$\boxed{\text{Lecture 6}}$ 2.A Linear Independence II

Last time: span

Today: linear independence

Def^{n}. A list of vectors $v_1, \ldots, v_m$ in V is linearly independent if only choice of $a_1, \ldots, a_m \in F$ that satisfy
$$a_1 v_1 + a_2 v_2 + \ldots + a_m v_m = 0$$
is $a_1 = a_2 = \cdots = a_m = 0$

- empty list is linearly independent
- A list of vectors that is not linearly independent is linearly dependent, i.e., there is a nontrivial choice of $a_1, \ldots, a_m$ such that
$$a_1 v_1 + a_2 v_2 + \ldots + a_m v_m = 0$$

Ex
1. $(1,0,0), (0,1,0), (0,0,1)$ in F^3 linearly independent
2. If $v \in V$, and if $v \neq 0$, then v is linearly independent.
3. If $v, w \in V$, $v, w \neq 0$ and $v \neq cw$ for any $c \in F$, then v, w are linearly independent.
4. In $P_m(F)$, $1, z, z^2, \ldots, z^m$ is linearly independent.

Ex $(1,0,0), (1,1,0), (a,b,0)$ in F^3 is linearly dependent

Since

$$1\cdot(a,b,0) + (-b)(1,1,0) + (b-a)(1,0,0) = (0,0,0)$$

Fact If one of $v_1, \dots, v_m$ is the 0 vector, then $v_1, \dots, v_m$ is linearly dependent

Proof: Suppose $v_1 = 0$. Then

$$0 = c\cdot 0 + 0\cdot v_2 + 0\cdot v_3 + \dots + 0\, v_m$$

is a nontrivial sol$^{\underline{n}}$ for all $0 \neq c \in F$

(Linear dependence lemma) Suppose $v_1, \dots, v_m$ linearly dependent. Then there exists $j \in \{1, \dots, m\}$ such that

1. $v_j \in \operatorname{span}(v_1, \dots, v_{j-1})$

2. $\operatorname{span}(v_1, \dots, v_{j-1}, v_{j+1}, \dots, v_m) = \operatorname{span}(v_1, \dots, v_m)$

Proof 1. We are given $a_1 v_1 + \dots + a_m v_m = 0$ with not all $a_i = 0$

Let j be the largest index such that $a_j \neq 0$. So

$$a_1 v_1 + \dots + a_j v_j = 0 \quad \text{with} \quad a_j \neq 0$$

Rearrange

$$v_j = \frac{(-a_1)}{a_j} v_1 + \frac{(-a_2)}{a_j} v_2 + \cdots + \frac{(-a_{j-1})}{a_j} v_{j-1} \quad (*)$$

So $v_j \in \operatorname{span}(v_1, \ldots, v_{j-1})$

(2) Since $v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m \in \operatorname{span}(v_1, \ldots v_m)$

$\operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m) \subseteq \operatorname{span}(v_1, \ldots, v_m)$

Let $u \in \operatorname{span}(v_1, \ldots, v_m)$. So

$u = b_1 v_1 + \cdots + b_j v_j + \cdots + b_m v_m$ for $b_i \in F$

By $(*)$ can replace v_j with an expression in $v_1, \ldots, v_{j-1}$.

$$u = b_1 v_1 + \cdots + b_j \left[\frac{(-a_1)}{a_j} v_1 + \cdots + \frac{(-a_{j-1})}{a_j} v_{j-1} \right] + \cdots + b_m v_m.$$

So $u \in \operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$.

Thus $\operatorname{span}(v_1, \ldots, v_m) \subseteq \operatorname{span}(v_1, \ldots, v_{j-1}, v_{j+1}, \ldots, v_m)$ $\blacksquare$

Thm In a finite dimensional v.s., every list of linearly independent vectors is less than or equal to the length of every spanning list of vectors

Proof Let $u_1, .., u_m$ be linearly independent in V
Let $v_1, .., v_n$ span V
Want to show $m \leq n$

Since $V = \text{span}(v_1, .., v_n)$, $u_1, v_1, .., v_n$ is linearly dependent.

Why? $a_1 v_1 + .. + a_n v_n = u_1$ for some $a_1, .., a_n$. So
$(-1)u_1 + a_1 v_1 + .. + a_n v_n = 0$ ← a linear dependence

By lemma, can remove one of $v_1, .., v_n$ so new list of u_1, and $n-1$ of $v_1, .., v_n$ is a list that spans V.

Now repeat process: At j^{th} step

$$u_j \in \text{span}(u_1, u_2, .., u_{j-1}, \underbrace{v_{i_1}, .., v_{i_t}}_{n-(j-1)}) = V$$

So $u_1, .., u_{j-1}, u_j, v_{i_1}, .., v_{i_t}$ linearly dependent. Since $u_1, u_2, .., u_j$ are linearly independent, when we remove an element, we remove one of the v_i's..

So $u_1, .., u_j, \overbrace{v_{k_1}, v_{k_2}, .., v_{k_t}}^{n-j}$ spans V

At each step, we remove one v_i and add one u_j.

So $m \leq n$ $\boxtimes$

Consequence: In F^n, no list of $p > n$ vectors is linearly independent.

Proof: We know $(1,0,\dots,0), (0,1,0\dots0), \dots (0,\dots,0,1)$ span F^n. So any linearly independent list of vectors is $\leq n$. $\boxtimes$

Remark: In 1B03, proved this using pivots of a matrix (don't need matrices)

Thm Every subspace of a finite dim v.s is also finite dimensional.

Proof Let $U \subseteq V$ be a subspace.

If $U = \{0\}$, then $U = \text{span}\{0\}$ is fin dim ← so done!

Suppose $\{0\} \subsetneq U$ and U not finite dim.

So, we can find $u_1 \in U$ such that $U \neq \text{span}(u_1)$.

and for each $j \geq 1$, there is $u_j \in U \setminus \text{span}(u_1, u_2, .., u_{j-1})$

So

$$u_2 \notin \text{span}(u_1),\ u_3 \notin \text{span}(u_1, u_2),\ u_4 \notin \text{span}(u_1, u_2, u_3), \ldots$$

(note: all $u_1, u_2, u_3,$ are in $U \subseteq V$)

By independence lemma, $u_1, u_2, u_3, \ldots$ is linearly independent

But V is finite dim, so a finite set spans V.

We now have a contradiction to the previous result, i.e. there is an infinite list of elements that are linearly indep and there is a finite list of elements that span V.

So, U must be finite dimensional $\square$

<u>Next lecture</u> Bases $\Rightarrow$ Combine two ideas of span & linear independence

Key ideas
* linear independence & dependence
* linear independence lemma