$\boxed{\text{Lecture 14}}$ Chapter 4 (crash course on polynomials)

Goal: Study properties of polynomials

$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n \quad a_i \in F$$

Motivation (Section 5.B).

Given $T \in \mathcal{L}(V) = \mathcal{L}(V,V)$, let $T^i = \underbrace{T \cdot T \cdots T}$

T composed i times

Relate operator to polynomial

$$a_0 I + a_1 T + a_2 T^2 + \dots + a_m T^m \longleftrightarrow a_0 + a_1 z + \dots + a_m z^m$$

$\boxed{\text{Properties of complex numbers}}$

Let $F = \mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$ where $i^2 = -1$

Given $z = a + bi$, real part of $z = \mathrm{Re}(z) = a$

imaginary part of $z = \mathrm{Im}(z) = b$

Complex conjugate of $z = a+bi$ is $\bar{z} = a-bi$
absolute value of $z = a+bi$ is $|z| = \sqrt{a^2+b^2}$

Ex If $z = 2+6i$

$\mathrm{Re}\, z = 2$, $\mathrm{Im}\, z = 6$, $\bar{z} = 2-6i$, $|z| = \sqrt{2^2+6^2} = \sqrt{40}$

Properties
- $\overline{w} + \overline{z} = \overline{w+z}$
- $\overline{w} \cdot \overline{z} = \overline{wz}$
- $\bar{\bar{z}} = z$
- $|\bar{z}| = |z|$
- $|wz| = |w||z|$
- $|w+z| \leq |w| + |z|$

$\boxed{\text{Polynomials}}$

Def$^{\underline{n}}$ A function $p : F \to F$ is a polynomial with coeffecients in F if there exists $a_0, \ldots, a_m \in F$ such that

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

Thm If $p(z) = a_0 + a_1 z + \cdots + a_m z^m = 0$ for all $z \in F$ (as functions), then $a_0 = \cdots = a_m = 0$

Recall
- $\deg p(z) = m$ if $a_m \neq 0$
  $= -\infty$ if $p(z) = 0$
- $\mathcal{P}(F)$ = all polynomials

(Division Algorithm for Polynomials)
Suppose $p, s \in \mathcal{P}(\mathbb{F})$ and $s \neq 0$. Then there exists unique polynomials q and $r \in \mathcal{P}(\mathbb{F})$ such that

$$p = sq + \underbrace{r}_{\text{remainder}} \quad \text{with } r = 0 \text{ or } \deg r < \deg s.$$

Ex $\quad p = 3z^3 + 2z^2 + 1 \qquad s = 6z^2 + 7z$

Find q and r

long division of polynomials

$$
\begin{array}{r|l}
 & \tfrac{1}{2}z - \tfrac{3}{12} \\
\hline
6z^2 + 7z & 3z^3 + 2z^2 + 0z + 1 \\
 & 3z^3 + \tfrac{7}{2}z^2 \\
\cline{2-2}
 & -\tfrac{3}{2}z^2 \qquad\qquad + 1 \\
 & -\tfrac{3}{2}z^2 - \tfrac{21}{12}z \\
\cline{2-2}
 & \tfrac{21}{12}z + 1
\end{array}
$$

$$\therefore \overbrace{3z^3 + 2z^2 + 1}^{p} = \overbrace{(6z^2 + 7z)}^{s}\overbrace{(\tfrac{1}{2}z - \tfrac{3}{12})}^{q} + \overbrace{(\tfrac{21}{12}z + 1)}^{r}$$

Division Alg justifies why there is always a unique sol$^{\underline{n}}$ to this high school problem

Roots

<u>Def$^{\underline{n}}$</u> A number $\lambda \in F$ is a <u>root</u> or <u>zero</u> of $p \in \mathcal{P}(F)$ if $p(\lambda) = 0$.

<u>Thm</u> Suppose $p \in \mathcal{P}(F)$ and $\lambda \in F$. Then λ is a root of p if and only if $p(z) = (z-\lambda)\, q(z)$ for some $q \in \mathcal{P}(F)$

<u>Proof</u> ($\Rightarrow$) Consider $p(z)$ and $s(z) = (z-\lambda)$

By the division alg, there exists $q, r \in \mathcal{P}(F)$ such that

$$p(z) = (z-\lambda)q(z) + r(z) \quad \text{with } r(z) = 0 \text{ or } \deg r(z) < \underbrace{\deg s(z)}_{=1}$$

But $0 = p(\lambda) = (\lambda-\lambda)q(\lambda) + r(\lambda) \Rightarrow 0 = r(\lambda)$

If $\deg r(z) = 0$, then $r(z) = r \in F$. If $r \neq 0$, then $0 = r(\lambda) = r \neq 0$. A contradiction. So $r = 0$.

