

Lecture 3 1.C Subspaces

Goal: Introduce Subspaces

Defⁿ Let V be a vector space. A subset $U \subseteq V$ is a subspace if U is also a vector space with the same scalar multiplication and addition of V

Ex (trivial subspaces) $\{0\}$ and V are subspaces of V

Thm U is a subspace of V if and only if

1. $0 \in U$
2. if $u, v \in U$, then $u+v \in U$ (closure under add.)
3. if $u \in U$, $c \in F$, then $cu \in U$ (closure under scalar mult.)

Proof (\Rightarrow) If U is a subspace, it is a vector space, so it satisfies 1, 2, & 3

(\Leftarrow) Need to show U satisfies all axioms of a v.s.

Note that

- $u+v = v+u$
- $a(u+v) = au+av$
- $u+(v+w) = (u+v)+w$
- $1v = v$
- $(ab)v = a(bv)$

hold for all elements in V , including those in U .

- U has an additive identity by 1.
- 2 and 3 imply U has scalar mult and add.
- By 3, if $u \in U$, then $(-1) \cdot u \in U$. But $(-1) \cdot u = -u$. So all additive inverses also in U . \square

Ex Let $V = F^3$ and $W = \{(x_1, x_2, x_3) \mid x_1 + x_2 + x_3 = 0\}$
 Show W is a subspace of V

Proof Check the three conditions

1. $0 \in W$ since $0 = (0, 0, 0)$ and $0 + 0 + 0 = 0$

2. Let $x, y \in W$. So $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$
 and $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0$

Then $x + y = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$

But $(x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) = (x_1 + x_2 + x_3) + (y_1 + y_2 + y_3) = 0$

So $x + y \in W$

3. Let $x \in W$ and $c \in F$. So $x = (x_1, x_2, x_3)$
and $x_1 + x_2 + x_3 = 0$. Then

$cx = (cx_1, cx_2, cx_3)$. Then

$$cx_1 + cx_2 + cx_3 = c(x_1 + x_2 + x_3) = c \cdot 0 = 0$$

So $cx \in W$.

Thus, W is a subspace of V . □

Ex Let $V = \mathbb{R}^{[0,1]}$ ← all real valued functions from $[0,1]$ to \mathbb{R}

Let $C \subseteq V$ with $C = \{f \in V \mid f \text{ continuous}\}$

Then C is a subspace of V .

"
 $\{f \mid f: [0,1] \rightarrow \mathbb{R}\}$
f continuous

Proof 1. $0 \in C$ since the 0 element of V
is $0: [0,1] \rightarrow \mathbb{R}$ is the function $0(x) = 0$
This is a continuous function.

2. Let $f, g \in C$. From calculus, if f, g continuous,
then $(f+g)(x) = f(x) + g(x)$ is continuous. So
 $f+g \in C$.

3. Let $f \in C$ and $\lambda \in F$. From calculus, if f continuous,
 $(\lambda f)(x) = \lambda(f(x))$ is continuous. So $\lambda f \in C$.

So, C is a subspace of V .

Sums of subspaces

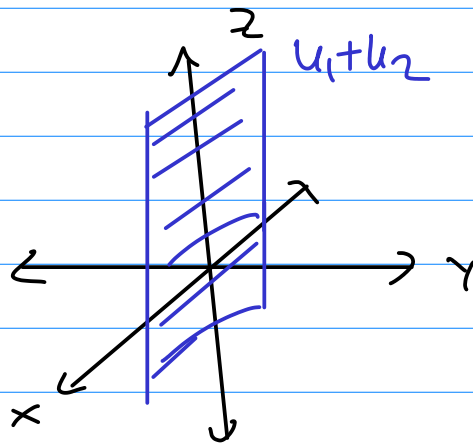
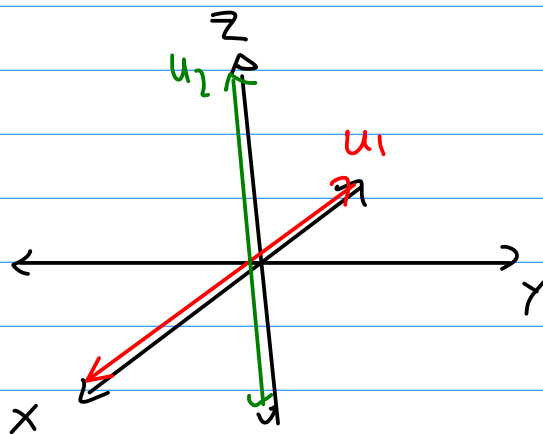
Given subspaces U_1, \dots, U_m of V , can build bigger subspaces

Defⁿ Suppose U_1, \dots, U_m subsets of V . The Sum of U_1, \dots, U_m , denoted $U_1 + \dots + U_m$ is
 $U_1 + \dots + U_m = \{u_1 + u_2 + \dots + u_m \mid u_i \in U_i\}$

Ex $V = F^3$ and $U_1 = \{(x, 0, 0) \mid x \in F\}$
 $U_2 = \{(0, 0, z) \mid z \in F\}$

Then $U_1 + U_2 = \{(x, 0, 0) + (0, 0, z) = (x, 0, z) \mid x, z \in F\}$

Picture



Note: $U_1 + U_2 \supsetneq U_1 \cup U_2$

Thm If U_1, \dots, U_m subspaces of V , then

- (A) $U_1 + \dots + U_m$ a subspace of V
- (B) $U_1 + \dots + U_m$ is the smallest subspace to contain U_1, \dots, U_m

Proof (A) Check three conditions of subspace

1. $0 \in U_1 + \dots + U_m$ since $0 \in U_i$ for all i and $0 + 0 + 0 + \dots + 0 = 0 \in U_1 + \dots + U_m$
2. Let $x, y \in U_1 + \dots + U_m$. So $x = x_1 + \dots + x_m$ with $x_i \in U_i$ and $y = y_1 + \dots + y_m$ with $y_i \in U_i$.

$$\begin{aligned} \text{Then } (x+y) &= (x_1 + \dots + x_m) + (y_1 + \dots + y_m) \\ &= (x_1 + y_1) + \dots + (x_m + y_m) \end{aligned}$$

Since $x_i, y_i \in U_i$ and U_i a subspace, $x_i + y_i \in U_i$

So $(x+y) \in U_1 + \dots + U_m$

3. Let $x \in U_1 + \dots + U_m$ and $c \in F$. So $x = x_1 + \dots + x_m$ with $x_i \in U_i$. Then

$$cx = c(x_1 + \dots + x_m) = cx_1 + \dots + cx_m \in U_1 + \dots + U_m$$

Since $cx_i \in U_i$ for all i . (since U_i is a subspace)
Thus $U_1 + \dots + U_m$ is a subspace of V .

(B) Note that $U_i \subseteq U_1 + \dots + U_m$. Now suppose $U_1, \dots, U_m \subseteq W$. Then for any $u_i \in U_i$, we have $u_1, u_2, \dots, u_m \in W$. But W is a subspace, so $u_1 + \dots + u_m \in W$.
Thus $U_1 + \dots + U_m \subseteq W$. □

Key ideas:

- * subspaces
- * 3 conditions for a subspace
- * sums of subsets/subspaces