$\boxed{\text{Lecture 22}}$ 8.B Decomposition of an Operator

Recall Given $T \in \mathcal{L}(V)$ want to decompose V such that
$$V = U_1 \oplus \cdots \oplus U_m$$
where T is invariant on U_i

If V has a basis of eigenvectors, say $v_1, \ldots, v_n$, can do this:
$$V = \operatorname{span}(v_1) \oplus \operatorname{span}(v_2) \oplus \cdots \oplus \operatorname{span}(v_n)$$

However T may not have eigenvectors (e.g. if T is not diagonalizable)

Over $\mathbb{C}$, we can "fix" this problem with generalized eigenvectors

Lemma If $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(F)$, then T invariant on $\operatorname{null}(p(T))$ and $\operatorname{range}(p(T))$

I.e. if $v \in \operatorname{null}(p(T))$, then $Tv \in \operatorname{null}(p(T))$
if $v \in \operatorname{range}(p(T))$, then $Tv \in \operatorname{range}(p(T))$

Thm Let V be a vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be distinct eigenvalues. Then

1. $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$
2. T invariant on $G(\lambda_i, T)$
3. $(T - \lambda_i I)\big|_{G(\lambda_i, T)}$ is nilpotent

Proof 2. Since $G(\lambda_i, T) = \text{Null}((T - \lambda_i I)^n)$
$= \text{Null}(p(T))$ when $p(z) = (z - \lambda_i)^n$

By the lemma, T is invariant on $G(\lambda_i, T)$

3. Let $v \in G(\lambda_i, T)$. So $(T - \lambda_i I)^n v = 0$

So $(T - \lambda_i I)$ is nilpotent on $G(\lambda_i, T)$.

1. (Main idea only)

Do induction on $\dim V = n$. If $\dim V = 1$, then T has an eigenvalue (since $F = \mathbb{C}$) and
$V = G(\lambda, T)$

Suppose $\dim V = n \geq 1$. Since $F = \mathbb{C}$, T has an eigenvalue, say λ_1

Then

$$V = \mathrm{Null}((T-\lambda_1 I)^n) \oplus \mathrm{Range}((T-\lambda_1 I)^n)$$
$$= G(\underline{\lambda_1}, T) \oplus U$$

Now $\dim U < n$ (since $\dim G(\lambda_1, T) \geq 1$)
Also, T invariant on U by lemma since

$$U = \mathrm{range}(p(T)) \quad \text{where} \quad p(z) = (z-\lambda_1)^n$$

Finally, eigenvalues of $T|_U$ are $\lambda_2, \dots, \lambda_n$ (to check)

So, apply induction to $T|_U \in \mathcal{L}(U)$, i.e.

$$U = G(\lambda_2, T|_U) \oplus \cdots \oplus G(\lambda_n, T|_U)$$

Need to check $G(\lambda_i, T) = G(\lambda_i, T|_U)$
for $i = 2, \dots, m$. $\Big\}$ details skipped

So $V = G(\lambda_1, T) \oplus U$
$$= G(\lambda_1, T) \oplus G(\lambda_2, T) \oplus \cdots \oplus G(\lambda_n, T)$$

$\square$

Cor Suppose $F = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors

Proof $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$. Combine the bases of the $G(\lambda_i, T)$'s to form a basis of V $\square$

Multiplicity

Def^n The multiplicity of λ is $\dim G(\lambda, T)$

Thm If $T \in \mathcal{L}(V)$ and $\lambda_1, .., \lambda_\ell$ distinct eigenvalues with multiplicities $m_1, .., m_\ell$, then

$$\dim V = m_1 + \cdots + m_\ell$$

Proof Follows from $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_\ell, T)$.

$\square$

Connection to 1B03

1B03   2R03

algebraic multiplicity of λ = mult of $\lambda = \dim G(\lambda, T)$
geometric multiplicity of $\lambda = \dim E(\lambda, T)$

Can now easily prove statements from 1B03

Thm For any eigenvalue λ,
$$1 \leq \text{geo. mul of } \lambda \leq \text{alg. mult of } \lambda$$

Proof Follows from fact
$$\{0\} \subsetneq E(\lambda, T) \subseteq G(\lambda, T) \qquad \square$$

Thm A diagonalizable $\iff$ geo. mult of λ_i = alg mul of λ_i
for all eigenvalues

Proof A diagonalizable $\iff$ linear map T associated to this matrix is diagonalizable
$\iff V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_n, T) = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_n, T)$
$\iff E(\lambda_i, T) = G(\lambda_i, T)$ for all i
$\iff$ geo mult of λ_i = alg mult of λ_i $\qquad \square$

Block Diagonal Matrices

Defn: A block diagonal matrix is an $n \times n$ matrix of form

$$\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & 0 & \ddots & \\ & & & A_\ell \end{bmatrix}$$

where each A_i is a square matrix on diagonal.

Ex

$$\begin{bmatrix} \boxed{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & \boxed{10} & \begin{matrix} 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} & \boxed{\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}} \end{bmatrix}$$

Thm Let $T \in \mathcal{L}(V)$ and $F = \mathbb{C}$. Let $\lambda_1, \ldots, \lambda_n$ be distinct eigenvalues with d_i = mult of λ_i. Then there is a basis of V such that

$$\mathcal{M}(T) = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_n \end{bmatrix} \quad \text{where } A_i = \underbrace{\begin{bmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}}_{d_i \times d_i}$$

Proof Recall $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_n, T)$.

For each λ_i, can write

$$T = (T - \lambda_i I) + \lambda_i I$$

Since $(T - \lambda_i I)\big|_{G(\lambda_i, T)}$ is nilpotent on $G(\lambda_i, T)$, there is a basis of $G(\lambda_i, T)$

such that

$$\mathcal{M}(T - \lambda_i I)\big|_{G(\lambda_i, T)} = \begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix} \leftarrow d_i \times d_i$$

Note that

$$\mathcal{M}(\lambda_i I|_{G(\lambda_i,T)}) = \begin{bmatrix} \lambda_i & & 0 \\ & \lambda_i & \\ & & \ddots \\ 0 & & & \lambda_i \end{bmatrix}$$ with respect to the same basis

So

$$\mathcal{M}(T|_{G(\lambda_i,T)}) = \mathcal{M}((T-\lambda_i I)|_{G(\lambda_i,T)}) + \mathcal{M}(\lambda_i I|_{G(\lambda_i,T)})$$

$$\begin{bmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}$$

Put the bases of the $G(\lambda_i,T)$'s together to get a basis of V. Since T invariant on $G(\lambda_i,T)$'s, the matrix of T is

$$\mathcal{M}(T) = \begin{bmatrix} \begin{bmatrix} \lambda_1 & & * \\ & \lambda_1 & \\ 0 & \ddots & \lambda_1 \end{bmatrix} & & 0 \\ & \begin{bmatrix} \lambda_2 & * \\ 0 & \lambda_2 \end{bmatrix} & \\ 0 & & \ddots \end{bmatrix}$$

Next lecture: A worked out example.

Key ideas:
- V has a basis of generalized eigenvectors (over $\mathbb{C}$)
- multiplicity
- with respect to basis of generalized eigenvectors $M(T)$ is block diagonal (over $\mathbb{C}$).