$\boxed{\text{Lecture 36}}$ 7.D Singular Value Decomposition

Last lecture ☺ ☹

Fundamental theorem in linear algebra
- singular value decomposition

<u>Def</u>

Def^{n} Given $T \in \mathcal{L}(V)$, the singular values of T are the eigenvalues of $\sqrt{T^*T}$, where each eigenvalue is repeated $\dim E(\lambda, \sqrt{T^*T})$ times

Ex From last lecture, let $T \in \mathcal{L}(V)$ be such that

$$\mathcal{M}(T) = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$$

Then $\mathcal{M}(\sqrt{T^*T}) = \begin{bmatrix} 2.8 & 0.4 \\ 0.4 & 2.2 \end{bmatrix}$ with eigenvalues $\underbrace{2 \text{ and } 3}_{\text{singular values of } T}$

Note These are not the eigenvalues of T

$$\det\begin{bmatrix} 2-\lambda & -1 \\ 2 & 2-\lambda \end{bmatrix} = (2-\lambda)^2+2 = 0$$

$$\iff \lambda^2-4\lambda+6=0$$

$$\lambda = \frac{4 \pm \sqrt{(4)^2-4\cdot 6}}{2} = \frac{4 \pm \sqrt{16-24}}{2} = 2 \pm \sqrt{2}i$$

Some facts

- Since $\sqrt{T^*T}$ positive, all its eigenvalues (i.e the singular values) are nonnegative
- Since $\sqrt{T^*T}$ is self-adjoint, by Spectral Theorem $T \in \mathcal{L}(V)$ has dim V singular values (counted w/ multiplicity)

Singular Value Decomposition Theorem

Suppose $T \in \mathcal{L}(V)$ has singular values $s_1, s_2, \ldots, s_n$. Then there exists an orthonormal basis $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ of V such that

$$Tv = \underline{s_1 \langle v, e_1 \rangle} f_1 + \underline{s_2 \langle v, e_2 \rangle} f_2 + \ldots + \underbrace{s_n \langle v, e_n \rangle}_{\text{constants}} f_n$$

for all $v \in V$.

<u>Proof</u> Since $\sqrt{T^*T}$ is self-adjoint, by the Spectral Theorem there exists an orthonormal basis $e_1, \ldots, e_n$ ← basis of eigenvectors

eigenvalue of e_i

such that $\sqrt{T^*T}\, e_i = s_i e_i$ for all i

Since $e_1, e_2, \ldots, e_n$ is an orthonormal basis of V

$$v = \langle v, e_1 \rangle e_1 + \ldots + \langle v, e_n \rangle e_n$$

Apply $\sqrt{T^*T}$ to both sides

$$\begin{aligned}(\sqrt{T^*T})v &= \sqrt{T^*T}\left(\langle v, e_1 \rangle e_1 + \ldots + \langle v, e_n \rangle e_n\right) \quad (*) \\ &= \langle v, e_1 \rangle (\sqrt{T^*T} e_1) + \ldots + \langle v, e_n \rangle (\sqrt{T^*T} e_n) \\ &= \langle v, e_1 \rangle s_1 e_1 + \ldots + \langle v, e_n \rangle s_n e_n\end{aligned}$$

By Polar Decomp Thm, there is an isometry S

such that $T = S(\sqrt{T^*T})$

Apply S to both sides of $*$

$$S(\sqrt{T^*T})v = s_1\langle v, e_1\rangle Se_1 + \cdots + s_n\langle v, e_n\rangle Se_n$$

$$\| \quad Tv$$

But since $e_1, \ldots, e_n$ is an orthonormal basis, and S is an isometry, so $Se_1, \ldots, Se_n$ is also an orthonormal basis of V ← a property about isometries

Call this basis $f_1, \ldots f_n$. So

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

$\square$

$\square$

$T \in \mathcal{L}(\mathbb{R}^2) \qquad T(x_1, x_2) = (2x_1 - x_2,\ 2x_1 + 2x_2)$

$\Updownarrow$

<u>Ex</u> Find SVD of $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$

Last Time: $A^TA = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}^{-1}$$ ← by spectral thm

$$\sqrt{A^TA} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$$

Singular values of A = eigenvalues of $\sqrt{A^TA}$
$= \lambda = 2$ and $\lambda = 3$

Polar Decomposition $A = \underbrace{\begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 2.8 & 0.4 \\ 0.4 & 2.2 \end{bmatrix}}_{\sqrt{A^TA}}$

From $\sqrt{A^TA}$, $\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ is a basis of V ← orthogonal, but not orthonormal

Turn into orthonormal basis

$e_1 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \qquad e_2 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$ ← first orthonormal basis

