$\boxed{\text{Lecture 33}}$ 7.B Spectral Theorem

In Chapter 5, showed that if $T \in \mathcal{L}(V)$, there might exists a "nice" basis for T so $\mathcal{M}(T)$ is "nice", i.e. upper triangular, diagonal

Show similar results when T is normal or self-adjoint

Note: different results if $F = \mathbb{C}$ or $\mathbb{R}$

$\boxed{\text{Complex Spectral Theorem}}$

Thm Suppose $F = \mathbb{C}$ and $T \in \mathcal{L}(V)$. The following are equivalent:

(a) T is normal, i.e $T^*T = TT^*$
(b) V has an orthonormal basis of eigenvectors of T
(c) $\mathcal{M}(T)$ is a diagonal matrix with respect to some orthonormal basis of V

Proof

(b) $\iff$ (c) is Thm 5.41

$(a) \Rightarrow (c)$ Since V is an inner product space over $\mathbb{C}$, by Schur's Theorem there is an orthonormal basis $e_1, e_2, \ldots, e_n$ of V

such that $\mathcal{M}(T)$ is upper triangular

i.e

$$\mathcal{M}(T) = \begin{bmatrix} a_{11} & \overbrace{a_{12} \quad \cdots \quad a_{1n}}^{=0} \\ 0 & a_{22} \quad \cdots \quad a_{2n} \\ 0 & \ddots \qquad \vdots \\ \vdots & \\ 0 & \qquad\qquad a_{nn} \end{bmatrix}$$

This means $Te_1 = a_{11} e_1$. So

$$\|Te_1\|^2 = |a_{11}|^2 \|e_1\|^2 = |a_{11}|^2$$

But T^* has $\mathcal{M}(T^*) = \mathcal{M}(T)^*$ So this means

$$T^* e_1 = a_{11} e_1 + a_{12} e_2 + \cdots + a_{1n} e_n$$

$$\Rightarrow \|T^* e_1\|^2 = |a_{11}|^2 + |a_{12}|^2 + \cdots + |a_{1n}|^2$$

Becase T is normal

$$|a_{11}|^2 = \|Te_1\|^2 = \|T^*e_1\|^2 = |a_{11}|^2 + |a_{12}|^2 + \cdots + |a_{1n}|^2$$

So $|a_{12}| = \cdots = |a_{1n}| = 0 \Rightarrow a_{12} = \cdots = a_{1n} = 0$

Repeat for other entries on diagonal

(c) $\Rightarrow$ (a) If T has $\mathcal{M}(T)$ diagonal with respect to orthonormal basis, then

$$\mathcal{M}(T^*) = \mathcal{M}(T)^* \text{ is a diagonal matrix}$$

Then
$$\begin{aligned} \mathcal{M}(T^*T) &= \mathcal{M}(T^*)\mathcal{M}(T) \\ &= \mathcal{M}(T)\mathcal{M}(T^*) \\ &= \mathcal{M}(TT^*) \end{aligned}$$

diagonal matrices commute

So $T^*T = TT^*$, i.e T is normal operator $\square$

Ex Let $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ with entries in $\mathbb{R}$ but viewed in $\mathbb{C}$.

Then $A^* = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

and $A^*A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix} = AA^*$

So, A corresponds to normal $T \in \mathcal{L}(\mathbb{C}^2)$. I.e

$$Tx = T(x_1, x_2) = (ax_1 + bx_2, -bx_1 + ax_2)$$
$$= \left(\begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \right)^T$$

Eigenvalues are

both have norm 1

$\lambda = a+bi$ with eigenvector $\begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$

$\lambda = a-bi$ with eigenvector $\begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix}$

$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -i/\sqrt{2} \end{bmatrix} \right\}$ is an orthonormal basis of $\mathbb{C}^2$

W.R.T this basis $\mathcal{M}(T) = \begin{bmatrix} a+bi & 0 \\ 0 & a-bi \end{bmatrix}$

$\boxed{\text{Real Spectral Theorem}}$

<u>Thm</u> Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. The following are equivalent:

(a) T is self-adjoint, i.e. $T^* = T$
(b) V has an orthonormal basis of eigenvectors of T
(c) $\mathcal{M}(T)$ is diagonal w.r.t. to some orthonormal basis

<u>Proof</u> (b) $\Longleftrightarrow$ (c) from Thm 5.41

(c) $\Rightarrow$ (a) We have $\mathcal{M}(T)$ diagonal. So $\mathcal{M}(T)^T = \mathcal{M}(T)$

Since $\mathbb{F} = \mathbb{R}$, $\mathcal{M}(T)^T = \mathcal{M}(T)^* = \mathcal{M}(T^*)$

So $T^* = T$, i.e. T is self adjoint

(a) $\Rightarrow$ (c) See text for a proof.

Note, there is a real version of Schur's Theorem that can be used, but not proved in the text

$\square$

<u>Cor</u> (from 2LA3) Let A be a symmetric matrix. Then A can be diagonalized (in fact, orthogonally diagonalized)

<u>Note</u> In general, hard to "look" @ a matrix to determine if it can be diagonalized (even saw this in 1B03). But the corollary gives a large class where it is easy to determine if a matrix can be diagonalized

Spectral Decomposition

From above, if A is a symmetric matrix, can write

eigenvalues

$$A = [u_1\, u_2\, u_3 \ldots u_n] \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} [u_1\, u_2 \ldots u_n]^{-1}$$

$u_1, \ldots, u_n$ is the orthonormal basis of eigenvectors of $V = \mathbb{R}^n$

Fact Suppose $U = [u_1\, u_2 \cdots u_n]$ is an $n \times n$ matrix with orthonormal columns. Then

$$U^{-1} = U^T$$

Proof Because inverses of matrices unique, enough to show $U^T U = I_n$

$$\begin{array}{ccccc} U^T & & U & & \\ \begin{bmatrix} u_{11} & u_{21} & \cdots & u_{n1} \\ u_{12} & u_{22} & & u_{n2} \\ \vdots & & & \\ u_{1n} & & & u_{nn} \end{bmatrix} & & \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & & & \vdots \\ \vdots & & & \\ u_{n1} & & & u_{nn} \end{bmatrix} & = & \begin{bmatrix} u_1 \cdot u_1 & u_1 \cdot u_2 & u_1 \cdot u_n \\ & & \\ u_n \cdot u_1 & \cdots & u_n \cdot u_n \end{bmatrix} \\ & & \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ u_1 & u_2 & u_n \end{array} & & \end{array}$$

dot product

But $u_i \cdot u_j = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases}$

So $U^T U = I_n$

$\square$

If A is symmetric

$$A = [u_1\ u_2 \cdots u_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ & & \ddots \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= \begin{bmatrix} \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \end{bmatrix} \begin{bmatrix} u_1^T \\ u_2^T \\ \vdots \\ u_n^T \end{bmatrix}$$

$$= \boxed{\lambda_1 \underbrace{u_1 u_1^T}_{n \times n} + \lambda_2 \underbrace{u_2 u_2^T}_{n \times n} + \cdots + \lambda_n \underbrace{u_n u_n^T}_{n \times n}}$$

called the spectral decomposition of A

"spectral" refers to eigenvalues

NOTE If A is not symmetric, cannot find a basis of eigenvectors that is orthonormal

Key ideas: * Complex Spectral Thm
* Real Spectral Thm
* Spectral Decomposition