$\boxed{\text{Lecture 27}}$ Midterm II

$\boxed{\text{Lecture 28}}$ 6.A Inner Products II

Continue to develop prop of inner products

Recall An inner product space is a vector space V with an inner product, i.e., a function that associates $u,v \in V$ with $\langle u,v\rangle \in F$ such that

1. $\langle u,u\rangle \geq 0$
2. $\langle u,u\rangle = 0$ if and only if $u=0$
3. $\langle u+w,v\rangle = \langle u,v\rangle + \langle w,v\rangle$
4. $\langle \lambda u,v\rangle = \lambda\langle u,v\rangle$
5. $\langle u,v\rangle = \overline{\langle v,u\rangle}$

norm of $v \Rightarrow \|v\| = \sqrt{\langle v,v\rangle}$

u,v orthogonal $\Rightarrow \langle u,v\rangle = 0$

Facts ① Pythagorean Thm: If $\langle u,v\rangle = 0$, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

② $\|\lambda v\| = |\lambda|\,\|v\|$

if $\lambda = a+bi$, $|\lambda| = \sqrt{a^2+b^2}$

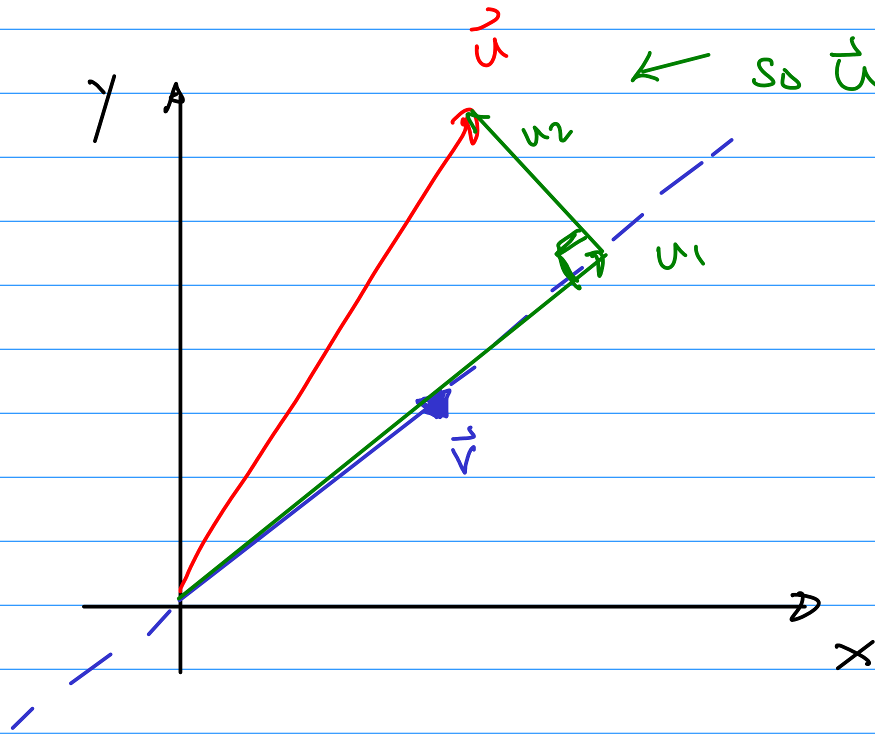
Orthogonal Decomposition

Fix $v \in V$. Given $u \in V$, want to write u as

$$u = u_1 + u_2$$

u_1 is a multiple of v — orthogonal to v

Picture in $\mathbb{R}^2$ (e.g. from 1B03)



so $\vec{u} = \vec{u}_1 + \vec{u}_2$

Notation from 1B03 $\quad \vec{u}_1 = \mathrm{proj}_{\vec{v}}\, \vec{u}$

Although picture is in $\mathbb{R}^2$, works in <u>any</u> inner product space.

