$\boxed{\text{Lecture 23}}$ 8.B A worked out Example

Last time: Let $T \in \mathcal{L}(V)$ and $F = \mathbb{C}$. Let $\lambda_1, \dots, \lambda_n$ be the distinct eigenvalues with

d_i = multiplicity of λ_i = $\dim G(\lambda_i, T)$

Then there is a basis of V such that

$$\mathcal{M}(T) = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{bmatrix} \quad \text{with } A_i = \begin{bmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}$$

$d_i \times d_i$ matrix

upper triangular matrix with λ_i on the diagonal

TODAY: A worked out example to find $\mathcal{M}(T)$

Let $T \in \mathcal{L}(\mathbb{C}^4)$ given by

$$T(x_1, x_2, x_3, x_4) = (10x_1 - 7x_2 + 15x_3 - 2x_4,\ 10x_2 + 5x_3 - 5x_4,\ -4x_2 + 25x_3 + x_4,\ 10x_2 - 5x_3 + 25x_4)$$

W.R.T. standard basis of $\mathbb{C}^4$, we have

$$\mathcal{M}(T) = \begin{bmatrix} 10 & -7 & 15 & -2 \\ 0 & 10 & 5 & -5 \\ 0 & -4 & 25 & 1 \\ 0 & 10 & -5 & 25 \end{bmatrix}$$

$M(T) = $ (see above)

Need the eigenvalues. Can use approach of 1B03 $\Rightarrow$ by finding char. equation or Octave/Matlab

If we do all the work, get two distinct eigenvalues

$$\lambda = 10 \qquad \text{and} \qquad \lambda = 20$$

So

$$\boxed{V = G(10,T) \oplus G(20,T)}$$

$\boxed{\text{Compute multiplicity of each } \lambda}$

Recall multiplicity of $\lambda = \dim G(\lambda, T)$

$$= \dim \operatorname{Null}\left((T-\lambda I)^{\dim V}\right)$$

$$= \dim \operatorname{Null}\left(\mathcal{M}(T-\lambda I)^{\dim V}\right)$$

= # of free variables in $\mathcal{M}(T-\lambda I)^{\dim V}$

$\boxed{\lambda = 10}$

$$(\text{mult of } \lambda = 10) = \dim\left(\operatorname{Null}\left(\mathcal{M}(T-10I)^{4}\right)\right) \quad \leftarrow \dim \mathbb{C}^4 = 4$$

$$\mathcal{M}(T-10I)^4 = \left[\mathcal{M}(T) - 10 I_4\right]^4 = \begin{bmatrix} 0 & -7 & 15 & -2 \\ 0 & 0 & 5 & -5 \\ 0 & -4 & 15 & 1 \\ 0 & 10 & -5 & 15 \end{bmatrix}^4$$

$$= \begin{bmatrix} 0 & -1000 & 30000 & 19000 \\ 0 & -12000 & 20000 & -2000 \\ 0 & 2000 & 30000 & 22000 \\ 0 & 22000 & -20000 & 12000 \end{bmatrix} \sim \begin{bmatrix} 0 & \underline{1} & 0 & 0 \\ 0 & 0 & \underline{1} & 0 \\ 0 & 0 & 0 & \underline{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ mult of $\lambda = 10 = \dim G(\lambda, T) =$ # of free variables $= \underline{1}$

$\boxed{\lambda = 20}$ Use the same approach, ie, find

\# of free variables in $\left[M(T) - 20I_4\right]^4$

$$\left[M(T)-20I_4\right]^4 = \begin{bmatrix} -10 & -7 & 15 & -2 \\ 0 & -10 & 5 & -5 \\ 0 & -4 & 5 & 1 \\ 0 & 10 & -5 & 5 \end{bmatrix}^4$$

$$= \begin{bmatrix} 10{,}000 & 3{,}000 & -10{,}000 & 3{,}000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \underline{1} & .3 & -1 & .3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ mult of $\lambda = 20$ = $\dim G(20, T)$ = # of free variables
$= 3$

Story so far ..... $\lambda = 10$ eigenvalue of mult 1
$\lambda = 20$ " " " 3

So, can find basis such that

$$M(T) = \begin{bmatrix} \underbrace{[10]}_{1\times 1} & 0 & 0 & 0 \\ 0 & 20 & * & * \\ 0 & 0 & 20 & * \\ 0 & 0 & 0 & 20 \end{bmatrix} \quad \leftarrow 3\times 3$$

← But what is the basis and what is $*$?

