Lecture 21 8.A Generalized Eigenvectors / Nilpotent Operators

Last time • If $T \in \mathcal{L}(V)$ with $n = \dim V$ and λ an eigenvalue, then generalized eigenspace of λ is

$$G(\lambda, T) = \text{Null}((T-\lambda I)^n)$$

• $v \in G(\lambda, T) \rightarrow v$ is generalized eigenvector

Thm Let $T \in \mathcal{L}(V)$ and $\lambda_1, .., \lambda_m$ distinct eigenvalues of T. If $v_1, .., v_m$ are corresponding generalized eigenvectors, then $v_1, .., v_m$ are linearly independent

Proof Suppose $0 = a_1 v_1 + \cdots + a_m v_m$ (*)

Let k be the largest integer such that $(T-\lambda_1 I)^k v_1 \neq 0$ (k can be zero)

Let $w = (T-\lambda_1 I)^k v_1$. So

$$(T-\lambda_1 I) w = (T-\lambda_1 I)(T-\lambda_1 I)^k v_1$$
$$= (T-\lambda_1 I)^{k+1} v_1 = 0$$

$\Rightarrow Tw = \lambda_1 w$

But then, for any $\lambda \in F$

$$(T-\lambda I)w = Tw - \lambda I w = \lambda_1 w - \lambda w = (\lambda_1 - \lambda) w$$

Hence

$$(T-\lambda I)^n w = (\lambda_1 - \lambda)^n w \quad \text{for all } \lambda \in F \text{ and } n = \dim V$$

Apply operator $(T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n$ to $\circledast$

We get

$$\begin{aligned} 0 = \; & a_1 (T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n v_1 \\ & + a_2 (T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n v_2 \\ & + \cdots + a_m (T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n v_m \end{aligned}$$

Operators commute. Note that

$$(T-\lambda_j I)^n v_j = 0 \qquad \text{since } v_j \in G(\lambda_j, T).$$

So all terms with $v_2, \ldots, v_m$ are killed and we are left with

$$0 = a_1 (T-\lambda_1 I)^k (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n v_1$$

$$= a_1 (T-\lambda_2 I)^n \cdots (T-\lambda_m I)^n \underbrace{(T-\lambda_1 I)^k v_1}_{=w}$$

$$= a_1 (T-\lambda_2 I)^n \cdots (T-\lambda_m I^n) w$$

$$= a_1 (\lambda_1-\lambda_2)^n (\lambda_1-\lambda_3)^n \cdots (\lambda_1-\lambda_m)^n w$$

Since $w \neq 0$ and $\lambda_1 \neq \lambda_i$ for $i \neq 1$, have $a_1 = 0$.

Same argument shows $a_2 = a_3 = \cdots = a_m = 0$

So $v_1, \ldots, v_n$ are linearly independent. $\square$

Nilpotent Operators

<u>Def</u>

Def$^{\underline{n}}$ $N \in \mathcal{L}(V)$ is nilpotent if $N^\ell = 0$ for some ℓ

Ex If $V = \mathcal{P}_2(\mathbb{R})$, the differential operator D is nilpotent. In fact $D^3 = 0$

I.e. if $p = a_0 + a_1 x + a_2 x^2 \in V$, then

$$Dp = a_1 + 2a_2 x$$
$$D^2 p = D(a_1 + 2a_2 x) = 2a_2$$
$$D^3 p = D(2a_2) = 0$$

Thm If N is nilpotent, then $N^{\dim V} = 0$

Proof $N^{\ell} = 0 \iff \operatorname{Null}(N^{\ell}) = V$

$$\Rightarrow \quad G(0,N) = V = \operatorname{Null}(N^{\dim V}) = V \qquad \boxtimes$$

only one direction

Thm Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then there is a basis of V such that

$$\mathcal{M}(N) = \begin{bmatrix} 0 & & * \\ & 0 & \\ & & \ddots & \\ 0 & & & 0 \end{bmatrix}$$

Proof We have

N is nilpotent

$$\{0\} \subseteq \operatorname{Null}(N) \subseteq \operatorname{Null}(N^2) \subseteq \cdots \subseteq \operatorname{Null}(N^{\ell}) = V$$

