$\boxed{\text{Lecture 2}}$ 1.B Vector Spaces

Goal: Introduce vector spaces over $F = \mathbb{R}$ or $\mathbb{C}$

Briefly: a vector space is a set V with two operations that satisfy some special properties

Note: $\mathbb{R}^n$ is a standard example of a vector space (use this example first to develop your intuition)

Def^n: · an addition on a set V is a function that assigns to a $v, w \in V$ an element $v + w \in V$

· a scalar multiplication on a set V is a function that assigns to any $\lambda \in F$, $v \in V$ an element $\lambda v \in V$

Def^{n} A vector space V is a set V with an addition and scalar multiplication such that

1. $u+v = v+u$ (addition commutes)

2. $u+(v+w) = (u+v)+w$ (addition associative)

3. there is a $0 \in V$ such that (additive identity)
   $v+0 = 0+v = v$ for all $v \in V$

4. for all $v \in V$, there is a $w \in V$ (additive inverse)
   such that $v+w = 0$

5. for all $a,b \in F$, $a(bv) = (ab)v$

6. $1 \cdot v = v$ for all $v \in V$

7. $a(u+v) = au + av$ for all $a \in F$, $u,v \in V$
   $(a+b)u = au + bu$ for all $a,b \in F$, $u \in V$

Remark
- elements of V are vectors
- V is a real vector space if $F = \mathbb{R}$

  a complex vector space if $F = \mathbb{C}$.

Note saw this def^{n} in 1B03 with $F = \mathbb{R}$

Examples

1. (zero vector space)

$V = \{0\}$ ← vector space with only 0

add: $0+0=0$ scal. mult: $c \cdot 0 = 0$

2. $F^n = \{(x_1,..,x_n) \mid x_i \in F\}$ is a vector space with

$(x_1,\dots,x_n) + (y_1,\dots,y_n) = (x_1+y_1, x_2+y_2,\dots, x_n+y_n)$

$c(x_1,\dots,x_n) = (cx_1,\dots, cx_n)$

studied in 1B03 $\mathbb{R}^n$

3. $F^\infty = \{(x_1,x_2,x_3,\dots) \mid x_i \in F\}$

$(x_1,x_2,\dots)+(y_1,y_2,\dots) = (x_1+y_1, x_2+y_2,\dots)$

$c(x_1,x_2,\dots) = (cx_1, cx_2,\dots)$

4. Let S be any set and

$$F^S = \{g \mid g: S \to F\}$$

← g is any function from S to F

e.g. $S=[0,1]$ and $F=\mathbb{R}$ $\Rightarrow$ $\mathbb{R}^{[0,1]} = \{g \mid g:[0,1]\to\mathbb{R}\}$

F^S is a vector space with

$g,h \in F^S \Rightarrow (g+h)(x) = g(x)+h(x)$ ← add two functions together

$c\in F, g \in F^S \Rightarrow (cg)(x) = c(g(x))$ ← scale function by c.

1B03 notation

5. (from 1B03)

$F^{m,n} = \{\text{all } m \times n \text{ matrices with entries in } F\} \; (= M_{m,n}(F))$

6. (from 1B03)

$P_d(F) = \{a_0 + a_1 x + a_2 x^2 + \dots + a_n x^d \mid a_i \in F\}$

all polynomials with deg $\leq d$ with coeff in F

Properties via Axioms

Can derive results about all v.s. from axioms

Thm The additive identity $0 \in V$ is unique

Proof: Suppose $0, 0'$ are additive identies of V. Then

$$0 = 0 + 0' = 0' + 0 = 0'$$

since $0'$ additive identity

addition commutes

0 is an additive identity.

So $0 = 0'$ ∎

Thm The additive inverse of each $v \in V$ (i.e. $v+w=0$) is unique.

Proof Let $v \in V$ and suppose $w, w' \in V$ both satisfy $v+w=0$ and $v+w'=0$. Then

$$w = w+0 = w+(v+w') = (w+v)+w' = (v+w)+w'$$
$$= 0+w' = w' \qquad \boxtimes$$

Notation: • If $v \in V$, write $-v$ to mean the unique inverse of v, i.e. $v+(-v)=0$

• $v-w$ is short for $v+(-w)$

scalar $0 \in F$

Thm For any $v \in V$, $0 \cdot v = 0$ ← zero vector of V

Proof $\underline{0} \cdot v = \underline{(0+0)} v = \underline{0v + 0v}$ by axioms.

Add $-(0 \cdot v)$ to both sides:

$$0 \cdot v + (-0 \cdot v) = (0v + 0 \cdot v) + (-0 \cdot v)$$

So $$0 = (0v + (0 \cdot v + (-0 \cdot v)) = 0 \cdot v. \qquad \boxtimes$$

zero vector

Thm For all $\lambda \in F$, $\lambda \cdot 0 = 0$

Proof $\lambda \cdot 0 = \lambda(0+0) = \lambda 0 + \lambda \cdot 0$.

Now add $(-\lambda 0)$ to both sides

$$\lambda \cdot 0 + (-\lambda 0) = (\lambda 0 + \lambda 0) + (-\lambda 0).$$

This simplifies to $0 = \lambda 0$. ☒

Thm For all $v \in V$, $(-1) \cdot v = -v$

scalar -1 additive inverse of v.

Proof: We have

$$0 = 0 \cdot v = (1 + (-1))v = v + (-1)v$$

proved above

So $(-1)v$ is an additive inverse of v. But since inverses are unique, this means $(-1)v = -v$

Problem Suppose $0 \neq c \in F$ and $c \cdot v = 0$. Then $v = 0$.

Sol^n Since $c \neq 0$, $c^{-1} = 1/c \in F$. So

$$1/c\,(c \cdot v) = 1/c \cdot 0 = 0.$$

Now $0 = 1/c \cdot 0 = 1/c\,(c \cdot v) = (1/c \cdot c) \cdot v = 1 \cdot v = v$. ☒

Key ideas:

* def^n of vector spaces
* examples
* basic properties