$\boxed{\text{Lecture 31}}$ 7.A Adjoint Operators

Chapter 7 looks at special operators on inner product spaces

(this material generalizes results on symmetric and orthogonal matrices in 2LA3)

$\boxed{\text{Adjoint}}$

<u>Def$^{\underline{n}}$</u> Let $T \in \mathcal{L}(V,W)$. The <u>adjoint</u> of T is the function $T^*: W \to V$ such that for any $v \in V$ and $w \in W$

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

"Unpack" the definition

Note $\langle Tv, w \rangle$ $\langle v, T^* w \rangle$

inner product of W inner product of V

For any fixed vector $\underline{w} \in W$,

$\varphi : V \longrightarrow F$ defined by $\varphi(v) = \langle Tv, w \rangle$
is a linear functional

By Reisz Representation Thm, there is $u \in V$ such that

$$\langle v, u \rangle = \varphi(v) = \langle Tv, w \rangle$$

We define $T^* w = u$

i.e. $T^* w$ is the unique vector so that

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

Remark Probably most complicated $\text{def}^{\underline{n}}$ in book!

$\underline{\text{Ex}}$ Let $T: \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$T(x_1, x_2) = (3x_1, 4x_2, 6x_1 + 7x_2)$$

Find $T^*: \mathbb{R}^3 \to \mathbb{R}^2$. (using standard inner prod in $\mathbb{R}^2$ and $\mathbb{R}^3$)

$$\underbrace{\langle T(x_1,x_2), (y_1,y_2,y_3) \rangle}_{\langle Tv, w\rangle} = \langle (3x_1, 4x_2, 6x_1+7x_2), (y_1,y_2,y_3) \rangle \quad \leftarrow \text{inner prod in } \mathbb{R}^3$$

$$= 3x_1y_1 + 4x_2y_2 + (6x_1+7x_2)y_3$$

$$= (3y_1+6y_3)x_1 + (4y_2+7y_3)x_2$$

$$= \langle (x_1,x_2), (3y_1+6y_3, 4y_2+7y_3) \rangle$$

$\uparrow$ inner prod in $\mathbb{R}^2$

$$= \langle v, T^* w \rangle$$

$$\Rightarrow T^*(y_1,y_2,y_3) = (3y_1+6y_3, 4y_2+7y_3)$$

$\underline{\text{Observations}}$

- $T^* \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$
- $M(T) = \begin{bmatrix} 3 & 0 \\ 0 & 4 \\ 6 & 7 \end{bmatrix}$ and $M(T^*) = \begin{bmatrix} 3 & 0 & 6 \\ 0 & 4 & 7 \end{bmatrix}$

$\uparrow$ transpose!

Thm If $T \in \mathcal{L}(V, W)$, then $T^* \in \mathcal{L}(W, V)$

Proof Need to show $T^*(w_1 + w_2) = T^*w_1 + T^*w_2$
and $T^*(\lambda w) = \lambda T^* w$

Let $T \in \mathcal{L}(V, W)$, and $w_1, w_2 \in W$. If $v \in V$

$$\langle v, \underline{T^*(w_1 + w_2)} \rangle = \langle Tv, w_1 + w_2 \rangle$$

defn of adjoint (useful trick, swap order so replace T^* with a T)

$$= \langle Tv, w_1 \rangle + \langle Tv, w_2 \rangle$$
$$= \langle v, T^* w_1 \rangle + \langle v, T^* w_2 \rangle$$
$$= \langle v, \underline{T^* w_1 + T^* w_2} \rangle$$

By Reisz Representation Thm, $T^*(w_1 + w_2)$ is unique

$$\Rightarrow T^*(w_1 + w_2) = T^* w_1 + T^* w_2$$

Let $T \in \mathcal{L}(V, W)$ and $w \in W$ and $\lambda \in \mathbb{F}$. Then

$$\langle v, \underline{T^*(\lambda w)} \rangle = \langle Tv, \lambda w \rangle$$
$$= \bar{\lambda} \langle Tv, w \rangle$$
$$= \bar{\lambda} \langle v, T^* w \rangle$$
$$= \langle v, \underline{\lambda (T^* w)} \rangle$$

