$\boxed{\text{Lecture 32}}$ 7.A Self-Adjoint and Normal Operators

Last lecture: introduced adjoints

roughly $\Rightarrow$ operators associated with transposes

Today two special adjoints: self-adjoint & normal

$\boxed{\text{Self-Adjoint}}$

Def^n: $T \in \mathcal{L}(V)$ is self-adjoint if $T = T^*$, i.e.
$\langle Tv, w \rangle = \langle v, Tw \rangle$ for all $v, w \in V$.

Remark At "matrix level", says
$$M(T^*) = M(T)^* = M(T)$$

If $F = \mathbb{R}$, and A $n \times n$ matrix, $A = A^T \Leftrightarrow A$ is symmetric matrix

If $F = \mathbb{C}$, and A $n \times n$ matrix, $A = A^* \Leftrightarrow A$ is Hermitian matrix

Thm If $T \in \mathcal{L}(V)$ is self-adjoint, every eigenvalue of T is real

Proof Let λ be an eigenvalue with nonzero eigenvector v, i.e. $Tv = \lambda v$.

Then

$$\lambda \|v\|^2 = \lambda \langle v,v \rangle = \langle \lambda v, v \rangle$$
$$= \langle Tv, v \rangle$$
$$= \langle v, Tv \rangle \quad \text{self adjoint}$$
$$= \langle v, \lambda v \rangle$$
$$= \bar{\lambda} \langle v,v \rangle = \bar{\lambda} \|v\|^2$$

Since $\|v\| \neq 0$, $\lambda = \bar{\lambda}$, i.e. $\lambda \in \mathbb{R}$ $\square$

Cor Let A be a real symmetric matrix (i.e. $A = A^T$). Then every eigenvalue of A is real

Proof. Let T be the operator given by $Tx = Ax$. Then $M(T) = A$, so $M(T^*) = A^T = A = M(T)$. I.e $T^* = T$. So apply above result

(Specialized proof given in 2LA3)

Over $F=\mathbb{C}$, use

$$u \cdot v = u_1\bar{v}_1 + \dots + u_n\bar{v}_n \quad \text{where } V=\mathbb{C}^n$$

Note $u \cdot v = u^T \bar{v}$ If λ is an eigenvalue of A

$$\lambda \|v\|^2 = \lambda(v \cdot v) = (\lambda v \cdot v)$$
$$= (\lambda v)^T \bar{v} = (Av)^T \bar{v} \quad \text{transpose prop}$$
$$= v^T A^T \bar{v} \quad \text{symmetric}$$
$$= v^T A \bar{v}$$
$$= v^T \bar{A} \bar{v} = v \cdot (Av) = v \cdot \lambda v = \bar{\lambda} \cdot v \cdot v = \bar{\lambda} \|v\|^2.$$

A is real

(Now proof continues from above)

<u>Fact</u> Suppose V is an inner product space <u>over $\mathbb{C}$</u> and $T \in \mathcal{L}(V)$. If
$\langle Tv, v \rangle = 0$ for all $v \in V$, then $T = 0$

<u>Remark</u> Need $\mathbb{C}$. If $F=\mathbb{R}$, then

$$T \in \mathcal{L}(\mathbb{R}^2) \text{ with } T(x,y) = (-y, x)$$

This has the prop that $\langle T(x,y), (x,y) \rangle = \langle (-y,x), (x,y) \rangle = 0$ but $T \neq 0$

In this case, T maps every vector v to an orthogonal element. Not possible over $\mathbb{C}$.

Thm Let $T \in \mathcal{L}(V)$ over $\mathbb{C}$. Then

T is self-adjoint iff $\langle Tv, v \rangle \in \mathbb{R}$ for all $v \in V$

Thm If $T \in \mathcal{L}(V)$ is self-adjoint, and if $\langle Tv, v \rangle = 0$ for all v, then $T = 0$

Ex Above example $T(x,y) = (-y, x)$ satisfies $\langle Tv, v \rangle = 0$ for all v, but $T \neq 0$

So, T can't be self adjoint. Indeed

$$\mathcal{M}(T) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$ ← not symmetric

Normal Operators

Def$^{\underline{n}}$ An operator $T \in \mathcal{L}(V)$ is <u>normal</u> if $TT^* = T^*T$