Second orthonormal basis given by

$$f_1 = S e_1 \qquad\qquad f_2 = S e_2$$

$$= \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \qquad\qquad = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$= \begin{bmatrix} 10/5\sqrt{5} \\ -5/5\sqrt{5} \end{bmatrix} \qquad\qquad = \begin{bmatrix} 5/5\sqrt{5} \\ 10/5\sqrt{5} \end{bmatrix}$$

So $$Av = 2 \underbrace{\left\langle v, \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix} \right\rangle}_{\text{normal dot product}} \begin{bmatrix} 10/5\sqrt{5} \\ -5/5\sqrt{5} \end{bmatrix} + 3 \left\langle v, \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\rangle \begin{bmatrix} 5/5\sqrt{5} \\ 10/5\sqrt{5} \end{bmatrix}$$

Rewrite in terms of matrices

$$\underset{\substack{\parallel \\ A}}{\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}} = \underset{\substack{\uparrow \\ \text{columns are} \\ f_1, f_2}}{\begin{bmatrix} 10/5\sqrt{5} & 5/5\sqrt{5} \\ -5/5\sqrt{5} & 10/5\sqrt{5} \end{bmatrix}} \underset{\substack{\uparrow \\ \text{diagonal matrix} \\ \text{w/ singular values}}}{\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}} \underset{\substack{\uparrow \\ \text{columns} \\ \text{are } e_1, e_2}}{\begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}^T}$$

$\boxed{\text{Alternative P.O.V.}}$

Let $T \in \mathcal{L}(\mathbb{R}^2)$ and let e_1, e_2 and f_1, f_2 be the orthonormal basis that come from SVD

Note $Te_1 = s_1 \langle e_1, e_1 \rangle f_1 + s_2 \langle e_1, e_2 \rangle f_2 = s_1 f_1$

$Te_2 = s_1 \langle e_2, e_1 \rangle f_1 + s_2 \langle e_2, e_2 \rangle f_2 = s_2 f_2$

Then $\mathcal{M}(T, (e_1, e_2), (f_1, f_2)) =$

$$\begin{array}{ccc} e_1 & e_2 & \\ \left[\begin{array}{c} s_1 \\ 0 \end{array}\right. & \left.\begin{array}{c} 0 \\ s_2 \end{array}\right] & \begin{array}{c} f_1 \\ f_2 \end{array} \end{array}$$

In general, for <u>any</u> $T \in \mathcal{L}(V)$ can find orthonormal basis $e_1, \ldots, e_n$ and $f_1, \ldots, f_n$ so

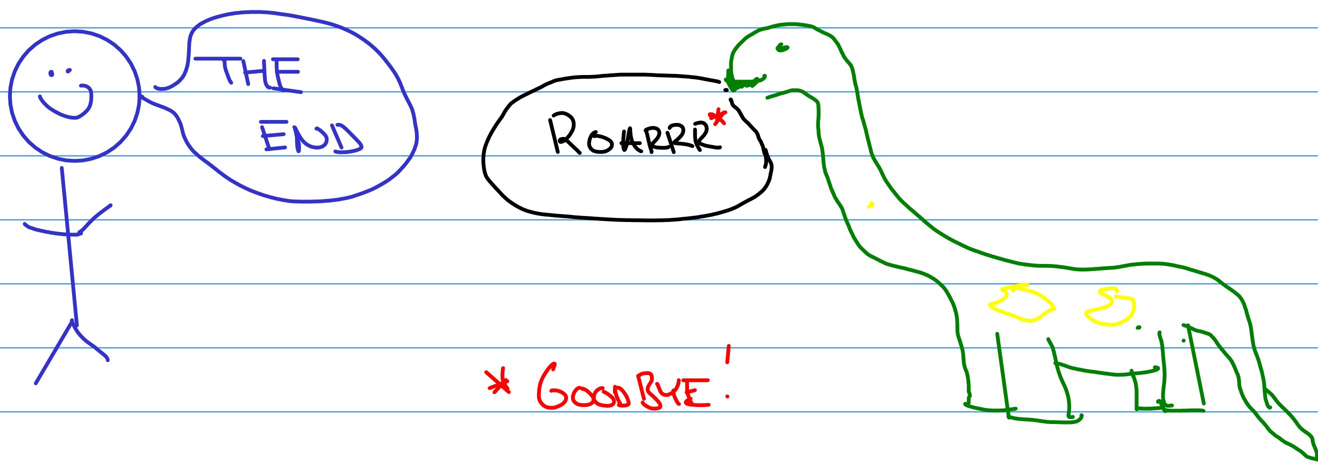
Singular values

$$\mathcal{M}(T, (e_1, \ldots, e_n), (f_1, \ldots, f_n)) = \begin{bmatrix} s_1 & & 0 \\ & \ddots & \\ 0 & & s_n \end{bmatrix}$$

Recall Lecture 1

"Linear algebra is the study of linear maps on finite dimensional vector spaces"

$\Rightarrow$ SVD "breaks apart" any linear map!



Key ideas SVD