Lecture 9   3.A Vector Spaces of linear maps

Theme in mathematics: introduce objects and maps b/w objects

In Linear Algebra: objects = vector spaces
maps = linear maps

Today: introduce linear maps

Defⁿ A linear map (or linear transformation) from a vector space V to v.s. W is a function $T: V \longrightarrow W$ such that

- $T(u+v) = T(u) + T(v)$ for all $u, v \in V$ (additivity property)
- $T(\lambda u) = \lambda T(u)$ for all $u \in V$, $\lambda \in F$ (homogeneity property).

Notation write $T(u)$ as Tu

Defⁿ $\mathcal{L}(V, W) = \{$ all linear maps from V to $W \}$

Ex 1 For all V, W, the zero map $0 \in \mathcal{L}(V, W)$ is the function $0: V \longrightarrow W$ defined by
$$0(v) = 0$$
zero vector in W

Fact 0 is a linear map

$$\mathcal{L}(V,V)$$
$$\shortparallel$$

<u>Ex 2</u> If $V=W$, the <u>identity</u> map $I \in \mathcal{L}(V,W)$ is the function

$$I : V \to V \quad \text{defined by}$$
$$I(v) = v$$

<u>Fact</u>: I is a linear map.

<u>Ex 3</u> Let $V=W=\mathcal{P}(\mathbb{R})$ ← all poly w/ coeff in $\mathbb{R}$

Then $D \in \mathcal{L}(\mathcal{P}(\mathbb{R}), \mathcal{P}(\mathbb{R}))$ defined by

$$D(p) = p' \quad \text{← take } p \text{ to its derivative}$$

This is a linear map via prop of calculus! Since

$$D(p+q) = (p+q)' = p' + q' = D(p) + D(q)$$

$$D(\lambda p) = (\lambda p)' = \lambda (p') = \lambda D(p)$$

Ex 4 Fix $a \leq b$ in $\mathbb{R}$. Let $T \in \mathcal{L}(\mathcal{P}(\mathbb{R}); \mathbb{R})$ defined by

$$T(p) = \int_a^b p(x)\,dx \leftarrow \text{an element of } \mathbb{R}$$

Ex $a=0, b=1$ $\quad T(x^2+x) = \int_0^1 x^2+x\,dx = \frac{x^3}{3} + \frac{x^2}{2}\Big|_0^1 = \frac{1}{3} + \frac{1}{2}$

This is a linear map by calculus since

$$T(p+q) = \int_a^b (p(x)+q(x))\,dx = \int_a^b p(x)\,dx + \int_a^b q(x)\,dx$$

$$= T(p) + T(q)$$

$$T(\lambda p) = \int_a^b (\lambda p(x))\,dx = \lambda \int_a^b p(x)\,dx = \lambda T(p)$$

So T is a linear map.

Thm $T \in \mathcal{L}(F^n, F^m)$ if and only if exists constants A_{ij} with $1 \le i \le m$ and $1 \le j \le n$ such that

$$T(x_1, \ldots, x_n) = (A_{11}x_1 + \cdots + A_{1n}x_n, \ldots, A_{m1}x_1 + \cdots + A_{mn}x_n)$$

Not new! In 1303, $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if

Same

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ A_{21} & & A_{2n} \\ \vdots & & \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_{11}x_1 + \cdots + A_{1n}x_n \\ \vdots \\ A_{m1}x_1 + \cdots + A_{mn}x_n \end{bmatrix}$$

Linear maps determined by where a basis is sent.

Thm Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T: V \to W$ such that

$$T(v_i) = w_i$$

(Idea of proof) Since $v_1, \ldots, v_n$ is a basis, can write $v \in V$ as

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n \quad \text{with } c_i \in F$$

Define $T: V \to W$ by
$$T(v) = T(c_1v_1 + \cdots + c_nv_n) = c_1w_1 + c_2w_2 + \cdots + c_nw_n$$

Now check (A) T has the additivity prop
(B) T has the homogeneity prop } T is a linear map

(C) If T' is any linear map with $T'(v_i) = w_i$,
Then $T'(v) = T(v)$ for all v. $\blacksquare$

Structure of $\mathcal{L}(V,W)$

We can define operations on elements of $\mathcal{L}(V,W)$.

Let $S, T \in \mathcal{L}(V,W)$ and $\lambda \in F$. Define

(A) $(S+T): V \to W$ by
$$(S+T)(v) = S(v) + T(v)$$

(B) $(\lambda S): V \to W$
$$(\lambda S)(v) = \lambda(S(v))$$

Call $(S+T)$ the <u>sum</u> λS the <u>scalar product</u>

Fact $S+T, \lambda S$ linear maps, i.e. $S+T, \lambda S \in \mathcal{L}(V,W)$

So $\mathcal{L}(V,W)$ is a set with a sum and scalar mult!

Thm $\mathcal{L}(V,W)$ is a vector space with these operations

Proof (Some properties)

- The zero vector is the zero map $0 : V \to W$ given by $0(v) = 0$

- Let $T \in \mathcal{L}(V,W)$. Then $(-T) : V \to W$ is the function $(-T)(v) = -(T(v))$. This is a linear map, so $(-T) \in \mathcal{L}(V,W)$

But then $(T+(-T))(v) = T(v) + (-T(v)) = 0$ for all $v \in V$.

