$\boxed{\text{Lecture 25 - (Extra)}}$ Jordan Form
- worked out example

In last lecture I an example of Jordan Basis for $T \in \mathcal{L}(\mathbb{C}^4)$ where

$$T(x_1, x_2, x_3, x_4) = (10x_1 - 7x_2 + 15x_3 - 2x_4, \; 10x_2 + 5x_3 - 5x_4, \; -4x_2 + 25x_3 + x_4, \; 10x_2 - 5x_3 + 25x_4)$$

In this "extra" lecture, I want to show how I got this basis

W.R.T standard basis

$$\mathcal{M}(T) = \begin{bmatrix} 10 & -7 & 15 & -2 \\ 0 & 10 & 5 & -5 \\ 0 & -4 & 25 & 1 \\ 0 & 10 & -5 & 25 \end{bmatrix}$$

From Lecture 23:

eigenvalue $\lambda = 10$ with multiplicity 1
$\lambda = 20$ with multiplicity 3

So

$$\boxed{V = G(10, T) \oplus G(20, T)}$$

$\boxed{\lambda = 10}$

Since multiplicity of $\lambda = 10$ is 1

$$E(10,T) = G(10,T)$$

Using 1B03 methods, can show

$$E(10,T) = G(10,T) = \text{Null}\left(\begin{bmatrix} 0 & -7 & 15 & -2 \\ 0 & -10 & 5 & -5 \\ 0 & -4 & 15 & 1 \\ 0 & 10 & -5 & 15 \end{bmatrix}\right)$$

$$= \text{Null}\left(\mathcal{M}(T - 10I)\right)$$

$$= \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

$\boxed{\lambda = 20}$

Since multiplicity of $\lambda=20$ is 3,

$$\dim G(20,T) = 3$$

We need to find 3 basis elements of $G(20,T)$ (but we want "good" basis elements so that the associated matrix has Jordan form)

Start by finding a bases for

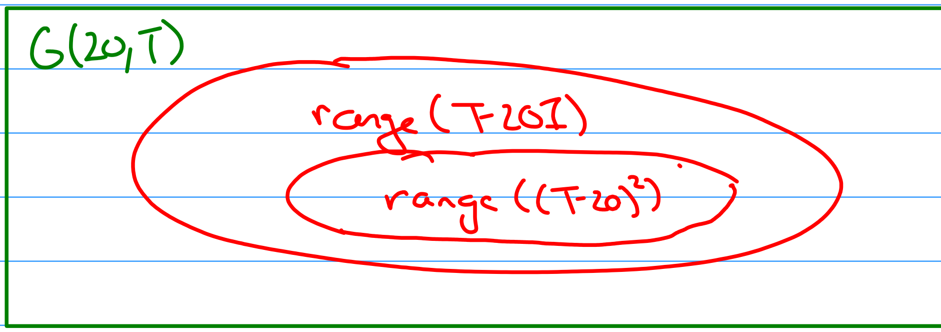
- $G(20,T) = \mathrm{Null}((T-20I)^3)$
- $\mathrm{range}((T-20I))$
- $\mathrm{range}((T-20I)^2)$

<u>Note</u> $G(20,T) \supseteq \mathrm{range}(T-20I) \supseteq \mathrm{range}((T-20I)^2) \supseteq \{0\}$

$$\underbrace{\dim \mathrm{range}((T-20I)^3)}_{0} + \underbrace{\dim \mathrm{Null}((T-20I)^3)}_{3} = \underbrace{\dim G(20,T)}_{3}$$

$\Rightarrow \dim \mathrm{range}((T-20I)^3) = 0$

Picture



Recall $T-20I$ is invariant on $G(20,T)$

A basis of $\text{Null}((T-20I)^3) = G(20,T)$

$$\text{span}\left(\begin{bmatrix} .3 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} .3 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right)$$

$$= \text{span}(u_1, u_2, u_3)$$

Basis of $\text{range}((T-20I))$ given by linear indep elements in

$$\text{span}\left(M(T-20I)u_1, M(T-20I)u_2, M(T-20I)u_3 \right)$$

$$= \text{span}\left(\begin{bmatrix} 4 \\ 10 \\ 4 \\ -10 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \\ -5 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -1 \\ -5 \end{bmatrix} \right) = \text{span}\left(\begin{bmatrix} 4 \\ 10 \\ 4 \\ -10 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 5 \\ -5 \end{bmatrix} \right)$$

$$\operatorname{span}\left(\begin{bmatrix}4\\10\\4\\-10\end{bmatrix}, \begin{bmatrix}5\\5\\5\\-5\end{bmatrix}\right) = \operatorname{span}(w_1, w_2)$$

Basis for range$((T-20I)^2)$ given by linearly independent elements of

$$\operatorname{span}(M(T-20I)w_1, M(T-20I)w_2)$$

$$= \operatorname{span}\left(\begin{bmatrix}-30\\-30\\-30\\30\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right) = \operatorname{span}\left(\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}\right)$$

We now "pull back" the bases.