<u>Thm</u> Let $u, v \in V$ with $v \neq 0$. If

$$c = \frac{\langle u,v\rangle}{\|v\|^2} \quad \text{and} \quad w = u - \frac{\langle u,v\rangle}{\|v\|^2} v$$

then $\langle w, v\rangle = 0$ and $u = cv + w$

<u>Proof</u>

- $$cv + w = \cancel{\frac{\langle u,v\rangle}{\|v\|^2}} v + u - \cancel{\frac{\langle u,v\rangle}{\|v\|^2}} v = u$$

- $$\langle w, v\rangle = \left\langle u - \frac{\langle u,v\rangle}{\|v\|^2} v, v \right\rangle$$

$$= \langle u, v\rangle - \left\langle \overbrace{\frac{\langle u,v\rangle}{\|v\|^2}}^{\text{a constant}} v, v\right\rangle$$

$$= \langle u, v\rangle - \frac{\langle u,v\rangle}{\|v\|^2} \langle v, v\rangle$$

$$= \langle u, v\rangle - \frac{\langle u,v\rangle}{\cancel{\|v\|^2}} \cancel{\|v\|^2} = 0$$

$\square$

Cauchy Schwarz

(Cauch-Schwarz) Let $u, v \in V$. Then

$$|\langle u,v\rangle| \le \|u\|\,\|v\|$$

Furthermore

$$|\langle u,v\rangle| = \|u\|\,\|v\| \iff u \text{ and } v \text{ scalar multiples of each other}$$

Proof If $v=0$, both sides equal 0

If $v \neq 0$, write

$$u = \frac{\langle u,v\rangle}{\|v\|^2} v + w \quad \text{(as by previous result)}$$

Since $\langle w,v\rangle = 0$, by Pythagorean Thm

$$\|u\|^2 = \left\| \frac{\langle u,v\rangle}{\|v\|^2} v + w \right\|^2 = \left\| \frac{\langle u,v\rangle}{\|v\|^2} v \right\|^2 + \|w\|^2$$

$$= \frac{|\langle u,v\rangle|^2}{\|v\|^4} \|v\|^2 + \|w\|^2 \qquad (\text{note } \|w\|^2 \geq 0)$$

$$\geq \frac{|\langle u,v\rangle|^2}{\|v\|^4} \|v\|^2 = \frac{|\langle u,v\rangle|^2}{\|v\|^2}$$

So $|\langle u,v\rangle|^2 \leq \|u\|^2 \|v\|^2$

$\Rightarrow |\langle u,v\rangle| \leq \|u\| \, \|v\|$

(see text for other part) $\square$

<u>Ex 1</u> Show $|x_1y_1 + \cdots + x_ny_n|^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)$
for any $2n$ real numbers $x_1, \ldots, x_n, y_1, \ldots, y_n$

<u>Proof</u> Let $V = \mathbb{R}^n$ and use standard Euclidean inner product.
Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$

So by Cauchy-Schwarz $|\langle x,y\rangle|^2 \leq \|x\|^2 \|y\|^2$

$\Rightarrow |x_1y_1 + x_2y_2 + \cdots + x_ny_n|^2 \leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2)$

want to rewrite this as $|\langle x,y\rangle| \le \|x\|\,\|y\|$

Ex 2 For all positive real numbers a,b,c,d

$$16 \le (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

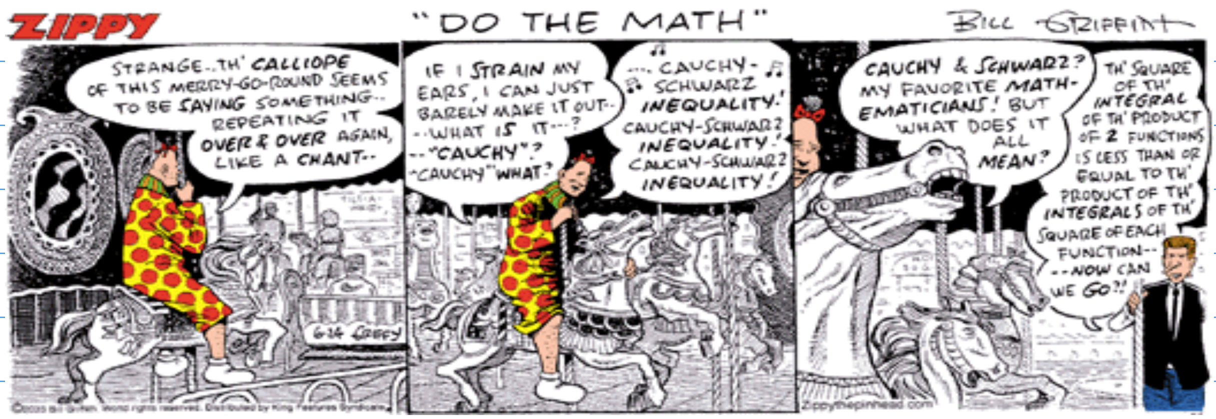
Let $V=\mathbb{R}^4$ and use standard inner product

Let $x=(\sqrt{a},\sqrt{b},\sqrt{c},\sqrt{d})$ $\quad y=(1/\sqrt{a},\, 1/\sqrt{b},\, 1/\sqrt{c},\, 1/\sqrt{d})$

