$\boxed{\text{Lecture 35}}$ 7.D Polar Decomposition

Last time: $T \in \mathcal{L}(V)$ is positive if T is self-adjoint and $\langle Tv, v\rangle \geq 0$

Fact $T \in \mathcal{L}(V)$ positive $\iff$ exists $R \in \mathcal{L}(V)$ such that $R^*R = T$

Cor For any $T \in \mathcal{L}(V)$, T^*T is a positive operator.

Proof Let $S = T^*T$, so S is positive by Fact $\square$

$\boxed{\text{1B03/2LA3 P.O.V}}$ Given any matrix A, (over $\mathbb{R}$)

A^TA is symmetric and A^TA has nonnegative eigenvalues

Fact If $T \in \mathcal{L}(V)$ is positive, then exists a unique positive $R \in \mathcal{L}(V)$ such that $R^2 = T$ ← R is called the square root of T

Def$^{\underline{n}}$ If $T \in \mathcal{L}(V)$, let
$\sqrt{T^*T}$ denotes the "unique" positive square root of T^*T

a positive op

Cor For any $T \in \mathcal{L}(V)$, $\sqrt{T^*T}$ exists

(Polar Decomposition)
For any $T \in \mathcal{L}(V)$, there exists an isometry $S \in \mathcal{L}(V)$ such that

$$T = \underbrace{S}_{\text{isometry}} \underbrace{\sqrt{T^*T}}_{\text{positive operator}}$$

Analogy with $\mathbb{C}$

Very roughly, $\mathbb{C}$ and $\mathcal{L}(V)$ have similar properties

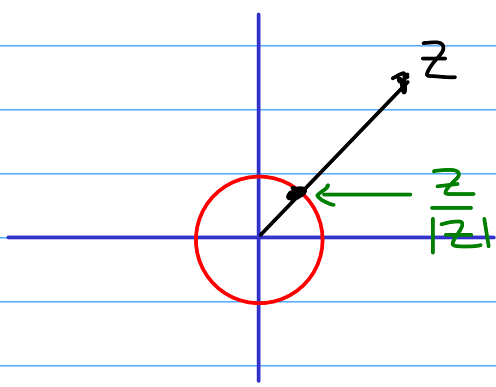
$\mathbb{C}$	$\mathcal{L}(V)$
complex number $z = a+bi$	operator T
conjugate $\bar{z} = a-bi$	adjoint T^*
$z = \bar{z} \iff z$ is real	self-adjoint $T^* = T$
$\lvert z\rvert = \sqrt{a^2+b^2} = \sqrt{z\cdot\bar{z}}$	
$\lvert z\rvert = 1 \iff z\cdot\bar{z} = 1$	isometry $T^*T = I$
$z = \left(\frac{z}{\lvert z\rvert}\right)\lvert z\rvert = \left(\frac{z}{\lvert z\rvert}\right)\sqrt{\bar{z}z}$	Polar Dec Thm $T = S\sqrt{T^*T}$