<u>Step 1</u> Find an element in range$(T-20I)$ that maps to $\begin{bmatrix}1\\1\\1\\-1\end{bmatrix}$

$$\iff \text{solve } M(T-20I)x = \begin{bmatrix}1\\1\\1\\-1\end{bmatrix} \iff x = \begin{bmatrix}1/5\\0\\1/5\\0\end{bmatrix}$$

Step 2 Since $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \in \text{Range}((T-20I)^2) \subseteq \text{Range}(T-20I)$

and $\begin{bmatrix} 1/5 \\ 0 \\ 1/5 \\ 0 \end{bmatrix} \in \text{Range}(T-20I)$, use this as this

as beginning of basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 0 \\ 1/5 \\ 0 \end{bmatrix} \right\}$ of $\text{Range}(T-20I)$

NOTE In our case, $\dim \text{Range}(T-20I) = 2$
So our vectors are a basis for $\text{Range}(T-20I)$ because they are linearly independent.

If this had not been a basis, would need to extend these linearly indep elements to a basis of $\text{Range}((T-\lambda I)^d)$ such that new basis elements are mapped to zero by $(T-\lambda I)$.

<u>Step</u> 3 "Pull back" $\begin{bmatrix} 1/5 \\ 0 \\ 1/5 \\ 0 \end{bmatrix}$ to $G(20, T)$

$\Longleftrightarrow$ Solve $\mathcal{M}(T-20I)\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 0 \\ 1/5 \\ 0 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 17/300 \\ 1/30 \\ 2/30 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

Now

$\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 0 \\ 1/5 \\ 0 \end{bmatrix}, \begin{bmatrix} 17/300 \\ 1/30 \\ 2/30 \\ 0 \end{bmatrix}$ is linearly indep in $G(20, T)$

and $\dim G(20, T) = 3$

So this is a basis for $G(20, T)$.

If $v = \begin{bmatrix} 17/300 \\ 1/30 \\ 2/30 \\ 0 \end{bmatrix}$, then $\mathcal{M}(T-20I)v = \begin{bmatrix} 1/5 \\ 0 \\ 1/5 \\ 0 \end{bmatrix}$

$$\mathcal{M}((T-20)^2)v = \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Our basis has the form $v_1, \mathcal{M}(T-20I)v_1, \mathcal{M}(T-20I)^2 v_1$

This gives the desired basis for $G(20, T)$

Combine with basis for $G(10, T)$ gives

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1/5 \\ 0 \\ 1/5 \\ 0 \end{bmatrix}, \begin{bmatrix} 17/300 \\ 1/30 \\ 2/30 \\ 0 \end{bmatrix}$$

WRT this basis

$$M(T) = \begin{bmatrix} [10] & 0 & 0 & 0 \\ 0 & 20 & 1 & 0 \\ 0 & 0 & 20 & 1 \\ 0 & 0 & 0 & 20 \end{bmatrix}$$

$\boxed{\text{General Strategy}}$ (this is what happens in proof of Thm 8.55)

Given $u_1, u_2, \ldots, u_t \in \operatorname{range}((T-\lambda I)^{d})$

- solve $M(T-\lambda I)w_i = u_i$ for all i
- find linearly independent elements of

  $\{w_1, u_1, w_2, u_2, \ldots\ w_t, u_t\} \subseteq \operatorname{range}((T-\lambda I)^{d-1})$
- Now extend to $w_1, u_1, \ldots, w_t, u_t, z_1, \ldots, z_\ell$ to a basis
- Then let $N = (T-\lambda I)$
- $Nz_i = a_1u_1 + \ldots + a_tu_t$

  $= a_1Nw_1 + \ldots + a_tNw_t = N(a_1w_1 + \ldots + a_tw_t)$
- Replace z_i with $z_i - (a_1w_1 + \ldots + a_tw_t)$
- Still have a basis of $\operatorname{Range}((T-\lambda I)^{d-1})$, but $N(z_i - (a_1w_1 + \ldots + a_tw_t)) = 0$
- Now <u>repeat</u>.