$\boxed{\text{Lecture 25}}$ 8.D Jordan Form

The story so far...

V is a v.s over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are the eigenvalues with multiplicities $d_1, \ldots, d_m$. Then we can find a basis so that

$$\mathcal{M}(T) = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix} \quad \text{when } A_i = \underbrace{\begin{bmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}}_{d_i \times d_i}$$

Note Under this basis, if $A_i = \begin{bmatrix} \lambda_i & & * \\ & \ddots & \\ 0 & & \lambda_i \end{bmatrix}$

the matrix $\begin{bmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$ comes from the nilpotent operator $(T - \lambda_i I)\big|_{G(\lambda_i, T)}$

To "improve" this result, want a basis for nilpotent operators so that the associated matrix has as few zeroes as possible $\Rightarrow$ need a "better" basis for nilpotent operators

Thm Suppose $N \in \mathcal{L}(V)$ is nilpotent. Then there exists $v_1, \ldots, v_n \in V$ and nonnegative integers $m_1, \ldots, m_n$ such that

- $N^{m_1}v_1, N^{m_1-1}v_1, \ldots, Nv_1, v_1, N^{m_2}v_2, N^{m_2-1}v_2, \ldots, Nv_2, v_2, \ldots, N^{m_n}v_n, N^{m_n-1}v_n, \ldots, Nv_n, v_n$ is a basis for V

- $N^{m_1+1}v_1 = N^{m_2+1}v_2 = \cdots = N^{m_n+1}v_n = 0$

This is a "better" basis.

Note $N(\underbrace{N^i v_j}_{\text{basis element of } V}) = N^{i+1}v_j \leftarrow N$ maps a basis element to another basis element or 0 (e.g. $N(N^{m_i}v_i) = 0$)

So, with respect to this basis, $M(N)=$

$$\begin{array}{cc}
 & \begin{array}{ccccccccccc} N^{m_1}v_1 & N^{m_1-1}v_1 & \cdots & Nv_1 & v_1 & \cdots\cdots & N^{m_n}v_n & N^{m_n-1}v_n & \cdots & Nv_n & v_n \end{array} \\
\begin{array}{c} N^{m_1}v_1 \\ N^{m_1-1}v_1 \\ \vdots \\ N^{2}v_1 \\ Nv_1 \\ v_1 \\ \vdots \\ N^{m_n}v_n \\ N^{m_n-1}v_n \\ \vdots \\ Nv_n \\ v_n \end{array} &
\begin{bmatrix}
0 & 1 & & 0 & 0 & & & & & & \\
0 & 0 & \ddots & & & & & & & & \\
\vdots & \vdots & & 1 & 0 & & & & & & \\
 & & & & & & & & & & \\
0 & 0 & & 0 & 1 & & & & & & \\
0 & 0 & & 0 & 0 & & & & & & \\
\vdots & & & & & \ddots & & & & & \\
0 & 0 & & & & & 0 & 1 & 0 & & 0 \\
0 & 0 & & & & & 0 & 0 & 1 & & 0 \\
\vdots & \vdots & & & & & & & \ddots & & \\
0 & 0 & & & & & & & & 0 & 1 \\
0 & 0 & & & & & 0 & 0 & & & 0
\end{bmatrix}
\end{array}$$

Other blocks

$M(N)$ is made up of $(m_j+1)\times(m_j+1)$ blocks of matrices with all 1's above diagonals and zeroes everywhere else

<u>Def$^{\underline{n}}$</u> A basis for V is called a <u>Jordan</u> <u>Basis</u> for $T \in \mathcal{L}(V)$ if respect to this basis

$$M(T) = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{bmatrix} \quad \text{when } A_i = \begin{bmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{bmatrix}$$

<u>Cor</u> If $N \in \mathcal{L}(V)$ is nilpotent, then V has a Jordan Basis

<u>Proof</u> See the matrix above. It has the correct form

Thm If V is a complex vector space, then every $T \in \mathcal{L}(V)$ has a Jordan Basis

Proof Let $T \in \mathcal{L}(V)$ with eigenvalues $\lambda_1, .., \lambda_m$
So

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$$

Each $(T - \lambda_i I)\Big|_{G(\lambda_i, T)}$ is nilpotent on $G(\lambda_i, T)$

So, can find a Jordan Basis of $G(\lambda_i, T)$

Put all the bases together. This becomes a basis for V that puts $\mathcal{M}(T)$ into Jordan form $\square$

Ex In Lec 23, looked

$T \in \mathcal{L}(\mathbb{C}^4)$ with

$T(x_1,x_2,x_3,x_4) = (10x_1 - 7x_2 + 15x_3 - 2x_4,$
$10x_2 + 5x_3 - 5x_4, -4x_2 + 25x_3 + x_4, 10x_2 - 5x_3 + 25x_4)$

