\frac{-1}{i}\right)^{2019} = i^{2019} = i^{4 \cdot 504}\, i^3 = (i^4)^{504}\, i^3 = 1^{504} \cdot i^3$

$= i^2(i) = (-1)i = \boxed{-i}$

11. Find a complex number z such that

$$\begin{vmatrix} 1 & 0 & 0 \\ 2+2i & z & 0 \\ 5-6i & 2019i & 1-i \end{vmatrix} = 2019.$$

a) $\dfrac{1-i}{2019}$.

b) $\dfrac{1+i}{2}$

c) $2019 - 2019i$

d) $\boxed{\dfrac{2019+2019i}{2}}$

e) $\dfrac{-2019+2019i}{2}$

Because our matrix is triangular,

$2019 = 1 \cdot z\,(1-i)$ ← product of diagonal entries.

So $z = \dfrac{2019}{1-i} = \dfrac{2019}{1-i} \cdot \dfrac{(1+i)}{(1+i)} = \dfrac{2019+2019i}{1-(-1)}$

$= \dfrac{2019+2019i}{2}$

12. The following four commands are entered into Matlab:

```
P = [2019 2019 2019 1 1 1 3 2 1]
T = reshape(P,3,3)
R = triu(T)
det(R)
```

$T = \begin{bmatrix} 2019 & 1 & 3 \\ 2019 & 1 & 2 \\ 2019 & 1 & 1 \end{bmatrix}$ $R = \begin{bmatrix} 2019 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

So $\det(R) = 2019$

What is the output of the last command?

a) `-2.2393e-13` b) `0` c) `1` $\boxed{\text{d) } 2019}$ e) `4038`

The following questions are all **TRUE-FALSE** questions.

13. Which of the following statements are true?

(1) If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists an $n \times m$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. FALSE: A must be m×n

(2) For all square matrices A and B of the same size, $\det(A + B) = \det(A) + \det(B)$.

$\boxed{\text{a) (1) is false and (2) is false.}}$ FALSE. Ex.

b) (1) is true and (2) is false.

c) (1) is false and (2) is true.

d) (1) is true and (2) is true.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, so

$A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Then $\det(A+B) = 0 \neq 1+1 = \det(A) + \det(B)$

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14. Which of the following statements are true?

(1) If λ is an eigenvalue of A, then λ^3 is an eigenvalue of A^3. TRUE:

(2) If $z_1 = r(\cos(\theta) + i\sin(\theta))$ and $z_2 = s(\cos(\varphi) + i\sin(\varphi))$ are complex numbers, then $z_1 z_2 = (rs)(\cos(\theta + \varphi) + i\sin(\theta + \varphi))$. TRUE:

a) (1) is false and (2) is false.

b) (1) is true and (2) is false.

c) (1) is false and (2) is true.

$\boxed{\text{d) (1) is true and (2) is true.}}$

Suppose $A\vec{x} = \lambda\vec{x}$. Then

$$A^3\vec{x} = A(A(A\vec{x})) = A(A(\lambda\vec{x})) = \lambda(A(A\vec{x})) = \lambda(A(\lambda\vec{x})) = \lambda^2(A\vec{x}) = \lambda^2\lambda\vec{x} = \lambda^3\vec{x}.$$

15. Which of the following statements are true?

(1) If a 5×5 matrix A has 5 distinct eigenvalues, then A is diagonalizable. TRUE

(2) If A is a diagonalizable matrix, then A is invertible. FALSE

a matrix can have a zero eigenvalue, but still be diagonalizable, e.g $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

a) (1) is false and (2) is false.

b) (1) is true and (2) is false.

c) (1) is false and (2) is true.

d) (1) is true and (2) is true.

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16. Which of the following statements are true?

(1) If A is a 3×3 matrix with real entries, then A can have three eigenvalues that are all complex numbers. FALSE Complex eigenvalues come in pairs

(2) If λ is an eigenvalue of a matrix A, then the geometric multiplicity of λ is less than or equal to the algebraic multiplicity of λ. TRUE

a) (1) is false and (2) is false.

b) (1) is true and (2) is false.

c) (1) is false and (2) is true.

d) (1) is true and (2) is true.

The following two questions are **SHORT ANSWER QUESTIONS**. Write your answer directly into the space provided. You will be graded for both your answer and your explanation.

Many ways to prove this. Here's two sol^n s

✓ **17.** **[2pts]** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and suppose that $a+b=0$ $c+d=0$. Show that $\det(A) = 0$.

Sol^n ①

Since $a=-b$ and $c=-d$, we can rewrite our matrix as

$A = \begin{bmatrix} -b & b \\ -d & d \end{bmatrix}$. So $\det(A) = -bd-(-bd) = -bd+bd = 0$.

Sol^n ② Consider $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$.

So $A\vec{x}=\vec{0}$ has a nontrivial sol^n, so A is not invertible, or equivalently, $\det(A)=0$.

18. **[2pts]** Let A be a matrix with eigenvalue $\lambda = 2019$. Show that if $\mathbf{w}$ and $\mathbf{y}$ are both eigenvectors of A corresponding to the eigenvalue $\lambda = 2019$, then $\mathbf{z} = 2\mathbf{w} - 3\mathbf{y}$ is also an eigenvalue corresponding to $\lambda = 2019$.

We need to show that $A\vec{z} = 2019\vec{z}$. Now

$$A\vec{z} = A(2\vec{w}-3\vec{y}) = A(2\vec{w}) - A(3\vec{y}) = 2A\vec{w} - 3A\vec{y}$$

by the properties of matrix multiplication. Since $\vec{w}$ and $\vec{y}$ are eigenvectors of $\lambda=2019$, $A\vec{w} = 2019\vec{w}$ and $A\vec{y} = 2019\vec{y}$.

So $A\vec{z} = 2(A\vec{w}) - 3(A\vec{y}) = 2(2019\vec{w}) - 3(2019\vec{y}) = 2019(2\vec{w}-3\vec{y}) = 2019\vec{z}$.

So $\vec{z}$ is also an eigen~~value~~vector of $\lambda=2019$.

$\boxed{2}$

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SCRAP WORK

END OF TEST PAPER
(NEXT PAGE CONTAINS BUBBLE SHEET FOR MULTIPLE CHOICE)