$\boxed{\text{Lecture 4}}$ 1.C Subspaces II

Writing Proofs

So far: introduced vector spaces, subspaces, and sums

Theme in course: decompose vector spaces into subspaces

Last time: $U_1 + U_2 + \cdots + U_m = \{u_1 + \cdots + u_m \mid u_i \in U_i\}$

Ex (Sums are not unique)

$V = F^3$ $\quad U_1 = \{(x,y,0) \mid x,y \in F\}$

$\quad\quad\quad U_2 = \{(0,y,z) \mid y,z \in F\}$

Have $U_1 + U_2 = F^3$ (check!)

Also
$$\left.\begin{aligned}(0,0,0) &= (0,0,0) + (0,0,0) \in U_1 + U_2 \\ &= (0,1,0) + (0,-1,0) \in U_1 + U_2\end{aligned}\right\} \text{two ways to write } (0,0,0) \text{ in } U_1 + U_2$$

Def^{n} Suppose $U_1, \ldots, U_m$ subspaces of V. Then $U_1 + \cdots + U_m$ is a direct sum if each element $U_1 + \cdots + U_m$ can be written uniquely as a sum
$$u_1 + u_2 + \cdots + u_m \quad \text{with } u_i \in U_i$$

In this case, write $U_1 \oplus U_2 \oplus \cdots \oplus U_m$

Ex Above example is not a direct sum since we can express $(0,0,0)$ in two different ways.

Ex $V = F^2$ $\quad U = \{(x,0) \mid x \in F\}$ and $W = \{(0,y) \mid y \in F\}$

$U+W$ is a direct sum since

$(a,b) \in U+W \Rightarrow \quad (a,b) = \underbrace{(x,0)}_{\text{in } U} + \underbrace{(0,y)}_{\text{in } W}$ for some $x \in F$, $y \in F$

But this can only happen if $a = x$ and $b = y$. So

$$(a,b) = (a,0) + (0,b) \in U+W$$

Note: $F^2 = U \oplus W$. Clear that $U \oplus W \subseteq F^2$. And if $(a,b) \in F^2$ then $(a,0) \in U$ and $(0,b) \in W$, so $(a,b) = (a,0) + (0,b) \in U \oplus W$.

Ex $V = F^n$ and let $U_j = \{(0,\ldots,\underbrace{u_j}_{j\text{th spot}},\ldots,0) \mid u_j \in F\}$. Then

$$F^n = U_1 \oplus \cdots \oplus U_n$$

Thm $U_1 + \cdots + U_m$ is a direct sum if and only if the only way to write 0 as a sum
$$0 = u_1 + \cdots + u_m$$
is when $u_1 = \cdots = u_m = 0$

need to only show that you write 0 uniquely.

Proof ($\Rightarrow$) If $U_1 + \cdots + U_m$ is a direct sum, then
$$0 = 0 + 0 + \cdots + 0 \quad \text{with } 0 \in U_i$$
and this is the only way to express it

($\Leftarrow$) Let $w \in U_1 + \cdots + U_m$. And now suppose

$w = a_1 + \cdots + a_m$ and $w = b_1 + b_2 + \cdots + b_m$ with $a_i, b_i \in U_i$.

Then

$$0 = w - w = (a_1 + \cdots + a_m) - (b_1 + \cdots + b_m)$$
$$= (a_1 - b_1) + (a_2 - b_2) + \cdots + (a_m - b_m)$$

Since each $a_i - b_i \in U_i$, the condition on 0 (i.e. our hypotheses) implies each $a_i - b_i = 0 \Leftrightarrow a_i = b_i$.

So, there is a unique way to express w so

$U_1 + \cdots + U_m$ is a direct sum $\blacksquare$

For two subspaces, alternate way to check

Thm Let U, W be subspaces of V. Then $U \oplus W$ is a direct sum if and only if $U \cap W = \{0\}$

<u>Proof</u> ($\Rightarrow$) Let $w \in U \cap W$. Then

$$0 = w + (-w) \quad \text{with } w \in U, -w \in W$$

Since $U \oplus W$ is a direct sum, this implies $w = (-w) = 0$.
So $U \cap W = \{0\}$

($\Leftarrow$) Let $0 = u + w \in U + W$. Then $u = -w$.
Since $u \in U$, $-u = -(-w) = w \in U$. So
$w \in U \cap W = \{0\}$. So $w = 0$, and thus
$u = 0$. So $0 = 0 + 0$ is the only way
to express 0, Thus $U \oplus W$ is a direct sum ☒

Writing Proofs (some tips)

work "forwards and backwards"

Most Mathematical statements have form
"If P, then Q"

From P, argue "forwards" ← what statements must be true

From Q, argue "backwards" ← what statements must be true to force Q to be true

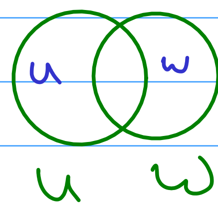
Try to meet in the middle

$$\underset{\text{hypotheses}}{P} \Rightarrow P_1 \Rightarrow P_2 \Rightarrow \cdots \;\; ?? \;\; \Rightarrow Q_2 \Rightarrow Q_1 \Rightarrow \underset{\text{conclusion}}{Q}$$

<u>Ex</u> If $\underbrace{U \not\subseteq W \text{ and } W \not\subseteq U}_{P}$, then $U \cup W$ is not a $\underbrace{\text{subspace}}_{Q}$.

both subspaces

(SCRAP)



P

$U \not\subseteq W \Rightarrow$ there is a $u \in U \setminus W$ ← in U but not W

$W \not\subseteq U \Rightarrow$ there is a $w \in W \setminus U$ ← in W but not U

$\Downarrow$

We have $u, w \in U \cup W$. Is $u + w \in U \cup W$?

<u>No</u> If $u+w \in U \cup W$, then $u+w \in U$ or $u+w \in W$.

If $u+w \in U$, then since $-u \in U$, $(u+w)+(-u) = w \in U$!!!

If $u+w \in W$, then since $-w \in W$, $(u+w)+(-w) = u \in W$!!!

$\Downarrow$

won't work		won't work since $U \cup W$ does satisfy this p/q
fails $0 \in U \cup W$	fails closed under addition	fails closed scalar mult
$\Downarrow$	$\Downarrow$	$\Downarrow$

$U \cup W$ fails one of 3 conditions of a subspace

$\Downarrow$

Q

$U \cup W$ is not a subspace

use complete English sentences.

Write "polished" proof:

see handout on webpage for proof writing tips!

<u>Proof</u> Since $U \not\subseteq W$ and $W \not\subseteq U$, there is $u \in U \setminus W$ and $w \in W \setminus U$.

Then $u, w \in U \cup W$, but $u+w \notin U \cup W$.

To see why, if $u+w \in U \cup W$, then $u+w \in U$ or $u+w \in W$. If $u+w \in U$, then since $u \in U$,

$(u+w)+(-u) = w \in U$, which is false! This shows $u+w \notin U$.

The same argument shows $u+w \notin W$. Since $U \cup W$ is not closed under addition, it is not a subspace $\boxtimes$

<u>Recall</u> Contrapositive:

"If P, then Q" $\iff$ "If not Q, then not P"

Contrapositive in our case

"If $U \cup W$ is a subspace, then $U \subseteq W$ or $W \subseteq U$"

Key ideas: * direct sums
* how to check if you have a direct sum
* proof writing