$\boxed{\text{Lecture 18}}$ 5.B Eigenvectors + upper triangular matrices

Fix a basis $v_1, \dots, v_n$ of V. If $T \in \mathcal{L}(V)$, then the matrix of T w.r.t. to the basis is the $n \times n$ matrix:

j^{th} column

$$\mathcal{M}(T) = \begin{bmatrix} A_{11} & \cdots A_{1j} & A_{1n} \\ A_{21} & \vdots & \vdots \\ \vdots & & \vdots \\ A_{n1} & A_{nj} & A_{nn} \end{bmatrix}$$

the coeffecients

where $Tv_j = A_{1j}v_1 + A_{2j}v_2 + \cdots + A_{nj}v_n$

GOAL: Pick $v_1, \dots, v_n$ so that $\mathcal{M}(T)$ is "simple"

lots of zeroes

Def$^{\underline{n}}$ A matrix is upper triangular if all entries below diagonal are 0

Ex
$$\begin{bmatrix} 2 & 7 & 8 \\ 0 & 0 & 9 \\ 0 & 0 & 10 \end{bmatrix} \qquad \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix}$$

0 = all zeroes
$*$ = any value

Today: If V is a vector space over $\mathbb{C}$, can find $v_1, \ldots, v_n$ such that $\mathcal{M}(T)$ is upper triangular

Thm Let $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ be a basis for V.
TFAE:

1. $\mathcal{M}(T)$ is upper triangular w.r.t. this basis
2. $Tv_j \in \operatorname{span}(v_1, \ldots, v_j)$ for $j = 1, \ldots, n$
3. $\operatorname{span}(v_1, \ldots, v_j)$ is invariant under T for $j = 1, \ldots, n$.

Ex $T \in \mathcal{L}(\mathbb{R}^2)$ $\quad T(x,y) = (3x+4y, 5y)$

Using standard basis $e_1 = (1,0)$ and $e_2 = (0,1)$

$T(e_1) = T(1,0) = (3,0) \in \operatorname{span}((1,0))$
$T(e_2) = T(0,1) = (4,5) \in \operatorname{span}((1,0),(0,1))$

So condition 2 is satisfied

So, $\mathcal{M}(T)$ is upper triangular w.r.t. e_1, e_2.
In fact

$$\mathcal{M}(T) = \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}$$

<u>Proof</u> $1 \Rightarrow 2$

$$\text{Given } M(T) = \begin{array}{c} \text{j}^{\text{th}} \text{ column} \\ \begin{pmatrix} A_{11} & & * \\ & \ddots A_{jj} \ddots & \\ 0 & & A_{nn} \end{pmatrix} \end{array}$$

j^{th} column records Tv_j,

This means

$$Tv_j = A_{1j} v_1 + \cdots + A_{jj} v_j + \underbrace{A_{j+1\,j}\, v_{j+1} + \cdots}_{0} + A_{nj} v_n$$

$$= A_{1j} v_1 + \cdots + A_{jj} v_j \in \operatorname{span}(v_1, \ldots, v_j)$$

$2 \Rightarrow 1$ Suppose $Tv_j \in \operatorname{span}(v_1, \ldots, v_j)$ for $j = 1, \ldots, n$

So

$$Tv_j = b_1 v_1 + b_2 v_2 + \cdots + b_j v_j$$
$$= b_1 v_1 + \quad\quad + b_j v_j + 0 v_{j+1} + \cdots + 0 \cdot v_n$$

So j^{th} column of $M(T)$ is:

$$\mathcal{M}(T) = \begin{array}{cc} & j^{\text{th}} \text{ column} \\ j \rightarrow & \begin{bmatrix} & b_1 & \\ & b_2 & \\ & \vdots & \\ & b_j & \\ & 0 & \\ & \vdots & \\ & 0 & \end{bmatrix} \end{array}$$

So $\mathcal{M}(T)$ is upper triangular since all entries below the diagonal are 0.

$3 \Longleftrightarrow 2$ (exercise) $\blacksquare$

<u>Thm</u> Suppose V is a fin. dim vector space over $\mathbb{C}$ and $T \in \mathcal{L}(V)$. Then V has a basis such that $\mathcal{M}(T)$ is upper triangular

<u>Proof</u> Do induction on $\dim V = n \geq 1$

If $\dim V = 1$, let v be a basis for V. (with $v \neq 0$) and $T \in \mathcal{L}(V)$. Suppose $Tv = cv$ for some $c \in F = \mathbb{C}$

Then $\mathcal{M}(T) = [\,c\,]$ ← an upper triangular matrix

Suppose $\dim V = n > 1$, and the result holds for all v.s. W with $1 \le \dim W < n$.

