$\boxed{\text{Lecture 12}}$ 3.D Invertibility and Isomorphisms I

Recap Let V have basis $v_1, \ldots, v_n$
W have basis $w_1, \ldots, w_m$
If $T \in \mathcal{L}(V, W)$, the matrix of T

$$\mathcal{M}(T) = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & & A_{mn} \end{bmatrix}$$

where

$$\underset{\substack{\text{image} \\ \text{of } v_k}}{Tv_k} = \underbrace{A_{1k}w_1 + A_{2k}w_2 + \cdots + A_{mk}w_m}_{\text{written in the basis of } W}$$

"Dictionary"

Linear map		matrices
$T \in \mathcal{L}(V, W)$	$\longleftrightarrow$	$\mathcal{M}(T)$
$T + S$	$\longleftrightarrow$	$\mathcal{M}(T+S) = \mathcal{M}(T) + \mathcal{M}(S)$
λT	$\longleftrightarrow$	$\mathcal{M}(\lambda T) = \lambda \mathcal{M}(T)$
TS ← composition	$\longleftrightarrow$	$\mathcal{M}(TS) = \mathcal{M}(T)\mathcal{M}(S)$
???	$\longleftrightarrow$	inverse of a matrix

Invertible linear maps

<u>Def</u>

Def^{n}: A linear map $T \in \mathcal{L}(V,W)$ is <u>invertible</u> if there exists $S \in \mathcal{L}(W,V)$ such that

- ST is the identity on V, i.e. $(ST)v = Iv = v$ for all $v \in V$
- TS is the identity on W, i.e. $(TS)(w) = Iw = w$ for all $w \in W$.

A <u>linear map</u> $S \in \mathcal{L}(W,V)$ is the <u>inverse</u> of T if

$$ST = I \quad \text{and} \quad TS = I$$

<u>Thm</u> If $T \in \mathcal{L}(V,W)$ is invertible, then its inverse is unique

<u>Proof</u>: Suppose $S_1, S_2 \in \mathcal{L}(W,V)$ are inverses of T. Then

$$S_1 = \underbrace{S_1 I}_{\substack{\text{composed}\\ \text{with the identity}\\ \text{on } W}} = \underbrace{S_1(TS_2) = (S_1 T) S_2}_{\substack{\text{associative}\\ \text{prop}}} = \underbrace{I \cdot S_2}_{\substack{\text{identity}\\ \text{on } V}} = S_2$$

$\square$

Def^n If $T \in \mathcal{L}(V,W)$ is invertible, let T^{-1} denote unique inverse.

Thm $T \in \mathcal{L}(V,W)$ is invertible iff T is injective and surjective

Proof $(\Longrightarrow)$ (see the textbook)

$(\Longleftarrow)$ Define a map $S: W \to V$ by
$w \longmapsto S(w)$
where $S(w)$ in V satisfies $T(S(w)) = w$

(since T is surjective, there is an $x \in V$ such that $T(x) = w$
Because T is injective, there is only one $x \in V$ such that
$T(x) = w$. The map S sends w to this x)

By definition of S, $(T \circ S)(w) = T(S(w)) = w$ for all $w \in W$

by above

For any $v \in V$, $T((S \circ T)(v)) = (T \circ S)(T(v)) = T(v)$.
Because T is injective, this means $(S \circ T)(v) = v$

So $S \circ T$ is the identity on V.

Need to show $S \in \mathcal{L}(W,V)$, i.e, S is linear map

For any $w_1, w_2 \in W$

$T(Sw_1 + Sw_2) = T(Sw_1) + T(Sw_2)$ ← additive prop of T

$= w_1 + w_2$ ← since $T(Sw_1) = w_1$ and $T(Sw_2) = w_2$

By the definition $S(w_1+w_2)$ is the unique element of V that maps to w_1+w_2. But by above, $Sw_1 + Sw_2$ also maps to w_1+w_2. We thus have

$$S(w_1+w_2) = Sw_1 + Sw_2.$$

For homogenity, $T(\lambda Sw_1) = \lambda T(Sw_1)$ ← by homog. of T

$= \lambda w_1$ ← since $TS = I$

On other hand, $S(\lambda w_1)$ is the unique element of V mapped to λw_1.

So $\lambda Sw_1 = S(\lambda w_1)$.

So $S \in \mathcal{L}(W,V)$. ∎

Ex Let $D \in \mathcal{L}(\mathcal{P}_2(\mathbb{R}), \mathcal{P}_1(\mathbb{R}))$ be defined by
$$D(a_0 + a_1 x + a_2 x^2) = a_1 + 2a_2 x$$

and $Int \in \mathcal{L}(\mathcal{P}_1(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ be defined by

$$Int(b_0 + b_1 x) = b_0 x + \frac{b_1}{2} x^2$$

These are <u>not</u> inverses of each other:

$$(Int \circ D)(a_0 + a_1 x + a_2 x^2) = Int(a_1 + 2a_2 x) = a_1 x + a_2 x^2$$

$$(D \circ Int)(b_0 + b_1 x) = D\left(b_0 x + \frac{b_1}{2} x^2\right) = b_0 + b_1 x$$

So $D \circ Int = I$ on $\mathcal{P}_1(\mathbb{R})$ but $Int \circ D \neq I$ on $\mathcal{P}_2(\mathbb{R})$

Isomorphisms

Def^n An invertible linear map is called an <u>isomorphism</u>. Two vector spaces U and W are <u>isomorphic</u> if there is an isomorph. $T: V \to W$.

Remark V and W are isomorphic means that V and W "same" vector space, but with different labels.

Ex Let $V = \mathbb{R}^2$ and $W = \mathcal{P}_1(\mathbb{R})$.

Define $T: \mathbb{R}^2 \to \mathcal{P}_1(\mathbb{R})$ by

$$T(a,b) = a + bx$$ ← this is an isomorphism!

Thm Two fin. dim. vect spaces V and W are isomorphic if and only if $\dim V = \dim W$.

Proof $(\Rightarrow)$. Let $T: V \to W$ be an isomorphism. Then

$$\dim V = \dim \text{Null}(T) + \dim \text{range}(T)$$

Since T is an isomorph, T is injective and surjective. So $\text{Null}(T) = \{0\}$ and $\text{range}(T) = W$. So

$$\dim V = 0 + \dim \text{range}(T) = \dim W.$$

Same n since $\dim W = \dim V$.

$(\Leftarrow)$ Let $v_1, .., v_n$ and $w_1, .., w_n$ be bases of V and W.

Define $T: V \to W$ by

$$Tv_i = w_i$$

So

$$T(c_1v_1 + .. + c_nv_n) = c_1w_1 + \cdots + c_nw_n$$

is a linear map

T is injective since

$$T(c_1v_1+\dots+c_nv_n)=0 \iff c_1w_1+\dots+c_nw_n=0$$
$$\iff c_1=\dots=c_n=0$$

since $w_1,\dots,w_n$ is a basis

So $c_1v_1+\dots+c_nv_n=0$.

T is surjective since for any $w\in W$,
$w=a_1w_1+\dots+a_nw_n$ for some $a_i\in F$
But then $v=a_1v_1+\dots+a_nv_n\in V$, and
$T(v)=T(a_1v_1+\dots+a_nv_n)=w$.

So T is invertible, i.e. an isomorphism. $\square$

Ex For all n, $\mathbb{R}^n$ is isomorphic to $\mathcal{P}_{n-1}(\mathbb{R})$

since $\dim\mathbb{R}^n=n=\dim\mathcal{P}_{n-1}(\mathbb{R})$

Key ideas
* invertible linear maps
* isomorphisms