Ex Every self-adjoint operator is normal since

$$\underbrace{TT^*}_{T \text{ is self adjoint}} = (T^*)(T^*)^* = T^*T$$

1B03/2LA3 "matrix" point-of-view:

- over $\mathbb{R}$ $\quad AA^T = A^TA$
- over $\mathbb{C}$ $\quad AA^* = A^*A$

Thm $T \in \mathcal{L}(V)$ is normal iff

$$\|Tv\| = \|T^*v\| \quad \text{for all } v \in V$$

T and T^* send v to a vectors of the same norm

Proof T normal $\iff TT^* - T^*T = 0$

$$\iff \langle (TT^* - T^*T)v, v \rangle = 0 \quad \text{for all } v \in V$$

$$\iff \langle TT^*v, v \rangle - \langle T^*Tv, v \rangle = 0$$

$$\iff \langle T^*v, T^*v \rangle - \langle Tv, (T^*)^*v \rangle = 0$$

$$\iff \langle T^*v, T^*v \rangle - \langle Tv, Tv \rangle = 0$$

$$\iff \|T^*v\|^2 = \|Tv\|^2 \iff \|Tv\| = \|T^*v\|$$

$\square$

Thm Suppose $T \in \mathcal{L}(V)$ and T normal. If λ is an eigenvalue of T, then $\bar{\lambda}$ is an eigenvalue of T^*

Proof: Claim For $\lambda \in F$, if T is normal, then $T-\lambda I$ normal

$$(T-\lambda I)(T-\lambda I)^* = (T-\lambda I)(T^* - (\lambda I)^*)$$
$$= (T-\lambda I)(T^* - \bar{\lambda} I^*)$$
$$= (T-\lambda I)(T^* - \bar{\lambda} I)$$

since $I = I^*$

Expand out

$$(T-\lambda I)(T^* - \bar{\lambda} I) = TT^* - \bar{\lambda} TI - \lambda I T^* - \lambda\bar{\lambda} I^2$$
$$= T^*T - \lambda T^* I - \bar{\lambda} IT - \bar{\lambda}\lambda I^2$$

normal

$$= (T^* - \bar{\lambda} I)(T - \lambda I)$$

(Finish the proof). Let v be an eigenvector λ for T.

Then since $(T-\lambda I)v=0 \iff Tv=\lambda I v$

$$0 = \|(T-\lambda I)v\| = \|(T-\lambda I)^* v\|$$ by the last result since $(T-\lambda I)$ is normal

$$= \|(T^*-\bar{\lambda} I)v\|$$

So $\bar{\lambda}$ is an eigenvalue of T^* $\square$

1B03/2LA3 Point-of-view: If λ is an eigenvalue of a symmetric A over $\mathbb{R}$, then $\bar{\lambda}=\lambda$ is an eigenvalue of A^T

Thm Suppose $T\in\mathcal{L}(V)$ is normal. Then eigenvectors corresponding to distinct eigenvalues are orthogonal

Proof Suppose $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$ with $\lambda_1 \neq \lambda_2$.

Since T is normal, $T^* v_1 = \bar{\lambda}_1 v_1$ and $T^* v_2 = \bar{\lambda}_2 v_2$

So $(\lambda_1 - \lambda_2)\langle v_1, v_2\rangle = \langle (\lambda_1 - \lambda_2) v_1, v_2 \rangle$

$$= \langle \lambda_1 v_1 - \lambda_2 v_1, v_2 \rangle$$

$$= \langle \lambda_1 v_1, v_2 \rangle - \langle \lambda_2 v_1, v_2 \rangle$$

$$= \langle T v_1, v_2 \rangle - \langle v_1, \bar{\lambda}_2 v_2 \rangle$$

$$= \langle T v_1, v_2 \rangle - \langle v_1, T^* v_2 \rangle$$

$$= \langle v_1, T^* v_2 \rangle - \langle v_1, T^* v_2 \rangle = 0$$

Since $\lambda_1 - \lambda_2 \neq 0 \Rightarrow \langle v_1, v_2 \rangle = 0$ ← i.e. orthogonal!

$\square$

Key ideas: * self-adjoint operators
* normal operators