Lecture 19 — 5.C Eigenspaces & Diagonal Matrices

Last time: If V is a finite dimensional v.s. over $\mathbb{C}$, we can find a basis for V such that $\mathcal{M}(T)$ is upper triangular for any $T \in \mathcal{L}(V)$

Q Can we do better? i.e. can $\mathcal{M}(T)$ have fewer 0's?

Def$^{\underline{n}}$ A diagonal matrix is an $n \times n$ matrix D where all the entries are 0 except possibly on the diagonal

Ex
$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Defⁿ Suppose $T \in \mathcal{L}(V)$ and $\lambda \in F$. The <u>eigenspace</u> of T corresponding to λ, is the subspace

$$E(\lambda, T) = \{ v \mid (T - \lambda I)v = 0 \} = \text{Null}(T - \lambda I)$$

<u>Note</u> 1 λ is an eigenvalue $\Longleftrightarrow E(\lambda, T) \neq \{0\}$

<u>Note</u> 2. If λ is an eigenvalue of T, then

$$T\Big|_{E(\lambda, T)} = \lambda v$$

← T restricted to $E(\lambda, T)$ is the same as multiplying by λ

Eigenspaces of distinct eigenvalues "disjoint"

<u>Thm</u> Suppose V is a finite dim vector space and $T \in \mathcal{L}(V)$ with distinct eigenvalues $\lambda_1, .., \lambda_m$. Then

- $E(\lambda_1, T) + .. + E(\lambda_m, T)$ is a direct sum
- $\dim(E(\lambda_1, T)) + .. + \dim(E(\lambda_m, T)) \leq \dim V$

Proof Dimension result follows from first

To prove direct sum, suppose

$$u_1 + u_2 + \cdots + u_m = 0 \quad \text{with } u_i \in E(\lambda_i, T)$$

Each u_i is an eigenvector of λ_i, i.e $Tu_i = \lambda_i u_i$. Since eigenvectors from distinct eigenvalues are linearly independent, then we have $u_1 = \cdots = u_m = 0$. $\boxed{\varepsilon}$

Def$^{\underline{n}}$ An operator $T \in \mathcal{L}(V)$ is <u>diagonalizable</u> if there is a basis of V such that $M(T)$ is a diagonal matrix w.r.t this basis

(Diagonalization in 1B03)

Consider $T \in \mathcal{L}(\mathbb{R}^2)$ given by

$$T(x,y) = (7x+2y, -4x+y) \quad \longleftrightarrow \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$

1B03 language

Using <u>standard basis</u> $\{e_1, e_2\}$,

$$M(T) = \begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$$

Eigenvalues of $\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$ are $\lambda = 5$ and $\lambda = 3$

Eigenvector for $\boxed{\lambda = 5}$

$$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -4 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow x_2 \text{ free}$$

$$x_1 = -x_2$$

So

$$\text{Null}\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right) = \left\{ x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \,\middle|\, x_2 \in \mathbb{R} \right\} = \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$$

$$\Rightarrow E(5, T) = \text{span}\left(\underline{(-1, 1)} \right)$$

Eigenvector for $\boxed{\lambda = 3}$ $\Longrightarrow$ $E(3, T) = \text{span}\left(\underline{(-1, 2)} \right)$

Use basis of eigenvectors for $\mathbb{R}^2$, i.e

$$\mathbb{R}^2 = \text{span}\left((-1, 1), (-1, 2) \right)$$

Then

$$T(-1, 1) = (-5, 5) = 5(-1, 1) + 0(-1, 2)$$

$$T(-1, 2) = (-3, 6) = 0(-1, 1) + 3(-1, 2)$$

So

$v_1 = (-1,1)$
$v_2 = (-1,2)$

$$M(T) = \begin{array}{c|c} & \begin{array}{cc} v_1 & v_2 \end{array} \\ \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} & \begin{array}{c} v_1 \\ v_2 \end{array} \end{array}$$

In 1B03, wrote this as

$$\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 2 \end{bmatrix}^{-1}$$

eigenvector of 5   eigenvector of 3

Q When can you diagonalize?

Thm Suppose $T \in \mathcal{L}(V)$ with V fin. dim.
Let $\lambda_1, .., \lambda_m$ be the distinct eigenvalues.
TFAE

1. T is diagonalizable
2. V has a basis of eigenvectors
3. $V = E(\lambda_1, T) \oplus E(\lambda_2, T) \oplus \cdots \oplus E(\lambda_m, T)$
4. $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$

Proof

1 $\Rightarrow$ 2 Let $v_1, .., v_n$ be a basis such that

$$M(T) = \begin{bmatrix} d_1 & & & 0 \\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n \end{bmatrix}$$

So d_i's are the eigenvalues. This means $Tv_i = d_i v_i$ for each i. So each v_i is <u>also</u> an eigenvector.

So $v_1, .., v_n$ is a basis of V of eigenvectors

2 $\Rightarrow$ 1 Let $v_1, .., v_n$ be a basis of eigenvectors of V.

So $\underline{Tv_i = \lambda_k v_i}$ for some $\lambda_k \in \{\lambda_1, .., \lambda_m\}$

$$\text{So } M(T) = \begin{array}{c} \\ \left(\begin{array}{ccc} \ddots & 0 & \\ & \vdots & \\ & 0 & \\ & \lambda_k & \\ & 0 & \ddots \\ & \vdots & \\ & 0 & \ddots \end{array} \right) \end{array} \begin{array}{l} i^{th} \text{ column} \\ \\ \\ \leftarrow i^{th} \text{ row} \\ \\ \\ \\ \end{array}$$

This is a diagonal matrix.

3 $\Leftrightarrow$ 4 Follows easily

[SEE TEXT 2 $\Leftrightarrow$ 3]

Cor Let $T \in \mathcal{L}(V)$ with $\dim V = n$. Suppose T has n distinct eigenvalues. Then T is diagonalizable.

Proof Let $\lambda_1, .., \lambda_n$ be the n distinct eigenvalues with $v_1, .., v_n$ the corresponding eigenvectors.

Since the λ_i's are distinct, then $v_1, .., v_n$ are n linearly independent vectors in V. Since $\dim V = n$, this means $v_1, .., v_n$ is a basis for V. So T is diagonalizable $\square$

NOTE Not every $T \in \mathcal{L}(V)$ is diagonalizable (even if $F = \mathbb{C}$)

Ex Let $T \in \mathcal{L}(\mathbb{C}^2)$ with $T(x,y) = (y,0)$

Claim 0 is the only eigenvalue

$$T(x,y) = (y,0) = \lambda(x,y) \iff \begin{array}{l} \lambda x = y \\ \lambda y = 0 \end{array}$$

not an eigenvector

If $\lambda \neq 0$, we have only ~~a~~ sol^n if $(x,y) = (0,0)$

Sol^{n}s if $\lambda = 0$, then we have a sol^{n}
if $(x,y) = (x,0)$

I.e $T(x,0) = (0,0) = 0(x,0)$

So $\lambda = 0$ is only eigenvalue and $E(0,T) = \text{span}((1,0))$

So $\mathbb{C}^2 \neq E(0,T)$ ← don't have the same dimension

<u>Observation</u> If we use basis $(1,0)$ and $(0,1)$,
still get upper triangular matrix

$$M(T) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$ ← upper triangular but not diagonal

CHAPTER 8 ⟹ How close to being diagonal can we get?

Key ideas
* diagonal matrices
* diagonalization
* eigenspace $E(\lambda, T)$
* characterization of diagonalization.