$\boxed{\text{Lecture 17}}$ 5.A Eigenvalues

5.B Polynomials applied to operators

Last time: • If $T \in \mathcal{L}(V)$, then λ is an <u>eigenvalue</u> if there exists some nonzero $v \in V$ such that $Tv = \lambda v$

• v is an <u>eigenvector</u> corresponding to λ if $v \neq 0$ and $Tv = \lambda v$

(Connection to 1B03) Suppose $T \in \mathcal{L}(\mathbb{R}^3)$ given by

$$T(x,y,z) = (3x+2y+4z,\ 4y+2z,\ 8z)$$

Let $\mathcal{M}(T)$ be the matrix of T (with respect to standard basis e_1, e_2, e_3 of $\mathbb{R}^3$)

$$\mathcal{M}(T) = \begin{bmatrix} 3 & 2 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 8 \end{bmatrix}$$

From 1B03, eigenvalues of $\mathcal{M}(T) = 3, 4, 8$ ← diagonal entries of an upper triangular matrix

These are the eigenvalues of T.

We check $\lambda = 3$

$$T(x,y,z) = (3x+2y+4z,\ 4y+2z,\ 8z) = 3(x,y,z)$$

$$\iff \begin{aligned} 3x &= 3x+2y+4z \\ 3y &= 4y+2z \\ 3z &= 8z \end{aligned}$$

Last eq implies $z=0$

So 2^{nd} eq implies $3y=4y$, ie $y=0$

So 1^{st} eq becomes $3x = 3x+0+0 \iff 3x=3x \iff$ for all x

So, for any $v=(x,0,0)$ with $x \in \mathbb{R}$,

$T(x,0,0) = (3x,0,0) = 3(x,0,0)$

If $x \neq 0$, then $(x,0,0)$ is an eigenvector with eigenvalue 3

"Bigger Picture" If $T \in \mathcal{L}(\mathbb{R}^n)$,

eigenvalues of T = eigenvalues of the $n \times n$ matrix $\mathcal{M}(T)$ as defined in 1B03

Polynomials applied to Operators

Observe If $T \in \mathcal{L}(V)$, then $T \circ T \in \mathcal{L}(V)$

Def$^{\underline{n}}$ If $T \in \mathcal{L}(V)$, then $T^m = \underbrace{T \circ T \circ T \circ \cdots \circ T}_{m \text{ times}}$

$$T^0 = I$$

If T invertible, then $T^{-m} = \underbrace{(T^{-1}) \circ \cdots \circ (T^{-1})}_{m \text{ times}}$

Note $T^{n+m} = T^n T^m$ and $(T^n)^m = T^{nm}$

Can now apply a polynomial to an operator

Def$^{\underline{n}}$ Suppose $T \in \mathcal{L}(V)$ and $p(z) = a_0 + a_1 z + \cdots + a_n z^n \in \mathcal{P}(F)$

Then

$$p(T) = \underbrace{a_0 I + a_1 T + \cdots + a_m T^m}_{\text{an element of } \mathcal{L}(V)}$$

Ex $T \in \mathcal{L}(V)$ $p(z) = 2 + 5z^2$, then
$p(T) = 2 \cdot I + 5T^2$

$$(2I + 5T^2)v = 2Iv + 5T^2 v = 2v + 5T(Tv)$$

Ex $T \in \mathcal{L}(\mathbb{R}^3)$ with

$$T(3x+2y+4z, 4y+2z, 8z).$$

Then

$$T^2(x,y,z) = T(T(x,y,z)) = T(3x+2y+4z, 4y+2z, 8z)$$

$$= (3(3x+2y+4z)+2(4y+2z)+4(8z), 4(4y+2z)+2(8z), 8(8z))$$

$$= (9x+14y+48z, 16y+24z, 64z)$$

So if $p(z) = 2+5z^2$, then

$$p(T) = (2I + 5T^2)$$

$$\Rightarrow (2I+5T^2)(x,y,z) = 2I(x,y,z) + 5T^2(x,y,z)$$

$$= 2(x,y,z) + 5(9x+14y+48z, 16y+24z, 64z)$$

So

$$p(T)(x,y,z) = (47x+70y+240z, 82y+120z, 322z)$$

Note

$$\mathcal{M}(T) = \begin{bmatrix} 3 & 2 & 4 \\ 0 & 4 & 2 \\ 0 & 0 & 8 \end{bmatrix} \Rightarrow \mathcal{M}(T)^2 = \begin{bmatrix} 9 & 14 & 48 \\ 0 & 16 & 24 \\ 0 & 0 & 64 \end{bmatrix}$$