$\langle x,y\rangle = x\cdot y = \sqrt{a}/\sqrt{a}+\dots+\sqrt{d}/\sqrt{d} = 4$

So, by Ex 1 $\Rightarrow$ $4^2 \le \left((\sqrt{a})^2+\dots+(\sqrt{d})^2\right)\left((1/\sqrt{a})^2+\dots+(1/\sqrt{d})^2\right)$

Ex 3



$V=\mathcal{P}(\mathbb{R})$ and $\langle p,q\rangle = \int_{-1}^{1} p(x)q(x)\,dx$

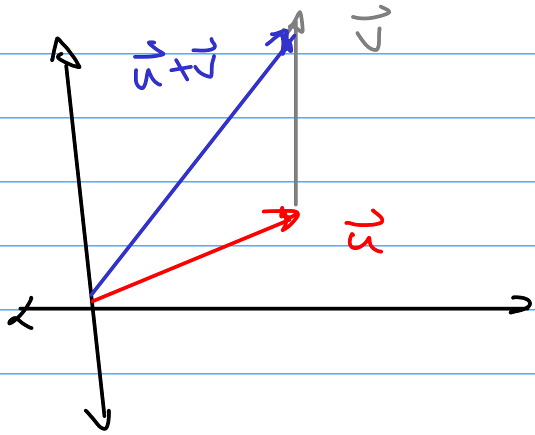
$$|\langle p,q\rangle|^2 = \left[\int_{-1}^{1} p(x)\,q(x)\,dx\right]^2$$

C.S $\le \|p\|^2\,\|q\|^2 = \langle p,p\rangle\langle q,q\rangle$

$$= \left(\int_{-1}^{1} p(x)^2\,dx\right)\left(\int_{-1}^{1} (q(x))^2\,dx\right)$$

Triangle Inequality

Inspiration from $\mathbb{R}^2$



Picture "shows"

$$\|\vec{u}+\vec{v}\| \leq \|\vec{u}\|+\|\vec{v}\|$$

Thm (triangle inequality theorem)

$\|u+v\| \leq \|u\| + \|v\|$ in any inner product space

Proof $\|u+v\|^2 = \langle u+v, u+v\rangle$

$$= \langle u,u\rangle + \langle u,v\rangle + \langle v,u\rangle + \langle v,v\rangle$$
$$= \|u\|^2 + \langle u,v\rangle + \overline{\langle u,v\rangle} + \|v\|^2$$
$$= \|u\|^2 + \underline{2\,\mathrm{Re}(\langle u,v\rangle)} + \|v\|^2$$

Note If $\lambda = a+bi$ and $\overline{\lambda} = a-bi$, $\lambda+\overline{\lambda} = 2\,\mathrm{Re}(\lambda) = 2a$

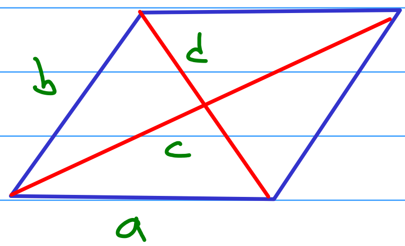
Also $\mathrm{Re}(\lambda) = a \leq |\lambda| = \sqrt{a^2+b^2}$

So $\|u+v\|^2 = \|u\|^2 + 2\,\mathrm{Re}(\langle u,v\rangle) + \|v\|^2$

$$\leq \|u\|^2 + 2|\langle u,v\rangle| + \|v\|^2$$
$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2 \quad \text{CS}$$
$$= (\|u\| + \|v\|)^2$$

$\square$

Another classical result from geometry



a, b, c, d lengths

Then $c^2 + d^2 = 2(a^2 + b^2)$

Theorem (Parallelogram equality)

$$\|u+v\|^2 + \|u-v\|^2 = 2\left(\|u\|^2 + \|v\|^2\right)$$

Proof

$$\|u+v\|^2 + \|u-v\|^2 = \langle u+v, u+v\rangle + \langle u-v, u-v\rangle$$

$$= \|u\|^2 + \langle u,v\rangle + \langle v,u\rangle + \|v\|^2$$
$$+ \|u\|^2 - \langle u,v\rangle - \langle v,u\rangle + \|v\|^2$$

$$= 2\left(\|u\|^2 + \|v\|^2\right)$$

$\square$

Key ideas
* orthogonal decomposition
* Cauchy-Schwarz
* triangle inequality