$\boxed{\text{Lecture 7}}$ 2.B Bases

Last time: span and linearly independent

Today : bases

$\boxed{\text{Basis}}$

Def^{n} A basis for V is a list of vectors $v_1, .., v_m \in V$ such that

- $v_1, .., v_m$ are linearly independent
- $\text{span}(v_1, .., v_m) = V$

Ex 1. $(1,0,..,0), (0,1,...,0), ..., (0,0,..,1)$ is a basis for F^n

2. $1, z, z^2, .., z^m$ is a basis for $\mathcal{P}_m(F)$

3. $((1,2), (1,-2))$ is a basis for F^2

4. $\{(1,1)\}$ is linearly independent in F^2 but does not span F^2, so not a basis

5. $\{(1,2), (1,-2), (1,1)\}$ spans F^2, but not linearly independent. So not a basis

Thm $v_1, .., v_m$ is a basis for V if and only if every $v \in V$ can be expressed uniquely in form

$$v = a_1 v_1 + a_2 v_2 + .. + a_m v_m \quad \text{for some } a_i \in F$$

Proof ($\Longrightarrow$) Since $v_1, \dots, v_m$ is a basis, for any $v \in V$,

$$v = a_1 v_1 + a_2 v_2 + \dots + a_m v_m \text{ with } a_i \in F \text{ since } V = \operatorname{span}(v_1, \dots, v_m)$$

Suppose we could also write

$$v = b_1 v_1 + b_2 v_2 + \dots + b_m v_m \text{ with } b_i \in F$$

Then

$$0 = v - v = (a_1 v_1 + \dots + a_m v_m) - (b_1 v_1 + \dots + b_m v_m)$$
$$= (a_1 - b_1) v_1 + \dots + (a_m - b_m) v_m$$

Since $v_1, \dots, v_m$ is linearly independent, this can only happen if $a_i = b_i \iff a_i - b_i = 0$ for all i.

($\Longleftarrow$) Let $v \in V$. Since $v = a_1 v_1 + \dots + a_m v_m$ for some $a_i \in F$

$v \in \operatorname{span}(v_1, \dots, v_m) \Longrightarrow V = \operatorname{span}(v_1, \dots, v_m)$. So $v_1, \dots, v_m$ is a spanning set of V.

Consider $0 \in V$. Since $0 = 0 \cdot v_1 + \dots + 0 \cdot v_n$. But since this is the only way to write 0 as a linear combination, then $v_1, \dots, v_m$ linearly independent. So $v_1, \dots, v_m$ is a basis $\blacksquare$

Given any spanning set, can "remove" vectors to get a basis.

<u>Thm</u> (spanning set to basis) Every spanning list of vectors can be reduced to a basis of V

<u>Proof</u> Suppose $V = \text{span}(v_1, .., v_m)$

Let $B = (v_1, .., v_m)$. Remove any v_i that is 0. In particular, relabel so $v_1, .., v_\ell \neq 0$, but $v_{\ell+1} = \cdots = v_m = 0$.
Then

$$V = \text{span}(v_1, .., v_m) = \text{span}(v_1, .., v_\ell)$$

For $j = 2, .., \ell$, if $v_j \in \text{span}(v_1, .., v_{j-1})$, we can remove v_j from B and let B now denote the list with v_j removed. If $v_j \notin \text{span}(v_1, .., v_{j-1})$, we keep it in B

After ℓ steps we finished. At each step, we have $\text{span}(B) = V$. Moreover, B contains only v_k such that $v_k \notin \text{span}(v_1, .., v_{k-1})$. So vectors left over are linearly independent. So B is a basis ☒

<u>Cor</u> Every finite dimensional vector space has a basis

<u>Proof</u> Finite dim $\Rightarrow V = \text{span}(v_1, .., v_m)$ for some $v_1, .., v_m$
By previous result, some subset of $v_1, .., v_m$ is a basis ☒

Opposite direction: "add" vectors to linearly indep. sets to get a basis.

Thm (linearly indep sets to bases) Every linearly independent set of vectors in a finite dimension vector space can be extended to a basis

Proof: Let $u_1, .., u_p$ be a linearly indep set of vectors and suppose
$$V = \text{span}(v_1, .., v_m)$$

Consider the list $u_1, .., u_p, v_1, ..., v_m$. This set spans V. If we apply the procedure of the last result to this list get, we get a basis of V. Moreover, no u_i is deleted since
$$u_j \notin \text{Span}(u_1, .., u_{j-1}) \text{ for } j = 2, .., p.$$
So we remove from among the v_i's. ∎

Ex
- $(1,2)$ in F^2 is linearly indep
- $F^2 = \text{span}((1,0), (0,1))$

Consider the list $B = ((1,2), (1,0), (0,1))$
- no 0 vectors to remove
- $(1,0) \notin \text{span}((1,2))$, so keep $(1,0)$ in B
- $(0,1) \in \text{span}((1,2), (1,0))$ since
$$\tfrac{1}{2}(1,2) - \tfrac{1}{2}(1,0) = (0,1).$$
We remove $(0,1)$ from B.