Showed that if we use basis

$$\begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\-1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}.3\\-1\\0\\0\end{bmatrix}$$

then $M(T) = \begin{bmatrix}10 & 0 & 0 & 0\\ 0 & 20 & -5 & -10\\ 0 & 0 & 20 & -6\\ 0 & 0 & 0 & 20\end{bmatrix}$

Extra video
I work out
the details
for this example

But if we use $\left\{ \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\-1\end{bmatrix}, \begin{bmatrix}1/5\\0\\1/5\\0\end{bmatrix}, \begin{bmatrix}17/300\\1/30\\2/30\\0\end{bmatrix} \right\}$

then $M(T) = \begin{bmatrix}\boxed{10} & 0 & 0 & 0\\ 0 & 20 & 1 & 0\\ 0 & 0 & 20 & 1\\ 0 & 0 & 0 & 20\end{bmatrix}$ ← Jordan Form

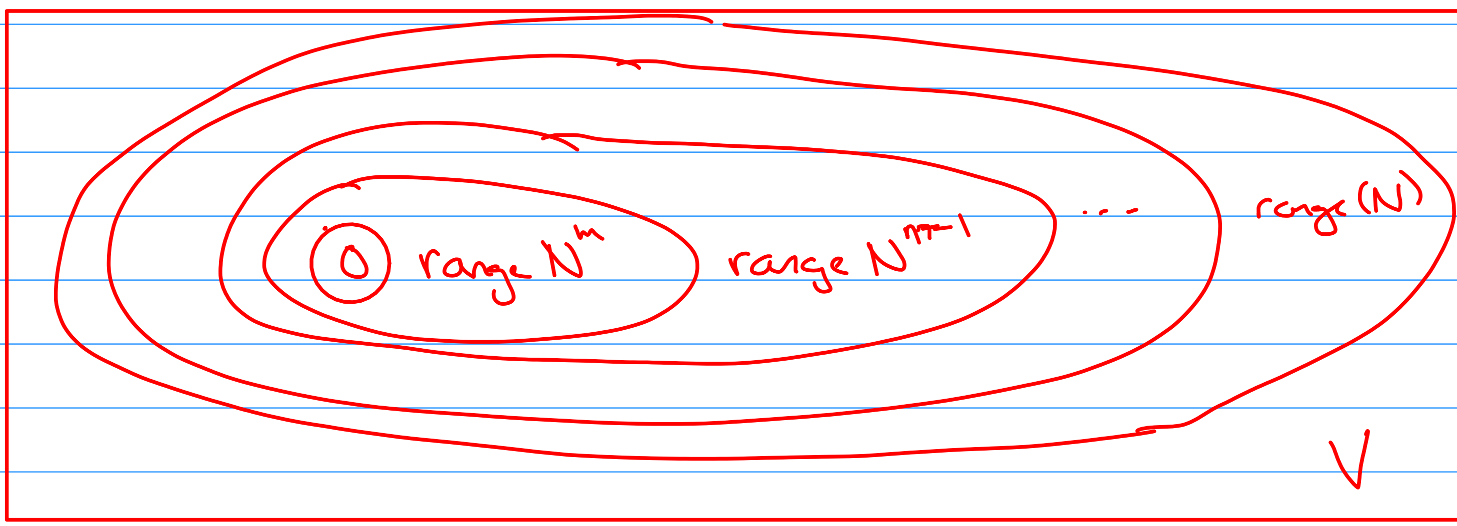
(Main Idea of Nilpotent Result)

If N is nilpotent on V, then we have

$$V \xrightarrow{N} \text{range}(N) \xrightarrow{N} \text{range}(N^2) \xrightarrow{N} \cdots \to \text{range}(N^t) = 0$$

for some range t

In particular, if m is the largest integer such that $\text{range}(N^m) \neq \{0\}$



The proof works via the following steps

1. Find a basis for range(N^m)
2. use this basis to make a linearly independent set in range(N^{m-1}). $u_1, \ldots, u_t$
3. extend this basis to a basis of range(N^{m-1})
   $u_1, \ldots, u_t, v_1, \ldots, v_r$
4. "tweak" the basis so extended elements get sent to zero by N. We get a new basis
   $u_1, \ldots, u_t, v_1', \ldots, v_r'$
5. Repeat this process in range(N^{m-1}), and so on
6. Stop when you get a basis of V

Compare to proof for basis with upper triangular matrices: There we had

$$\mathrm{Null}(N) \subseteq \mathrm{Null}(N^2) \subseteq \ldots \subseteq \mathrm{Null}(N^{m+1}) = V$$

There, we extended a basis of $\mathrm{Null}(N^i)$ to $\mathrm{Null}(N^{i+1})$

Key ideas * basis for nilpotent operator
* Jordan form
* every $T \in \mathcal{L}(V)$ with V over $\mathbb{C}$ gives a Jordan Basis of V

Extra lecture $\Rightarrow$ a worked out example of finding the Jordan form