Since V is a v.s. over $\mathbb{C}$, then $T \in \mathcal{L}(V)$ has an eigenvalue λ. Consider the linear map $(T-\lambda I): V \to V$ given $(T-\lambda I)v = Tv - \lambda I v$
$= Tv - \lambda v$

Let $U = \text{range}(T-\lambda I)$. Because λ is an eigenvalue, $\dim \text{Nul}(T-\lambda I) \geq 1$

So $\dim U \leq n-1$ since
$\dim \text{Nul}(T-\lambda I) + \dim U = n$

The subspace $U \subseteq V$ is invariant under T. Indeed, if $u \in U$

$$Tu = Tu - \lambda u + \lambda u$$
$$= (T-\lambda I)u + \lambda u$$

since U is a v.s

Then $\lambda u \in U$ and $(T-\lambda I)u \in U = \text{range}(T-\lambda I)$. So $Tu \in U$

So $T|_U$ is a linear operator, i.e. $T|_U \in \mathcal{L}(U)$

By induction, there is a basis
$u_1, \ldots, u_m$ of U
such that $\mathcal{M}(T|_U)$ is upper triangular

$\iff$ $Tu_j = T|_U u_j \in \operatorname{span}(u_1, \ldots, u_j)$ for $j = 1, \ldots, m$

Extend $u_1, \ldots, u_m$ to a basis of V, say
$u_1, \ldots, u_m, v_1, \ldots, v_p$

For each k,

$$Tv_k = Tv_k - \lambda I v_k + \lambda v_k$$

$$= (T - \lambda I)v_k + \lambda v_k$$

Now $(T - \lambda I)v_k \in U = \operatorname{span}(u_1, \ldots, u_m)$. So
$Tv_k \in \operatorname{span}(u_1, \ldots, u_m, v_1, \ldots, v_k)$

So the basis $u_1, \ldots, u_m, v_1, \ldots, v_p$ of V satisfies prop 2 of thm on upper triangular matrices. I.e.
$\mathcal{M}(T)$ is upper triangular with respect to this basis

$\square$

NOTE Proof is an existence proof $\Rightarrow$ does not show us how to find $v_1, v_2, .., v_n$.

Consequences

Suppose we have a basis $v_1, .., v_n$ of V such that $\mathcal{M}(T)$ is upper triangular

What else do you get?

Thm Suppose $T \in \mathcal{L}(V)$ and $\mathcal{M}(T)$ is upper triangular w.r.t. basis $v_1, .., v_n$ of T

(A) T is invertible $\Leftrightarrow$ all entries on the diagonal of $\mathcal{M}(T)$ are nonzero

(B) Eigenvalues of T are precisely the diagonal entries of $\mathcal{M}(T)$

Proof (A) see text

(B) We have $\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & & * \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

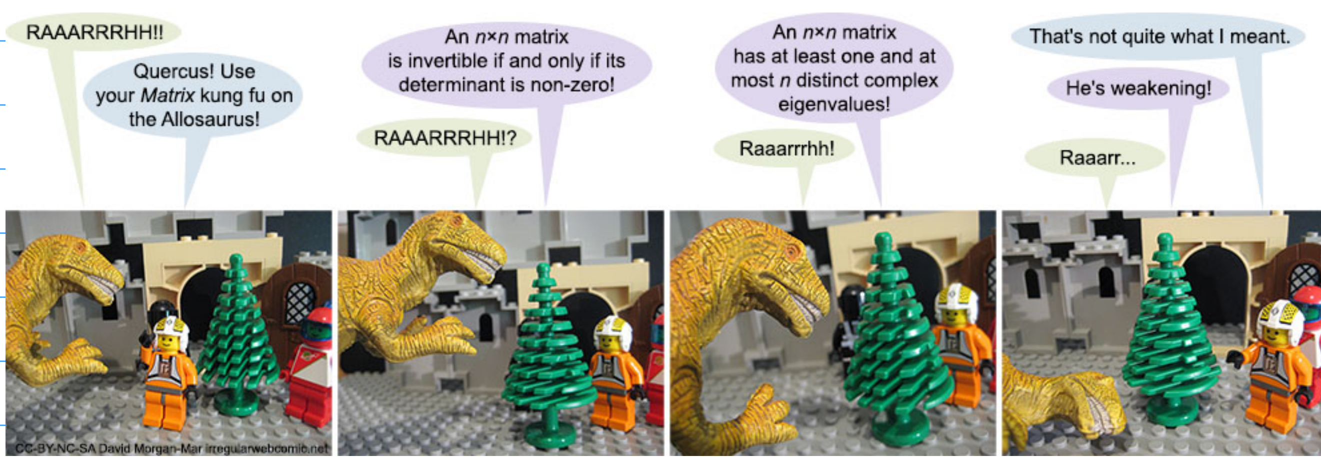
Let $\lambda \in F$. Then

$$M(T-\lambda I) = M(T) - M(\lambda I) = \begin{bmatrix} \lambda_1 - \lambda & & & * \\ & \lambda_2 - \lambda & & \\ & & \ddots & \\ 0 & & & \lambda_n - \lambda \end{bmatrix}$$

Then λ is eigenvalue $\Longleftrightarrow$ $T-\lambda I$ is not invertible

$\Longleftrightarrow$ $M(T-\lambda I)$ has a zero diagonal entry

$\Longleftrightarrow$ $\lambda = \lambda_i$ for some i $\square$



Key ideas:
* upper triangular matrices
* conditions for upper triangular
* If $F = \mathbb{C}$, can find basis for V so $M(T)$ upper triangular.