Let $u_{11}, u_{12}, \ldots, u_{1n_1}$ be a basis of $\operatorname{Null}(N)$

Extend to a basis of $\operatorname{Null}(N^2)$, i.e.

$$\underbrace{\underbrace{u_{11}, \ldots, u_{1n_1}}_{\text{basis of } \operatorname{Null}(N)} \; u_{21}, \ldots, u_{2n_2}}_{\text{basis of } \operatorname{Null}(N^2)}$$

Keep repeating

$$\underbrace{\underbrace{\underbrace{u_{11}, \dots, u_{1n_1}}_{\text{basis for Null}(N)}, u_{21}, \dots, u_{2n_2}}_{\text{basis for Null}(N^2)}, \dots \to u_{\ell 1}, \dots, u_{\ell n_\ell}}_{\text{basis for } V}$$

This gives a basis of V since $\text{Null}(N^\ell) = V$

This is the desired basis since

$$N u_{jk} \in \text{Null}(N^{j-1})$$

$$\Rightarrow N u_{jk} = \underbrace{c_{11} u_{11} + \dots + c_{1n_1} u_{1n_1} + \dots + c_{j-1\,1} u_{j-1\,1} + \dots + c_{j-1\,n_{j-1}} u_{j-1\,n_{j-1}}}_{\text{in terms of the basis for Null}(N), \dots, \text{Null}(N^{j-1})}$$

So all nonzero elements of $\mathcal{M}(N)$ are above the main diagonal in $\mathcal{M}(N)$

$$\mathcal{M}(N) = \begin{array}{c} \\ \left[\begin{array}{ccc} \ddots & c_{11} & \\ & \vdots & \\ & c_{j-1\,n_{j-1}} & \\ & 0 & \\ & 0 & \\ & \vdots & \\ & 0 & \ddots \end{array}\right] \end{array}
\begin{array}{l} u_{11} \\ \vdots \\ u_{j-1\,n_{j-1}} \\ u_{jk} \\ \\ \\ \\ \end{array}$$

with column u_{jk}.

Example The map $T \in \mathcal{L}(\mathbb{R}^3)$ given by

$T(x,y,z) = (2x+2y-2z,\ 5x+y-3z,\ x+5y-3z)$ is nilpotent

To see this, w.r.t. standard basis

$$M(T) = \begin{bmatrix} 2 & 2 & -2 \\ 5 & 1 & -3 \\ 1 & 5 & -3 \end{bmatrix}$$

So $M(T^3) = M(T)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow T^3 = 0$

Find basis so $M(T)$ has form of previous thm.

Since $M(T) \sim \begin{bmatrix} 1 & 5 & -3 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$, $\text{Null}(T) = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\}$

$M(T^2) = M(T)^2 = \begin{bmatrix} 12 & -4 & -4 \\ 12 & -4 & -4 \\ 24 & -8 & -8 \end{bmatrix} \sim \begin{bmatrix} 3 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

From IBO3

$$\mathcal{M}(T^2) = \mathcal{M}(T)^2 = \begin{bmatrix} 12 & -4 & -4 \\ 12 & -4 & -4 \\ 24 & -8 & -8 \end{bmatrix}$$

Via 1B03, $\text{Null}(T^2) = \text{span}\left\{ \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

However, want to extend basis of $\text{Null}(T)$ to $\text{Null}(T^2)$

$$\text{Null}(T^2) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 0 \\ 1 \end{bmatrix} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Now $\text{Null}(T^3) = \mathbb{R}^3$. So extend basis of $\text{Null}(T^2)$ to $\mathbb{R}^3$

$$\mathbb{R}^3 = \text{Null}(T^3) = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Null(T)

Null(T²)

Claim $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is the desired basis of V

