$\boxed{\text{Lecture 16}}$ 5.A Invariant Subspaces I

Recall from 1B03: If A is an $n\times n$ matrix, λ is an eigenvalue if there exists a nonzero $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$

Today: Extend this def^n beyond matrices

$\boxed{\text{Invariant subspaces}}$

Notation Suppose $T: V \to W$ is a function and $U \subseteq V$ is a subset. Then $T|_U$, the restriction of T to U, is the function

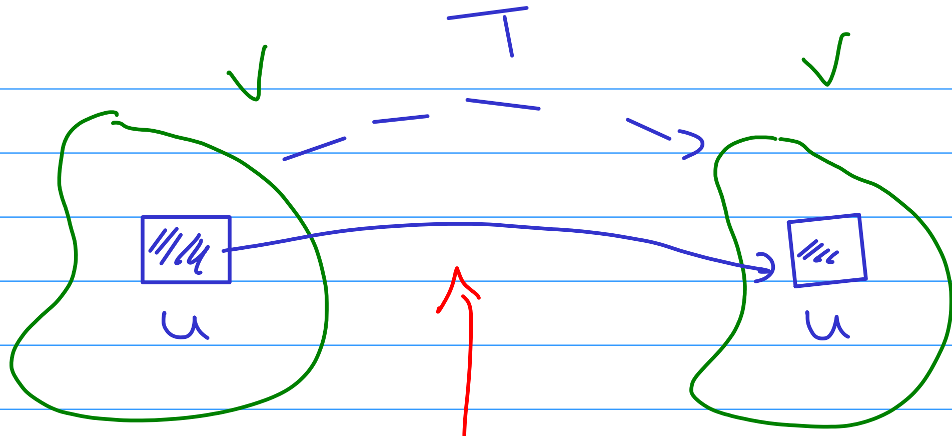
$$T|_U : U \to W \quad \text{where}$$

$$T|_U(u) = T(u)$$

(Restricting domain to U, ignore the elements $V \setminus U$)

Def^n Let $T \in \mathcal{L}(V)$. A subspace U of V is invariant under T if $T(u) \in U$ for all $u \in U$

Picture



every element of U is mapped back into U.

Ex For any $T \in \mathcal{L}(V)$,

1. $\{0\}$
2. $\text{Null}(T)$
3. V
4. $\text{range}(T)$

are invariant subspaces

For 2, Let $u \in \text{Null}(T)$. Then $Tu = 0$
But $0 \in \text{Null}(T)$. So $Tu \in \text{Null}(T)$

Fundamental Questions:

(A) Does T have any invariant subspaces?

(B) Can we find $U_1, .., U_t \subseteq V$ such that $V = U_1 \oplus .. \oplus U_t$ and T is invariant on each U_i?

Eigenvalues and Eigenvectors

A 1-dim subspace of V has the form

$$U = \text{span}(v) = \{\lambda v \mid \lambda \in F\} \text{ for some } v \neq 0$$

When is $T \in \mathcal{L}(V)$ invariant on a 1-dim subspace?

Lemma Let $T \in \mathcal{L}(V)$ and $v \neq 0$. Then T is invariant on $\text{span}(v) \iff Tv = \lambda v$ for some λ

Proof ($\Rightarrow$) Since T is invariant on $\text{span}(v)$ and $v \in \text{span}(v)$, then $Tv \in \text{span}(v)$, ie. $Tv = \lambda v$ for some λ.

($\Leftarrow$) Let $w \in \text{span}(v)$. So $w = kv$ for some $k \in F$.

$$Tw = T(kv) = k(Tv) = k(\lambda v) = (k\lambda)v \in \text{span}(v)$$

So T is invariant on $\text{span}(v)$. ∎

Def$^{\underline{n}}$ Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in F$ is an eigenvalue of T if there exists a nonzero vector v such that

$$Tv = \lambda v$$

Cor T has a 1-dimensional invariant subspace $\iff$ T has an eigenvalue

(Equivalent conditions to be eigenvalues)

Suppose V is finite dimensional, $T \in \mathcal{L}(V)$ and $\lambda \in F$

TFAE

(a) λ is an eigenvalue

(b) $T - \lambda I$ is not injective

(c) $T - \lambda I$ is not surjective

(d) $T - \lambda I$ is not invertible

Proof λ an eigenvalue

$\iff$ $Tv = \lambda v$ for some nonzero v

$\iff$ $Tv = \lambda I v$

$\iff$ $(T - \lambda I)v = 0$ for some nonzero v

$\iff$ $(T - \lambda I)$ not injective

$\left.\begin{array}{l} \iff (T-\lambda I) \text{ not surjective} \\ \iff (T-\lambda I) \text{ not invertible} \end{array}\right\}$ since V is fin. dim.

