$\boxed{\text{Lecture 34}}$ 7.C Positive Operators

We look @ two operators: positive operators and
isometries

$\boxed{\text{Positive Operators}}$

<u>Def</u>^{n} $T \in \mathcal{L}(V)$ is positive if T is self-adjoint
and $\langle Tv, v \rangle \geq 0$ for all $v \in V$

1B03/2LA3 Interpretation: Assume $F = \mathbb{R}$ and dim $V = n$

$$T \in \mathcal{L}(V) \longleftrightarrow A \quad n \times n \text{ matrix}$$

$$T \text{ self-adjoint} \longleftrightarrow A = A^{T} \quad \text{symmetric matrix}$$

$$\langle Tv, v \rangle \geq 0 \longleftrightarrow Av \cdot v \geq 0 \iff (Av)^{T} v$$

$$= v^{T} A^{T} v$$

$$= \underbrace{v^{T} A v}_{\text{this is a quadratic form}} \geq 0$$

dot product

So T is positive $\longleftrightarrow$ A is symmetric and
quadratic form $v^{T} A v \geq 0$ for all $v \in V$
$\iff$ A positive semidefinite matrix

Ex Show $T \in \mathcal{L}(\mathbb{R}^2)$ with
$T(x_1, x_2) = (5x_1 - 2x_2, -2x_1 + 5x_2)$ is positive

$M(T) = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$ So T is self-adjoint

By Spectral Theorem

$$\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{= P} \underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix}}_{= D} \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}^T}_{= P^{-1} = P^T}$$

Then $\langle T(x_1, x_2), (x_1, x_2) \rangle = (5x_1 - 2x_2, -2x_1 + 5x_2) \cdot (x_1, x_2)$
$= 5x_1^2 - 4x_1x_2 + 5x_2^2$

Set $x_1 = 1/\sqrt{2}\, y_1 + 1/\sqrt{2}\, y_2$ and $x_2 = 1/\sqrt{2}\, y_1 - 1/\sqrt{2}\, y_2$

$$\iff \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

make substitution

So $\langle T(x_1,x_2),(x_1,x_2)\rangle =$

$$5\left(\tfrac{1}{\sqrt2}y_1 + \tfrac{1}{\sqrt2}y_2\right)^2 + 4\left(\tfrac{1}{\sqrt2}y_1 + \tfrac{1}{\sqrt2}y_2\right)\left(\tfrac{1}{\sqrt2}y_1 - \tfrac{1}{\sqrt2}y_2\right) + 5\left(\tfrac{1}{\sqrt2}y_1 - \tfrac{1}{\sqrt2}y_2\right)^2$$

$$= 3y_1^2 + 7y_2^2 \quad \leftarrow \text{always } \geq 0$$

<u>Def$^{\underline{n}}$</u> An operator R is a square root of $T \in \mathcal{L}(V)$ if $R^2 = T$

<u>Thm</u> Let $T \in \mathcal{L}(V)$. TFAE

(a) T is positive
(b) T is self-adjoint and all eigenvalues of T are nonnegative
(c) T has a positive square root, $T = R^2$ and R positive
(d) T has a self-adjoint square root
(e) there exists $R \in \mathcal{L}(V)$ such that $T = R^* R$

Proof (see text)

Ex Previous example

$$T(x_1,x_2) = (5x_1 - 2x_2, -2x_1 + 5x_2)$$

A lot of work to prove positive by definition

$$M(T) = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}$$ ← so T is self-adjoint since $M(T)$ is symmetric

Eigenvalues of T are $\lambda = 3, 7 \geq 0$

So by Thm, T is positive!

Ex Thm tells we can find symmetric matrix R such that

$$R^2 = \begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix}.$$ What is R?

Recall $$\begin{bmatrix} 5 & -2 \\ -2 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{= P} \begin{bmatrix} 3 & 0 \\ 0 & 7 \end{bmatrix} P^{-1}$$

Using this orthonormal decomposition:

$$R = P \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{7} \end{bmatrix} P^{-1}.$$ Then $$R^2 = P \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{7} \end{bmatrix} P^{-1} P \begin{bmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{7} \end{bmatrix} P^{-1}$$

$$= P\begin{bmatrix}\sqrt{3} & 0\\ 0 & \sqrt{7}\end{bmatrix}\begin{bmatrix}\sqrt{3} & 0\\ 0 & \sqrt{7}\end{bmatrix}P^{-1} = P\begin{bmatrix}3 & 0\\ 0 & 7\end{bmatrix}P^{-1} = A$$

<u>Thm</u> Every positive $T \in \mathcal{L}(V)$ has a <u>unique</u> positive square root

Isometrics

norms are preserved by S

<u>Def</u>^{n} $S \in \mathcal{L}(V)$ is an isometry if $\|Sv\| = \|v\|$ for all v

(1B03/2LA3 connection:) Recall $\{\vec{u}_1, \ldots, \vec{u}_n\} \subseteq \mathbb{R}^n$ is an orthonormal set if $\vec{u}_i \cdot \vec{u}_j = 0$ if $i \neq j$ and $\|\vec{u}_i\| = 1$ for all i

An $n \times n$ matrix U whose columns are an orthonormal set is an <u>orthogonal matrix</u>

U has the property $\|U\vec{x}\| = \|\vec{x}\|$ for all $\vec{x} \in \mathbb{R}^n$

we want to generalize this

(Characterization of isometries) Let $S \in \mathcal{L}(V)$. TFAE

(a) S is an isometry
(b) $\langle Su, Sv \rangle = \langle u, v \rangle$ ← inner product is preserved
(c) if $e_1, \ldots, e_n$ orthonormal, then $Se_1, \ldots, Se_n$ is also orthonormal.
(d) there is an orthonormal basis of V such that $Se_1, \ldots, Se_n$ is orthonormal
(e) $SS^* = I$
(f) $S^*S = I$
(g) S^* is an isometry
(h) S is invertible and $S^{-1} = S^*$

Proof (a) ⇒ (b) Need a trick depending upon F

If $F = \mathbb{R}$ $\quad \langle a, b \rangle = \dfrac{\|a+b\|^2 - \|a-b\|^2}{4}$

If $F = \mathbb{C}$,

$$\langle a, b \rangle = \frac{\|a+b\|^2 - \|a-b\|^2 + \|a+ib\|^2 i - \|a-ib\|^2 i}{4}.$$

inner product expressed in terms of norms

Book does $F = \mathbb{R}$, I will do $F = \mathbb{C}$.

$$\langle Su, Sv\rangle = \frac{\|Su+Sv\|^2 - \|Su-Sv\|^2 + \|Su+iSv\|^2 i - \|Su-iSv\|^2 i}{4}$$

linear op

$$= \frac{\|S(u+v)\|^2 - \|S(u-v)\|^2 + \|S(u+iv)\|^2 i - \|S(u-iv)\|^2 i}{4}$$

S is an isometry

$$= \frac{\|u+v\|^2 - \|u-v\|^2 + \|u+iv\|^2 i - \|u-iv\|^2 i}{4}$$

$$= \langle u, v\rangle$$

(b) $\Rightarrow$ (a) $\quad \|Su\|^2 = \langle Su, Su\rangle$
$= \langle u, u\rangle = \|u\|^2$ for all u

$\blacksquare$

Note: If A is orthogonal matrix, then implies $A^{-1} = A^*$ ← conjugate transpose (part h)

Cor Every Isometry is normal

Proof $S^*S = I = SS^*$ if S is an isometry $\square$

Key ideas:
- positive operators
- isometries
- characterization of these operators