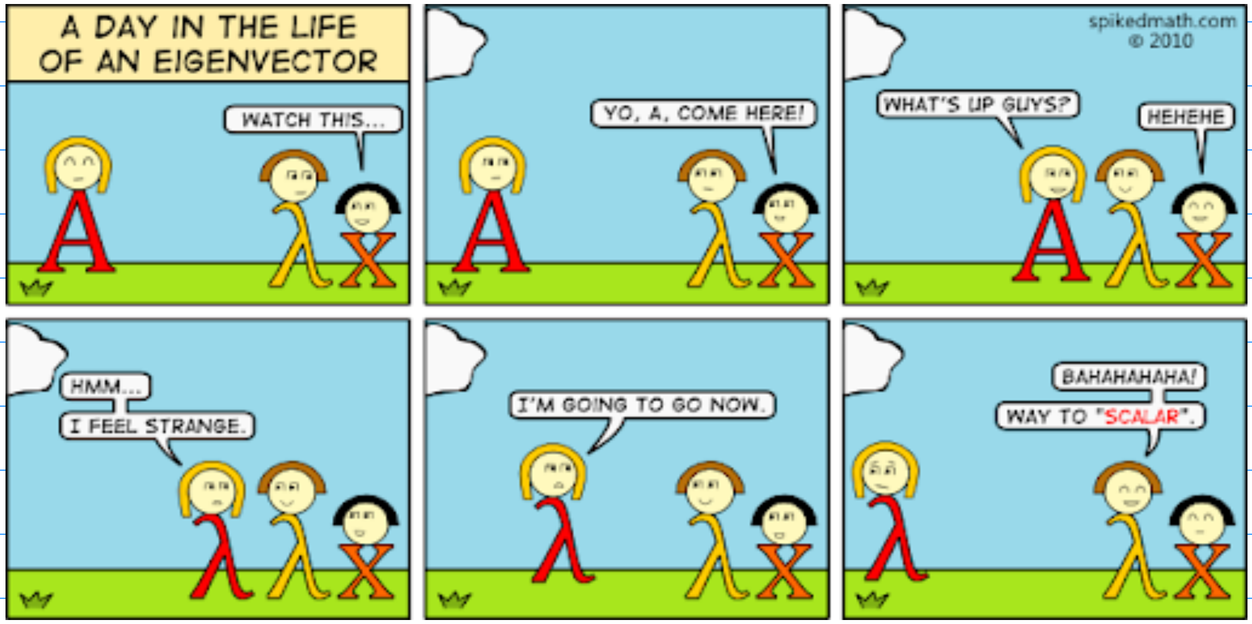
$$T\left(\begin{bmatrix}1\\1\\2\end{bmatrix}\right)=\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}1\\1\\2\end{bmatrix}+0\begin{bmatrix}1/3\\1\\0\end{bmatrix}+0\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}1/3\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2&2&-2\\5&1&-3\\1&5&-3\end{bmatrix}\begin{bmatrix}1/3\\1\\0\end{bmatrix}=\begin{bmatrix}8/3\\8/3\\16/3\end{bmatrix}=8/3\begin{bmatrix}1\\1\\2\end{bmatrix}+0\begin{bmatrix}1/3\\1\\0\end{bmatrix}+0\begin{bmatrix}1\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2&2&-2\\5&1&-3\\1&5&-3\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}2\\5\\1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1\\1\\2\end{bmatrix}+\frac{9}{2}\begin{bmatrix}1/3\\1\\0\end{bmatrix}+0\begin{bmatrix}1\\0\\0\end{bmatrix}$$

So
$$\mathcal{M}(T)=\begin{array}{cc} \begin{array}{ccc} u_1 & u_2 & u_3 \end{array} & \\ \begin{bmatrix}0&8/3&1/2\\0&0&9/2\\0&0&0\end{bmatrix} & \begin{array}{c}u_1\\u_2\\u_3\end{array}\end{array}$$

Key ideas: generalized eigenvectors & linear independence
- nilpotent operators
- basis and $\mathcal{M}(N)$ for nilpotent operators.


A DAY IN THE LIFE OF AN EIGENVECTOR
WATCH THIS...
YO, A, COME HERE!
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WHAT'S UP GUYS?
HEHEHE
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I FEEL STRANGE.
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