Then $2I_3 + 5\mathcal{M}(T)^2 =$

$$2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + 5\begin{bmatrix} 9 & 14 & 48 \\ 0 & 16 & 24 \\ 0 & 0 & 64 \end{bmatrix} = \begin{bmatrix} 47 & 70 & 240 \\ 0 & 82 & 120 \\ 0 & 0 & 322 \end{bmatrix}$$

$\mathcal{M}(p(T))$ the matrix of $p(T)$ is found by applying the polynomial to $\mathcal{M}(T)$

Fact Fix an operator T. Define a function

$$\phi : \mathcal{P}(F) \to \mathcal{L}(V)$$

by $\phi(p) = p(T)$.

Then ϕ is a linear map, i.e.

$$\phi(p+q) = (p+q)(T) = p(T) + q(T) = \phi(p) + \phi(q)$$
$$\phi(\lambda p) = (\lambda p)(T) = \lambda(p(T)) = \phi(p).$$

Fact (Multiplication Prop) Fix $T \in \mathcal{L}(V)$. Then for all $p, q \in \mathcal{P}(F)$,

$$(pq)(T) = p(T)\,q(T)$$

polynomial obtained by mult p and q and turning into an op

the composition of the operators from p and q

Existence

Does $T \in \mathcal{L}(V)$ have to have an eigenvalue?

<u>Thm</u> If $F = \mathbb{C}$ and V is a fin. dim v.s over F, then every $T \in \mathcal{L}(V)$ has an eigenvalue

<u>Proof</u> Assume $\dim V = n > 0$ and let $T \in \mathcal{L}(V)$. Consider any $v \in V$ with $v \neq 0$.

The vectors
$$v, Tv, T^2v, T^3v, .., T^n v$$
are $(n+1)$ vectors in an n-dim v.s., so linearly dependent

So there exists $a_0, .., a_n$ such that
$$a_0 v + \underline{a_1} Tv + \underline{a_2} T^2 v + .. + a_n T^n v = 0$$
with not all $a_1, .., a_n = 0$

(If they all were 0, then we have $a_0 v = 0$, i.e $a_0 = 0$, which forces $v, .., T^n v$ to be linearly independent)

Consider polynomial
$$p(z) = a_0 + a_1 z + a_2 z^2 + .. + a_n z^n.$$

By the Fund. Thm of Alg, can factor

$$p(z) = c(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)$$

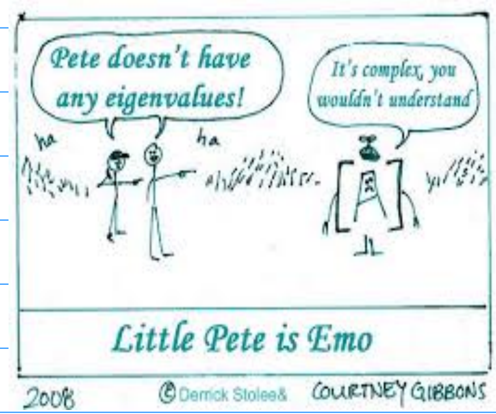
($m \leq n$ since a_n may equal 0)

So $$0 = a_0 I + a_1 Tv + a_2 T^2 v + \dots + a_n T^n v$$

$$= (a_0 I + a_1 T + \dots + a_n T^n) v$$
$$= (p(T)) v$$
$$= (c(T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_m I))v$$

One of the operators $(T-\lambda_1 I), \dots, (T-\lambda_m I)$ must be noninjective ($v \neq 0$, but it is sent to zero)

But $T-\lambda_i I$ not injective implies λ_i is an eigenvalue $\boxtimes$



Note existence depends upon F

If $F=\mathbb{R}$, consider $T \in \mathcal{L}(\mathbb{R}^2)$ given by

$$T(x,y) = (-y,x).$$

Apply approach of proof with $v=(1,0)$

$v=(1,0)$, $Tv=(0,1)$ $T^2 v = (-1,0)$

$1 \cdot v + 1 \cdot T^2 v = 0 \iff p(z) = 1+z^2$

Can't go any further since $p(z) = 1 + z^2$
does not factor over $\mathbb{R}$

Key ideas: * polynomials related to operators
* existence of eigenvalues