Thus $p(z) = (z-\lambda)\, q(z)$

($\Leftarrow$) If $p(z) = (z-\lambda)\, q(z)$, then

$$p(\lambda) = (\lambda-\lambda)q(\lambda) = 0 \cdot q(\lambda) = 0$$

So λ is a root of $p(z)$ ∎

WARNING! Existence of roots depends upon $\mathbb{F}$

$z^2+1=0 \iff z^2=-1$ ← has no roots in $\mathbb{R}$
← has roots $\pm i$ in $\mathbb{C}$

Thm (Fundamental Theorem of Algebra)
If $p \in \mathcal{P}(\mathbb{C})$ is a nonconstant polynomial of degree $m \geq 1$, then p has a unique factorization (up to order of factors) of the form

$$p(z) = c(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_m)$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$ need not be distinct

Remark
- Over $\mathbb{C}$, every polynomial can be factored into products of form $(z-a)$ with $a \in \mathbb{C}$.
- not true over $\mathbb{R}$,

ex $\quad (z^2+1) \neq (z-a)(z+b)$ for any $a,b \in \mathbb{R}$

Q: If $p \in \mathcal{P}(\mathbb{R})$, "how much" can we factor it?

Thm: $x^2+bx+c = (x-\lambda_1)(x-\lambda_2)$ with $b, c, \lambda_1, \lambda_2 \in \mathbb{R}$
$\iff b^2-4c \geq 0$ ← the discriminant from high school

Thm Suppose $p \in \mathcal{P}(\mathbb{R})$ is a nonconstant polynomial. Then p has a unique factorization (up to order)

$$p(z) = c(z-a_1)(z-a_2)\cdots(z-a_n)(z^2+b_1z+c_1)\cdots(z^2+b_mz+c_m)$$

with $c, a_1, \ldots, a_n, b_1, c_1, \ldots, b_m, c_m \in \mathbb{R}$ and $b_i^2-4c_i<0$.

Idea of proof:

(Fact) If $\lambda \in \mathbb{C}$ and $p \in \mathcal{P}(\mathbb{R})$ and $p(\lambda)=0$, then $p(\bar{\lambda})=0$

conjugate

Factor p over $\mathbb{C}$

$$p = c\underbrace{(z-a_1)\cdots(z-a_n)}_{\text{real roots}}\underbrace{(z-\lambda_1)\cdots(z-\lambda_{2m})}_{\text{all complex roots}}$$

(come in conjugate pairs by fact)

So

$$p(z) = c(z-a_1)\cdots(z-a_n)(z-\lambda_1)(z-\bar{\lambda}_1)\cdots(z-\lambda_m)(z-\bar{\lambda}_m)$$

But if $\lambda \in \mathbb{C}$,

$$(z-\lambda)(z-\overline{\lambda}) = z^2 - 2(\mathrm{Re}\,\lambda)z + |\lambda|^2$$ ← notice this is in $\mathcal{P}(\mathbb{R})$

with $(2(\mathrm{Re}\,\lambda))^2 - 4|\lambda|^2 < 0$

So

$$p(z) = c(z-a_1)\cdots(z-a_m)(z^2-(2\mathrm{Re}\,\lambda_1)z+|\lambda_1|^2)\cdots(z^2-(2\mathrm{Re}\,\lambda_n)z+|\lambda_n|^2)$$

$\square$

Cor If $p \in \mathcal{P}(\mathbb{R})$ has $\deg p$ odd, the p has a real root

Proof Since $p(z) = c(z-a_1)\cdots(z-a_m)(z^2+b_1z+c_1)\cdots(z^2-b_nz+c_n)$

has $\deg p(z) = m+2n$, and $\deg p(z)$ odd, then m is odd. So $m \geq 1$. So $(z-a_1)$ is a factor, i.e a_1 is a root.

$\blacksquare$

Key ideas

* Polynomials
* division algorithm
* factoring over $\mathbb{C}$ and $\mathbb{R}$

(for proofs, take Math 3GR3)

Have a good reading week