$\frac{z}{|z|}$: element on unit circle in complex plane

$\sqrt{\bar{z}z}$: scaling

S: rotation/scaler that preserves length

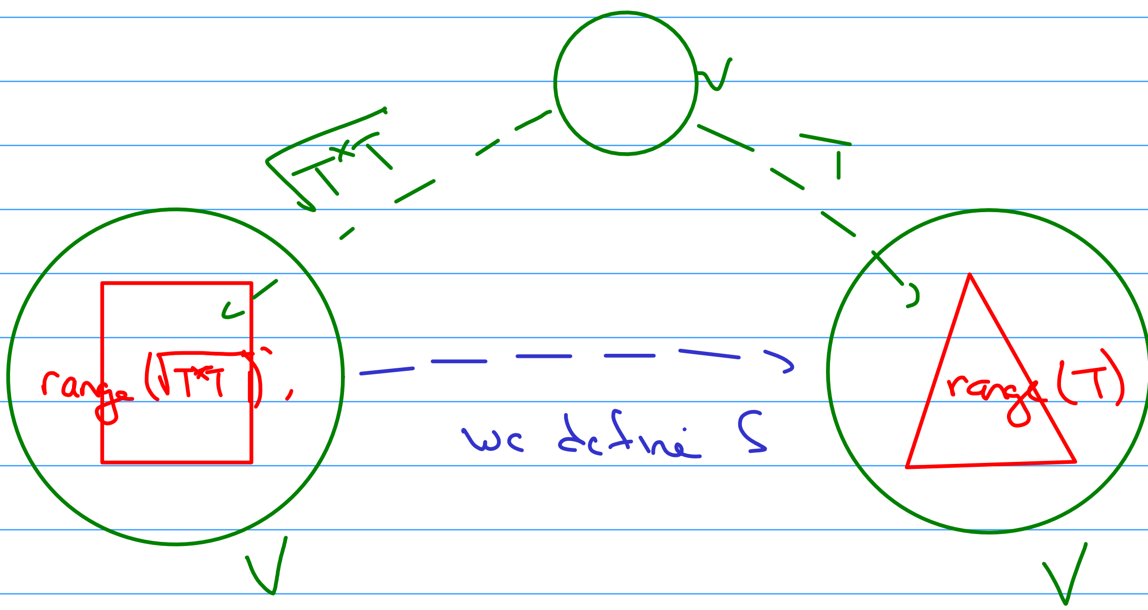
$\sqrt{T^*T}$: like scaling (a positive operator)



(Sketch of Proof)

Since $T, \sqrt{T^*T} \in \mathcal{L}(V)$

$\text{range}(\sqrt{T^*T}) \subseteq V$ $\qquad$ $\text{range}(T) \subseteq V$



Define a map $S_1 : \text{range}(\sqrt{T^*T}) \longrightarrow \text{range}(T)$ by

$$(\sqrt{T^*T})v \longmapsto Tv$$

(need to check this is well-defined)

Also define a map $S_2 : (\text{range}(\sqrt{T^*T}))^{\perp} \longrightarrow (\text{range}\, T)^{\perp}$
(Details skipped!)

orthogonal complement
= all elements orthogonal to the given space

<u>Fact</u> $V = \left(\operatorname{range} \sqrt{T^*T}\right) \oplus \left(\operatorname{range} \sqrt{T^*T}\right)^{\perp}$

So $v \in V \implies v = u + w$ with $u \in \operatorname{range}\left(\sqrt{T^*T}\right)$
and $w \in \left(\operatorname{range}\left(\sqrt{T^*T}\right)\right)^{\perp}$

We define $S : V \longrightarrow V$
$$v = u + w \longmapsto S_1 u + S_2 w$$

So $S\left(\sqrt{T^*T} v\right) = S_1\left(\sqrt{T^*T} v\right) + S_2 \cdot 0$

in range (T^*T)

$$= Tv$$

$\implies \left(S\sqrt{T^*T}\right) = T$

Many details to be checked:

- all maps are linear maps
- S is an isometry

Matrix Interpretation

In terms of matrices, the Polar Decomposition theorem tells us:

If A is an $n \times n$ matrix, can find an orthogonal matrix U and a positive semidefinite matrix R (← R has nonnegative eigenvalues)

such that $R^2 = A^TA$ and $A = UR$

matrix analog of isometry   matrix analog of sq-root

If A is an invertible matrix, "easy" to find this decomposition (over $\mathbb{R}$)

1. Compute A^TA and find the diagonalization
$$A^TA = PDP^{-1}$$

2. Let $R = P\begin{bmatrix} \sqrt{d_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{d_n} \end{bmatrix}P^{-1}$ where $D = \begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{bmatrix}$

3. Since A is invertible, $U = AR^{-1}$

Ex $\quad A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} \leftarrow$ find the polar decomposition

① $A^TA = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$

Diagonalize $A^TA = \underbrace{\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}}_{P}\underbrace{\begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}^{-1}}_{P^{-1}}$

② Let $\quad R = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 2.8 & 0.4 \\ 0.4 & 2.2 \end{bmatrix}$

③ $U = AR^{-1} = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 2.8 & 0.4 \\ 0.4 & 2.2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}$

Thus $\quad \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & 0.8 \end{bmatrix}\begin{bmatrix} 2.8 & 0.4 \\ 0.4 & 2.2 \end{bmatrix}$

Key Ideas: * Polar Decomposition