$\boxtimes$

Def^{n} Suppose $T \in \mathcal{L}(V)$ and $\lambda \in F$ is an eigenvalue. A vector $v \in V$ is an eigenvector of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

Remark Like the def^{n} in 1B03, but expressed in terms of T instead of A

Cor v an eigenvalue of T corresponding to λ $\iff$ $v \in \operatorname{null}(T - \lambda I)$

Thm Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues with corresponding eigenvectors $v_1, \ldots, v_m$. Then $v_1, \ldots, v_m$ are linearly independent.

Proof Suppose $v_1, \ldots, v_m$ are linearly dependent. Let k be the smallest index such that

$v_k \in \operatorname{span}(v_1, \ldots, v_{k-1})$ and $v_1, \ldots, v_{k-1}$ linearly independent.

So

$$v_k = a_1 v_1 + \cdots + a_{k-1} v_{k-1} \quad \text{with some } a_i \neq 0 \qquad (*)$$

Then $\lambda_k v_k = T v_k$ $\quad$ (**)

$$= T(a_1 v_1 + \dots + a_{k-1} v_{k-1})$$
$$= a_1 \lambda_1 v_1 + \dots + a_{k-1} \lambda_{k-1} v_{k-1}$$ } prop of lin. op.

Multiply (*) by λ_k to get

$$\lambda_k v_k = a_1 \lambda_k v_1 + \dots + a_{k-1} \lambda_k v_{k-1}$$ (***)

Subtract (***) from (**) to get

$$0 = (\lambda_k v_k - \lambda_k v_k) = a_1(\lambda_1 - \lambda_k) v_1 + \dots + a_{k-1}(\lambda_{k-1} - \lambda_k) v_{k-1}$$

Since $v_1, \dots, v_{k-1}$ linearly independent, $a_i(\lambda_i - \lambda_k) = 0$

Since $\lambda_i \neq \lambda_k$, this forces $a_i = 0$. This contradicts (*)

So our vectors <u>are</u> linearly independent $\blacksquare$

Cor Suppose $\dim V = n$. Then every $T \in \mathcal{L}(V)$ has at most n distinct eigenvalues

Proof Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T with eigenvectors $v_1, \ldots, v_m$.

Since $v_1, \ldots, v_m$ are linearly indep, we must have $m \le n = \dim V$. $\blacksquare$

Ex Let $T \in \mathcal{L}(\mathbb{R}^2)$ given by $T(x,y) = (3x, 4y)$.

Find eigenvalues.

In 1B03, used determinants ← we don't have these tools!

Using def^n, want λ such that

$$T(x,y) = (3x, 4y) = \lambda(x,y) = (\lambda x, \lambda y).$$

So $3x = \lambda x$ and $4y = \lambda y$. ← only sol^{n}s

1. $\lambda = 3$ and $y = 0$ and x arbitrary
2. $\lambda = 4$ and $x = 0$ and y arbitrary

Check: $T(x,0) = (3x, 0) = 3(x,0)$ ✓
$T(0,y) = (0, 4y) = 4(0,y)$ ✓

Note
- $(1,0)$ is an eigenvector of 3
- $(0,1)$ is an eigenvector of 4

Ex Let $D \in \mathcal{L}(P(\mathbb{R}))$ given by

$$Dp = p'$$ ← map p to its derivative

λ is an eigenvalue of $D \iff p' = Dp = \lambda p$

If $\deg p \geq 1$, $\deg p' = \deg p - 1$. Then no sol^{n} to $p' = \lambda p$ since both sides have different degree.

If $\deg p = 0$, then $p = c$ for some constant $c \in F$

Then $p' = 0$ and $p' = 0 = 0 \cdot p$

So $\lambda = 0$ is an eigenvalue of D.

Key ideas
* invariant subspaces
* eigenvalues
* eigenvectors