$\boxed{\text{Lecture 30}}$ 6.B Linear functionals and a worked out Example

Observation: F is an F-vector space. So, for any vector space V, $\mathcal{L}(V,F)$ makes sense

Def$^{\underline{n}}$ A linear functional on V is any element of $\mathcal{L}(V,F)$

Ex Let $V = F^3$ and define

$$\varphi : V \to F \quad \text{by} \quad \varphi((x,y,z)) = 3x + 12y - 2z$$

Note: If $\langle -,- \rangle$ denotes the standard Euclidean inner product on $F^3$, then

$$\varphi((x,y,z)) = 3x + 12y - 2z = \langle (x,y,z), (3,12,-2) \rangle$$

Ex Let $V = P_2(\mathbb{R})$ and define

$$\varphi : V \to \mathbb{R} \quad \text{by} \quad \varphi(p(x)) = \int_0^1 p(x)\,dx$$

So $$\varphi(a_0 + a_1 x + a_2 x^2) = \int_0^1 a_0 + a_1 x + a_2 x^2 \, dx$$

$$= a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} \Big|_0^1 = a_0 + \frac{a_1}{2} + \frac{a_2}{3}$$

Note Can define an inner product $\langle -,- \rangle$ on $P_2(\mathbb{R})$ by

$$\langle p(x), q(x) \rangle = \int_0^1 p(x) q(x)\,dx$$

So $$\varphi(p(x)) = \langle p(x), 1 \rangle = \int_0^1 p(x) \cdot 1 \, dx$$

(Reisz Representation Theorem) Let $V$ be a fin. dim. inner product space. If $\varphi$ is a linear functional, i.e. $\varphi \in \mathcal{L}(V, F)$, then there exists a unique $u \in V$ such that

$$\varphi(v) = \langle v, u \rangle \text{ for all } v \in V$$

exists by G.S.

Proof Let $e_1, \ldots, e_n$ be an orthonormal basis of $V$. Then

$$v = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \cdots + \langle v, e_n \rangle e_n \text{ for all } v \in V$$

So
$$\begin{aligned}
\varphi(v) &= \varphi(\langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n) \quad && \varphi \text{ is linear} \\
&= \langle v, e_1 \rangle \varphi(e_1) + \cdots + \langle v, e_n \rangle \varphi(e_n) && \text{prop of inner prod} \\
&= \langle v, \overline{\varphi(e_1)} e_1 \rangle + \cdots + \langle v, \overline{\varphi(e_n)} e_n \rangle && \text{"} \\
&= \langle v, \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n \rangle
\end{aligned}$$

Set $u = \overline{\varphi(e_1)} e_1 + \cdots + \overline{\varphi(e_n)} e_n$. So

$$\varphi(v) = \langle v, u \rangle \text{ for all } v \in V$$

To show $u$ is unique, suppose

$$\varphi(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle \quad \text{for some } u_1, u_2$$

So

$$\begin{aligned} 0 &= \langle v, u_1 \rangle - \langle v, u_2 \rangle \\ &= \langle v, u_1 - u_2 \rangle \qquad \text{for all } v \in V \end{aligned}$$

In particular, for $v = u_1 - u_2$,

$$0 = \langle u_1 - u_2, u_1 - u_2 \rangle \iff u_1 - u_2 = 0 \iff u_1 = u_2$$

$\blacksquare$

Very nice result! Proof gives procedure to find $u$

1. Find an orthonormal basis $e_1, \ldots, e_n$ for $V$ (via G-S)
2. Compute $\overline{\varphi(e_1)}, \ldots \overline{\varphi(e_n)}$
3. Return $u = \overline{\varphi(e_1)} e_1 + \overline{\varphi(e_2)} e_2 + \cdots + \overline{\varphi(e_n)} e_n$

## Worked out Example

Show that there is a polynomial $q(x) \in \mathcal{P}_1(\mathbb{R})$ such that

$$\int_0^1 p(x)q(x)\,dx = \int_0^1 p(x)\sin(\pi x)\,dx \text{ for all } p(x) \in \mathcal{P}_1(\mathbb{R})$$

(the power of Reisz Representation!)