So $(T+(-T)) = 0$ ← same as the zero map.

- (convince yourself that other props hold) ☒

$\boxed{\text{From two v.s } V, W, \text{ can create new v.s } \mathcal{L}(V,W).}$

Reframing 1B03

If $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$, then

$$T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \longleftrightarrow T(\vec{x}) = A\vec{x} \quad A \text{ } m\times n \text{ matrix}$$

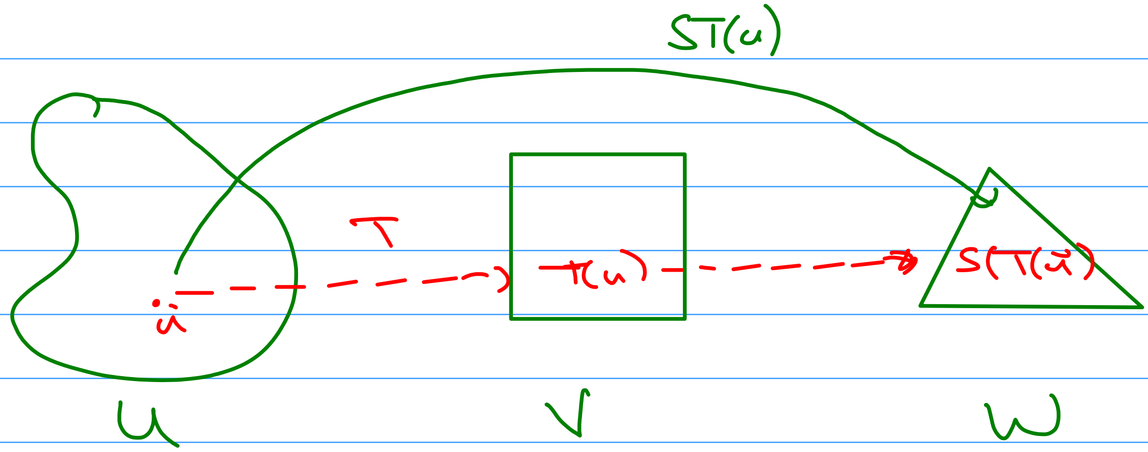
So $$\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \longleftrightarrow F^{m\times n}$$ ← $m \times n$ matrices

$$T \longleftrightarrow A$$

Thus, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ is the "same" as the v.s of $m\times n$ matrices

<u>Def</u>: If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, the <u>product</u> $ST \in \mathcal{L}(U,W)$ is the map

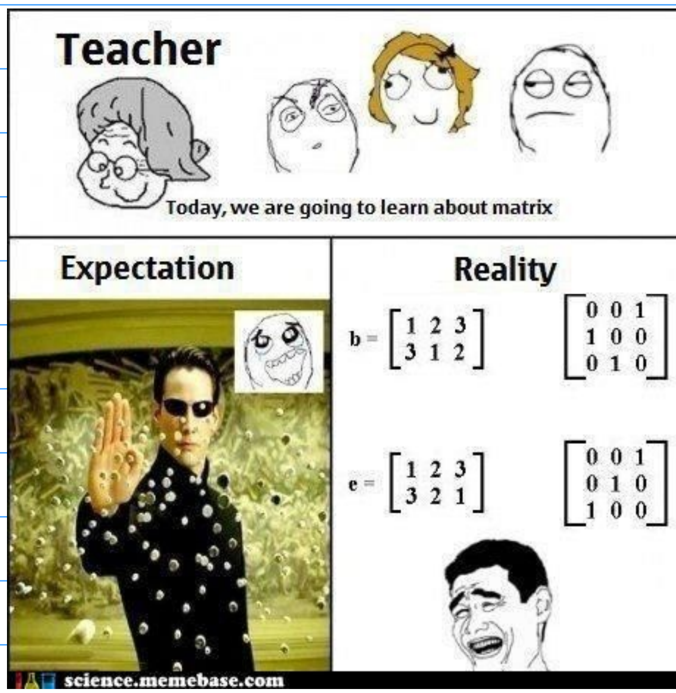
$$ST(u) = S(T(u))$$

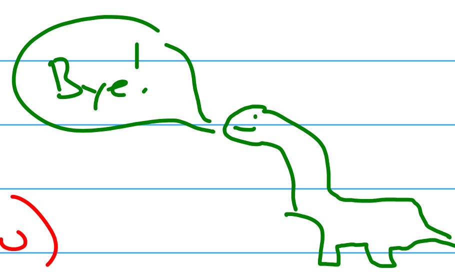


Product = composition

(Properties of Products)

1. $T_1(T_2T_3) = (T_1T_2)T_3$
2. $I_V T = T I_W = T$ where $I_V \in \mathcal{L}(V,V)$, $I_W \in \mathcal{L}(W,W)$
   identities
3. $(S_1+S_2)T = S_1T + S_2T$
   $S(T_1+T_2) = ST_1 + ST_2$





Key ideas:
- linear maps
- the vector space $\mathcal{L}(V,W)$
- products of linear maps