So by Reisz Representation $T^*(\lambda w) = \lambda(T^*w)$

Thus $T^* \in \mathcal{L}(W,V)$ $\square$

(Properties of adjoint) Let $S, T \in \mathcal{L}(V,W)$ and $\lambda \in F$

(a) $(S+T)^* = S^* + T^*$
(b) $(\lambda T)^* = \bar{\lambda} T^*$
(c) $(T^*)^* = T$
(d) $I^* = I$
(e) $(ST)^* = T^* S^*$ for $T \in \mathcal{L}(V,W)$, $S \in \mathcal{L}(W,Y)$

Proof see text use fact $\langle a, T^*b \rangle = \langle Ta, b \rangle$

Def$^{\underline{n}}$ If A is an $m \times n$ matrix, the conjugate transpose of A, denoted A^*, is the matrix formed by taking the transpose of A, and then the complex conjugate of each element

Ex $A = \begin{bmatrix} 3 & 2+i & 7 \\ -7i & 0 & 1-i \end{bmatrix}$

$A^* = \begin{bmatrix} 3 & 7i \\ 2-i & 0 \\ 7 & 1+i \end{bmatrix}$

<u>Note</u> If entries of A real, then $A^* = A^T$

<u>Thm</u> Let $T \in \mathcal{L}(V,W)$ and suppose that $e_1, \dots, e_n$ is an orthonormal basis of V and $f_1, \dots, f_m$ is an orthonormal basis of W. Then

$$\mathcal{M}(T^*, (f_1, \dots, f_m), (e_1, \dots, e_n)) = \mathcal{M}(T, (e_1, \dots, e_n), (f_1, \dots, f_m))^*$$

Observation: If we use standard bases of F^n and F^m, this says

$$\boxed{\mathcal{M}(T^*) = \mathcal{M}(T)^*}$$

matrix of adjoint

Conjugate transpose of $\mathcal{M}(T)$

Proof: k^{th} column of $\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_m))$ is given by Te_k written in terms of the basis $f_1, \ldots, f_m$

But $f_1, \ldots, f_m$ orthonormal, so

$$Te_k = \langle \underline{Te_k}, f_1 \rangle f_1 + \cdots + \langle \underline{Te_k}, f_m \rangle f_m$$

I.e. k^{th} column of $\mathcal{M}(T)$

$$\mathcal{M}(T) = \begin{array}{c} k^{th} \text{ column} \\ \begin{bmatrix} & \langle Te_k, f_1 \rangle & \\ & \langle Te_k, f_2 \rangle & \\ & \vdots & \\ & \langle Te_k, f_m \rangle & \end{bmatrix} \end{array}$$

On the other hand, entry in row k, column j of

$$M(T^*, (f_1,\dots,f_n), (e_1,\dots,e_n))$$

is given by $\langle T^* f_j, e_k \rangle$, i.e.

$$\text{So } M(T^*) = \begin{array}{c} \\ \\ k^{th} \text{ row} \\ \\ \\ \end{array} \begin{bmatrix} & & & \\ & & & \\ \langle T^* f_1, e_k \rangle & \langle T^* f_2, e_k \rangle & \cdots & \langle T^* f_m, e_k \rangle \\ \| & \| & & \| \\ \overline{\langle T e_k, f_1 \rangle} & & & \overline{\langle T e_k, f_m \rangle} \\ & & & \end{bmatrix}$$

But

$$\langle T^* f_j, e_k \rangle = \overline{\langle e_k, T^* f_j \rangle}$$

$$= \overline{\langle T e_k, f_j \rangle} \leftarrow \text{complex conjugate}$$

Thus, two matrices are conjugate transposes of each other $\square$

Key Ideas:

- adjoint
- conjugate transpose
- matrix of the adjoint