$\boxed{\text{Lecture 24}}$ 8.C Characteristic and Minimal Polynomials

COMMENT – most results hold over $\mathbb{R}$, but we only prove case $F = \mathbb{C}$

Defn Let V be a v.s. over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues with multiplicities $d_1, \ldots, d_m$. Then

$$(z-\lambda_1)^{d_1}(z-\lambda_2)^{d_2}\cdots(z-\lambda_m)^{d_m}$$

is the characteristic polynomial of T

Remark In 1B03, used the determinant to matrix $A - \lambda I_n$ to define this equation. Can prove the two defns are the same (see chapter 10)

Lemma If $q(z)$ is the characteristic polynomial of $T \in \mathcal{L}(V)$, then

① $\deg q(z) = \dim V$

② roots of $q(z)$ = eigenvalues of T

Proof ① $\deg q(z) = d_1 + \cdots + d_m = \dim V$

② λ is a root of $q(z) = (z-\lambda_1)^{d_1} \cdots (z-\lambda_m)^{d_m}$
$\iff \lambda = \lambda_i$ for some i

$\square$

Thm (Cayley-Hamilton) Let $T \in \mathcal{L}(V)$ with characteristic polynomial $q(z)$.
Then

$$q(T) = 0$$

Proof We have

$$V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$$

with $d_i = \dim G(\lambda_i, T)$

For each i, $(T - \lambda_i I)$ is nilpotent on $G(\lambda_i, T)$, i.e

$$(T - \lambda_i I)^{d_i} v = 0 \quad \text{for } v \in G(\lambda_i, T)$$

Since $q(z) = (z-\lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$, we have

$$q(T) = (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m}$$

Need to show $[q(T)]v = 0$ for $v \in V$

Since $v \in V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$

$v = v_1 + v_2 + \cdots + v_m$ with $v_i \in G(\lambda_i, T)$

Since $[q(T)]$ is a linear map

$[q(T)]v = [q(T)]v_1 + \cdots + [q(T)]v_m$

Look $[q(T)]v_j$

$[q(T)]v_j = (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m} v_j$

$= (T - \lambda_1 I)^{d_1} \cdots (T - \lambda_m I)^{d_m} (T - \lambda_j I)^{d_j} v_j$

operators commute, so "push" $(T - \lambda_j I)^{d_j}$ to the end

Some operator

So $[q(T)]v_j = (*)(T - \lambda_j I)^{d_j} v_j = 0$

Thus $[q(T)]v = 0$ for all $v \in V$ $\square$

Ex (from 1B03)

Let $A = \begin{bmatrix} 6 & 7 \\ 2 & 4 \end{bmatrix}$

Char poly $= \det(A - \lambda I) = \begin{bmatrix} 6-\lambda & 7 \\ 2 & 4-\lambda \end{bmatrix}$

$$= (6-\lambda)(4-\lambda) - 14 = \lambda^2 - 10\lambda + 10$$

Cayley-Hamilton implies when we "evaluate" A, get zero matrix, i.e.

$$\begin{bmatrix} 6 & 7 \\ 2 & 4 \end{bmatrix}^2 - 10\begin{bmatrix} 6 & 7 \\ 2 & 4 \end{bmatrix} + 10\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Minimal Polynomial

Def$^{\underline{n}}$ A monic polynomial is a polynomial where highest degree coeffecient is 1, i.e

$$p(z) = \underline{1} \cdot z^n + a_{n-1} z^{n-1} + \dots + a_1 z + a_0$$

Thm Suppose $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p(z)$ of smallest degree such that $p(T) = 0$

Proof Let $n = \dim V$ so $\dim \mathcal{L}(V) = n^2$

Consider the operators

$$I, T, T^2, T^3, \dots, T^{n^2} \in \mathcal{L}(V)$$

Since $\dim \mathcal{L}(V) = n^2$, these n^2+1 operators must be linearly dependent. Let m be the smallest integer such that

$$b_0 I + b_1 T + b_2 T^2 + \dots + b_m T^m = 0 \quad \text{with } b_m \neq 0$$

$$\Rightarrow \left(b_0/b_m\right) I + \left(b_1/b_m\right) T + \dots + \left(\frac{b_{m-1}}{b_m}\right) T^{m-1} + T^m = 0$$

Set $p(z) = (b_0/b_m) + (b_1/b_m) z + \dots + \left(\frac{b_{m-1}}{b_m}\right) z^{m-1} + z^m$

Claim $p(z)$ is the desired polynomial

We have $p(T)=0$ and $p(z)$ monic, and by construction, $p(z)$ has smallest degree.

Suppose $r(z)$ also monic, same degree, and $r(T)=0$. If $r(z) \neq p(z)$, then $(r-p)(z)$ is a polynomial of degree $< \deg p(z)$ and $(r-p)(T)=0$. This is a contradiction. So $p(z)$ is unique $\square$

<u>Def^n</u> Given $T \in \mathcal{L}(V)$, the <u>minimal polynomial</u> of T is the unique polynomial of the above theorem.

<u>Cor</u> If p is a minimal polynomial of T, then

$$\deg p \leq \dim V$$

<u>Proof</u> Since char. poly $q(z)$ is a monic poly such that $q(T)=0$,

$$\deg p(z) \leq \deg q(z) \leq n = \dim V.$$

$\square$

Additional Properties

<u>Thm</u> Suppose $T \in \mathcal{L}(V)$ and $q(z) \in \mathcal{P}(F)$ such that $q(T) = 0$. Then minimal polynomial $p(z)$ divides $q(z)$, i.e. $q(z) = p(z)r(z)$ for some polynomial $r(z)$

<u>Cor</u> If $q(z)$ is the characteristic polynomial of $T \in \mathcal{L}(V)$, then the minimal poly $p(z)$ of T divides $q(z)$

<u>Thm</u> If $p(z)$ is minimal poly of $T \in \mathcal{L}(V)$, then $p(\lambda) = 0$ if and only if $\lambda = \lambda_i$ is an eigenvalue

<u>Cor</u> Suppose $\lambda_1, \ldots, \lambda_m$ are the distinct eigenvalues with d_i the multiplicity of λ_i. Then

$$p(z) = (z-\lambda_1)^{a_1}(z-\lambda_2)^{a_2} \cdots (z-\lambda_m)^{a_m}$$

with $1 \le a_i \le d_i$

<u>Note</u> do not know a_i's exactly

Ex $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Eigenvalue $\lambda_1 = 1$ of mult 1
$\lambda_2 = 2$ of mult 3

$\Rightarrow$ char. poly $q(z) = (z-1)^1(z-2)^3$

So the minimal polynomial is one of

$(z-1)(z-2)$ or $(z-1)(z-2)^2$ or $(z-1)(z-2)^3$

To figure out which one, see which polynomial has the prop $p(T)=0$

$$(A-I_4)(A-2I_4) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

not the mini poly

$$(A-I_4)(A-2I_4)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 \end{bmatrix}$$

So $(z-1)(z-2)^2$ is the mini poly

Key ideas * characteristic polynomial
* minimal polynomial
* Cayley-Hamilton Theorem