Note: • $\mathcal{P}_1(\mathbb{R})$ is an inner product space with

$$\langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx$$

• The map $\varphi: \mathcal{P}_1(\mathbb{R}) \to \mathbb{R}$ given by

$$\varphi(p(x)) = \int_0^1 p(x)\sin(\pi x)\,dx \text{ is a linear functional}$$

hypotheses of Reisz Rep. Thm are met

Reisz Rep Thm $\Rightarrow$ exists some $q(x) \in \mathcal{P}_1(\mathbb{R})$ such that

$$\int_0^1 p(x)\sin(\pi x)\,dx = \varphi(p(x)) \underset{\text{Reisz Repres}}{=} \langle p(x), q(x)\rangle = \int_0^1 p(x)q(x)\,dx$$

We use the procedure to find $q(x)$

$\boxed{\text{Step 1}}$ Find an orthonormal basis of $\mathcal{P}_1(\mathbb{R})$

A basis for $\mathcal{P}_1(\mathbb{R})$ is $\{1, x\}$. Now apply Gram-Schmidt. Let $v_1 = 1$, $v_2 = x$

- $e_1 = \dfrac{v_1}{\|v_1\|} = 1$

- $e_2 = \dfrac{v_2 - \langle v_2, e_1\rangle e_1}{\|v_2 - \langle v_2, e_1\rangle e_1\|}$

Calculations: $\langle v_2, e_1\rangle = \langle x, 1\rangle = \int_0^1 x\,dx = \dfrac{x^2}{2}\Big|_0^1 = \dfrac{1}{2}$

So $v_2 - \langle v_2, e_1\rangle e_1 = x - \tfrac{1}{2}\cdot 1 = x - \tfrac{1}{2}$

$\|v_2 - \langle v_2, e_1\rangle e_1\| = \|x - \tfrac{1}{2}\| = \sqrt{\langle x - \tfrac{1}{2}, x - \tfrac{1}{2}\rangle}$

$= \sqrt{\int_0^1 (x - \tfrac{1}{2})^2\,dx} = \sqrt{\int_0^1 x^2 - x + \tfrac{1}{4}\,dx}$

$= \sqrt{\dfrac{x^3}{3} - \dfrac{x^2}{2} + \dfrac{1}{4}x\Big|_0^1} = \dfrac{1}{\sqrt{12}}$

So $e_2 = \dfrac{x - \tfrac{1}{2}}{1/\sqrt{12}} = \sqrt{12}\,(x - \tfrac{1}{2})$

Orthonormal basis of $\mathcal{P}_1(\mathbb{R})$ is $e_1 = 1$, $e_2 = \sqrt{12}\,(x - \tfrac{1}{2})$

$e_1 = 1$ $\qquad$ $e_2 = \sqrt{12}\left(x - \frac{1}{2}\right)$ ← orthonormal basis for $P_1(\mathbb{R})$

$\boxed{\text{Step 2}}$ Compute $\overline{\varphi(e_1)}, \overline{\varphi(e_2)}$

Note $F = \mathbb{R}$, $\overline{\varphi(e_1)} = \varphi(e_1)$, $\overline{\varphi(e_2)} = \varphi(e_2)$

$$\varphi(1) = \int_0^1 \sin(\pi x)\,dx = -\frac{\cos(\pi x)}{\pi}\Bigg|_0^1$$

$$= -\frac{\cos(\pi)}{\pi} + \frac{\cos(0)}{\pi} = \boxed{\frac{2}{\pi}}$$

$$\varphi\left(\sqrt{12}\left(x-\tfrac{1}{2}\right)\right) = \int_0^1 \sqrt{12}\left(x-\frac{1}{2}\right)\sin(\pi x)\,dx$$

$$= \sqrt{12}\underbrace{\int_0^1 x\sin(\pi x)\,dx}_{\downarrow} - \frac{\sqrt{12}}{2}\underline{\int_0^1 \sin(\pi x)\,dx}$$

$$= \sqrt{12}\left[-\frac{x\cos(\pi x)}{\pi} + \frac{\sin \pi x}{\pi}\Bigg|_0^1\right] - \frac{\sqrt{12}}{2}\left[-\frac{\cos \pi x}{\pi}\Bigg|_0^1\right]$$

$$= \sqrt{12}\left[\frac{-\cos(\pi)}{\pi} + 0\right] - \frac{\sqrt{12}}{2}\cdot\frac{2}{\pi}$$

$$= \sqrt{12}\left(\frac{-1(-1)}{\pi}\right) - \frac{\sqrt{12}}{\pi} = 0$$

So $\overline{\varphi(e_2)} = \varphi(e_2) = 0$

$\boxed{\text{Step 3}}$ Return . $u = \overline{\varphi(e_1)}e_1 + \overline{\varphi(e_2)}e_2$

$$u = \frac{2}{\pi}\cdot 1 + 0\cdot e_2 = \frac{2}{\pi}$$

Conclusion: For all $p(x) \in \mathcal{P}_1(\mathbb{R})$

$$\boxed{\int_0^1 p(x)\sin(\pi x)\,dx = \int_0^1 p(x)\cdot\frac{2}{\pi}\,dx}$$

Key ideas

* linear functionals
* Reisz Representation