Lecture 29 — 6.B Orthonormal Bases (and Gram-Schmidt)

In 1B03 you saw Gram-Schmidt. We extend to any inner product space

Orthonormal Vectors / Bases

Def^n A list of vectors $e_1, \dots, e_m$ is orthonormal if

- $\langle e_i, e_j \rangle = 0$ for all $i \neq j$
- $\|e_i\| = 1$ for all i

Ex Standard basis in F^n is orthonormal

(Orthonormal Prop) If $e_1, \dots, e_m$ are orthonormal, then

1. $\|a_1 e_1 + \dots + a_m e_m\|^2 = |a_1|^2 + |a_2|^2 + \dots + |a_m|^2$
2. $e_1, \dots, e_m$ are linearly independent

Proof 1. (case $m=2$)

$$\begin{aligned}\|a_1 e_1 + a_2 e_2\|^2 &= \|a_1 e_1\|^2 + \|a_2 e_2\|^2 \quad \text{(Pythagorean Thm)}\\ &= |a_1|^2 \|e_1\|^2 + |a_2|^2 \|e_2\|^2 \\ &= |a_1|^2 + |a_2|^2\end{aligned}$$

2\. Suppose $a_1e_1 + \cdots + a_me_m = 0$

Then

$$0 = \|0\|^2 = \|a_1e_1 + \cdots + a_me_m\|^2$$

$$= |a_1|^2 + \cdots + |a_m|^2$$

So $|a_1| = \cdots = |a_m| = 0 \Rightarrow a_i = 0$ $\square$

<u>Def^n</u> A basis of V is an <u>orthonormal</u> <u>basis</u> if the basis is also an orthonormal set

<u>Ex</u> $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ orthonormal basis in $\mathbb{R}^2$ with the standard inner product

(Orthonormal basis prop.)

Let $e_1, \ldots, e_n$ be an orthonormal basis of V

For all $v \in V$

1. $v = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \cdots + \langle v, e_n \rangle e_n$
2. $\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$

<u>Proof</u> 2 follows from 1 and previous result

1. Since $e_1, \ldots, e_n$ is a basis,

$$v = a_1 e_1 + \cdots + a_n e_n \quad \text{for some } a_1, \ldots, a_n$$

But then

$$\begin{aligned}
\langle v, e_i \rangle &= \langle a_1 e_1 + \cdots + a_n e_n, e_i \rangle \\
&= a_1 \langle e_1, e_i \rangle + \cdots + a_i \langle e_i, e_i \rangle + \cdots + a_n \langle e_n, e_i \rangle \\
&= a_i \langle e_i, e_i \rangle \qquad \text{since } \langle e_j, e_i \rangle = 0 \\
&= a_i
\end{aligned}$$

$\boxtimes$

<u>Consequence</u> Very easy to express v in terms of basis

We can find an orthonormal basis for <u>any</u> inner product space using Gram-Schmidt

$\Rightarrow$ in 1B03 only did this for $V = \mathbb{R}^n$ with dot-product

<u>Thm</u> (Gram-Schmidt) Suppose $v_1, \dots, v_m$ is a linearly independent list of vectors in V. Let

$$e_1 = \frac{v_1}{\|v_1\|} \quad \text{and for } j = 2, \dots, m$$

$$e_j = \frac{v_j - (\langle v_j, e_1\rangle e_1 + \dots + \langle v_j, e_{j-1}\rangle e_{j-1})}{\| v_j - (\langle v_j, e_1\rangle e_1 + \dots + \langle v_j, e_{j-1}\rangle e_{j-1}) \|}$$

Then $e_1, \dots, e_m$ is orthonormal and

$$\text{span}(e_1, \dots, e_j) = \text{span}(v_1, \dots, v_j) \text{ for } j = 1, \dots, m$$

<u>Note</u> different than 1B03 since we are making vectors of $\|e_i\| = 1$ at each step, but in 1B03 we first found orthogonal vectors

Proof Do induction on j

For $j=1$, $\|e_1\|=1$ and $\operatorname{span}(e_1)=\operatorname{span}(v_1)$
since e_1 is a scalar multiple of v_1

Assume we have found $e_1,\dots,e_{j-1}$ orthonormal such that

$$\operatorname{span}(e_1,\dots,e_{j-1}) = \operatorname{span}(v_1,\dots,v_{j-1})$$

Since $v_j \notin \operatorname{span}(e_1,\dots,e_{j-1}) = \operatorname{span}(v_1,\dots,v_{j-1})$

$$\|v_j - (\langle v_j,e_1\rangle e_1 + \dots + \langle v_j,e_{j-1}\rangle e_{j-1})\| \neq 0$$

denominator of e_j $\neq 0$

So $\|e_j\|=1$ (i.e, has correct norm)

$(*)$ = constant

Furthermore

$$\langle e_j, e_k\rangle = \langle \underbrace{v_j - (\langle v_j,e_1\rangle e_1 + \dots + \langle v_j,e_{j-1}\rangle e_{j-1}}_{(*)}, e_k\rangle$$

$$= \langle \underbrace{v_j}_{(*)}, e_k\rangle - \langle \langle \underbrace{v_j, e_k}_{(*)}\rangle e_k, e_k\rangle$$

$$= \langle \underset{*}{\underline{v_j}}, e_h \rangle - \langle \underset{*}{\underline{v_j}}, e_h \rangle \langle e_k, e_h \rangle = 0$$

So $e_1, \dots, e_j$ orthonormal

Note that $v_j \in \operatorname{span}(e_1, \dots, e_j)$ by rearranging the definition of e_j

So $\operatorname{span}(v_1, \dots, v_j) \subseteq \operatorname{span}(e_1, \dots, e_j)$

Both sides have same dimension since $v_1, \dots, v_j$ and $e_1, \dots, e_j$ are linearly independent.

Thus
$$\operatorname{span}(v_1, \dots, v_j) = \operatorname{span}(e_1, \dots, e_j)$$

$\square$

$\boxed{\text{Consequences}}$

Cor Every finite dimensional inner product space has an orthonormal basis

Proof Apply G-S to any basis of V.

Cor Every orthonormal set of vectors can be extended to an orthonormal basis of V

Proof Say $e_1, .., e_m$ orthonormal, and extend to basis

$$e_1, .., e_m, f_1, .., f_\ell$$

Apply G-S to this list. By construction G-S return $e_1, .., e_m, g_1, .., g_m$ ∎

Thm Let $T \in \mathcal{L}(V)$ and suppose V has a basis such that $\mathcal{M}(T)$ is upper triangular. Then V has an orthonormal basis such that $\mathcal{M}(T)$ upper triangular

Proof (Idea) Apply the G-S to the given basis and show new orthonormal basis has prop that $\mathcal{M}(T)$ is upper triangular.

Thm (Schur's Theorem) If $T \in \mathcal{L}(V)$ and $\mathbb{F} = \mathbb{C}$, then there is an orthonormal basis of V such that $\mathcal{M}(T)$ is upper triangular

<u>Proof</u> By Thm 5.27, over $\mathbb{C}$ exists a basis such that $\mathcal{M}(T)$ is upper triangular. Now apply the previous result.

<u>Remark</u> (Schur's Thm also holds over $\mathbb{R}$, but needs different approach)

Key ideas
* orthonormal
* Gram-Schmidt
* Schur's Theorem