$\therefore$ $B = ((1,2), (1,0))$ is a basis for F^2

Thm Let V be a fin. dim v.s. and $U \subseteq V$ a subspace. Then there is a subspace $W \subseteq V$ such that $V = U \oplus W$.

Proof U is fin dim, so has a basis $u_1, \ldots, u_p$ ← so are also linearly indep in V

By previous result, can extend this list to a basis of V

$$\underbrace{\underbrace{u_1, \ldots, u_p}_{\text{basis for } U}, w_1, \ldots, w_t}_{\text{basis for } V}$$

Let $W = \mathrm{span}(w_1, \ldots, w_t)$

Claim W is the desired subspace

To check (A) $V = U + W$

(B) $U \cap W = \{0\}$

See details in the textbook.

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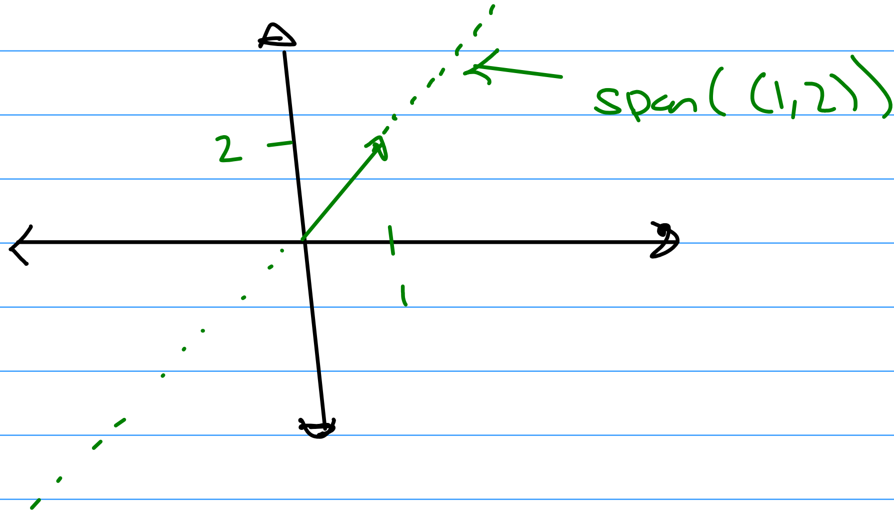
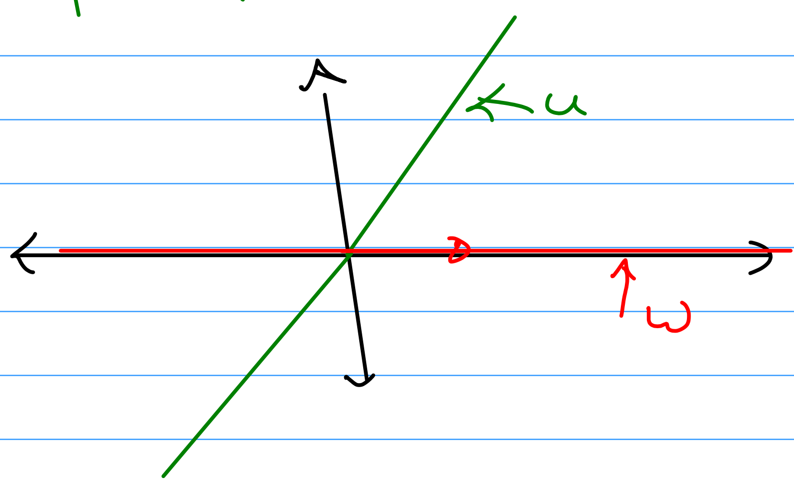
Ex $\quad U = \mathrm{span}((1,2)) \subseteq \mathbb{R}^2$



From last example, $\mathbb{R}^2 = \mathrm{span}((1,2),(1,0))$

Let $W = \mathrm{span}((1,0))$. So $\mathbb{R}^2 = U \oplus W$



Key ideas: · def$^{\underline{n}}$ of basis

- "shrink" spanning sets to bases
- "expand" linearly indep sets to bases