$\boxed{\text{Main idea to find } * \text{ and basis}}$

$(T-\lambda I)$ is nilpotent on $G(\lambda,T)$. So, will use nilpotent strategy, i.e. find a basis for $Null(T-\lambda I)$ and extend to $Null((T-\lambda I)^2)$. Then extend this basis to $Null((T-\lambda I)^3)$, and so on, until we have a basis for $G(\lambda,T)$ for each λ

$\boxed{\lambda=10}$ Since mult of $\lambda=1$,

$$G(10,T)=E(10,T)$$

Since $M(T-10I_4) \sim \begin{bmatrix} 0&1&0&0\\ 0&0&1&0\\ 0&0&0&1\\ 0&0&0&0 \end{bmatrix}$

$$\Rightarrow Null(T-10I_4)=Null(M(T-10I_4))$$
$$= \left\{ x_1 \begin{bmatrix}1\\0\\0\\0\end{bmatrix} \;\middle|\; x_1 \in \mathbb{C} \right\}$$

1BOJ

So $G(10,T)=\operatorname{span}\left(\begin{bmatrix}1\\0\\0\\0\end{bmatrix}\right)$

$\boxed{\lambda = 20}$

Basis for $\mathrm{Null}(T-20I) = \mathrm{Null}(\mathcal{M}(T-20I))$

$$\mathcal{M}(T-20I_4) \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $\mathrm{Null}(T-20I_4) = \left\{ x_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \;\middle|\; x_4 \in \mathbb{C} \right\}$

Basis for $\mathrm{Null}(T-20I_4) = \mathrm{span}\left(\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right)$

Now extend this basis to a basis of $\mathrm{Null}((T-20I_4)^2)$

$$\mathcal{M}((T-20I)^2) \sim \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Via IBoS, would say

$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ is a basis

Instead, want our basis to be

$$\underbrace{\left\{ \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}}$$

basis of Null(T-20I)   basis of Null((T-20I)²)

Extend this to a basis of $\text{Null}((T-20I)^3)$

$$\mathcal{M}((T-20I)^3) \sim \begin{bmatrix} 1 & 0.3 & -1 & 0.3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Already have two basis elements of null space.
The vector $\begin{bmatrix} 0.3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ is also in the null space and not in the span of the other two.

desired basis since multiplicity is 3

So, $\text{Null}((T-20I)^3) = G(20, T) = \text{span}\left(\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} .3 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right)$

$\boxed{\text{Putting pieces together}}$

We have

$$u_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad u_4 = \begin{bmatrix} 0.3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

basis for $G(10, T)$

basis for $G(20, T)$

Claim: u_1, u_2, u_3, u_4 desired basis of V

$$Tu_1 = \begin{bmatrix} 10 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \underline{10}\, u_1 + \underline{0} \cdot u_2 + \underline{0} \cdot u_3 + \underline{0} \cdot u_4$$

$$Tu_2 = \begin{bmatrix} 20 \\ -20 \\ -20 \\ 20 \end{bmatrix} = \underline{0}\, u_1 + \underline{20}\, u_2 + \underline{0} \cdot u_3 + \underline{0} \cdot u_4$$

$$Tu_3 = \begin{bmatrix} 25 \\ 5 \\ 25 \\ -5 \end{bmatrix} = 0u_1 - 5u_2 + 20u_3 + 0\cdot u_4$$

$$Tu_4 = \begin{bmatrix} 10 \\ -10 \\ 4 \\ -10 \end{bmatrix} = 0\cdot u_1 - 10u_2 - 6u_3 + 20u_4$$

So our matrix with respect to this basis is

$$M(T) = \begin{array}{c c} & \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \end{array} & \begin{bmatrix} [10] & 0 & 0 & 0 \\ 0 & 20 & -5 & -10 \\ 0 & 0 & 20 & -6 \\ 0 & 0 & 0 & 20 \end{bmatrix} \end{array}$$

<u>Note</u> Can verify basis of $\mathbb{C}^4$ to find

$$\text{original matrix} = \begin{bmatrix} 1 & -1 & 1 & 0.3 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 20 & -5 & -10 \\ 0 & 0 & 20 & -6 \\ 0 & 0 & 0 & 20 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 0.3 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1}$$

Original matrix <u>not</u> diagonalizable, but this is a diagonalization-like result

Key ideas * worked out example
of finding basis so that $M(T)$
is a block diagonal matrix