$\boxed{\text{Lecture 11}}$ 3.B Null spaces and range
3.C matrices

Last time: (Fundamental Thm of Linear Maps)
Suppose V is finite dimensional. If $T \in \mathcal{L}(V,W)$, then
$$\dim V = \dim \operatorname{Null}(T) + \dim \operatorname{range}(T)$$

Some consequences

<u>Thm</u> Let V, W be finite dim. vector spaces

(A) If $\dim V > \dim W$, then no $T \in \mathcal{L}(V,W)$ injective

(B) If $\dim V < \dim W$, then no $T \in \mathcal{L}(V,W)$ is surjective

<u>Proof</u> (A) note $\operatorname{range}(T) \subseteq W$. So
$\dim \operatorname{range}(T) \leq \dim W$

Then

$$\dim \operatorname{Null}(T) = \dim V - \dim \operatorname{rang}(T) \geq \dim V - \dim W > 0$$

So $\operatorname{Null}(T) \neq \{0\} \Rightarrow T$ is not injective.

(B) $\dim \operatorname{range}(T) = \dim V - \dim \operatorname{Null}(T)$.
$\leq \dim V < \dim W$

So $\operatorname{range}(T) \subsetneq W \Rightarrow T$ is not surjective. $\boxtimes$

Relation to 1B03: Any homogeneous SLE

$$a_{11}x_1 + \cdots + a_{1n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + \cdots + a_{mn}x_n = 0$$

with $n > m$ has a nontrivial soln

Why? In 1B03, coefficient matrix

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

has more columns than rows, it has a column corresponding to a free variable

But A defines linear map $T: \mathbb{R}^n \to \mathbb{R}^m$ with $Tx = Ax$ and $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. Since $n > m \iff \dim \mathbb{R}^n > \dim \mathbb{R}^m$ we have T is not injective, so this implies $\operatorname{Nul}(T) = \operatorname{Nul}(A) \neq \{0\}$. I.e a nontrivial soln to $Ax = 0$

Matrices

Matrices naturally arise when we study linear maps:

Notation A is $m\times n$ matrix $\Leftrightarrow$ m rows and n columns

$A_{ij} \Leftrightarrow$ entry in row i and column j

$A_{\cdot j} \Leftrightarrow$ all entries in column j

$A_{i\cdot} \Leftrightarrow$ all entries in row i

Def$^{\underline{n}}$ Suppose $T\in\mathcal{L}(V,W)$ and $v_1,..,v_n$ is a basis for V and $w_1,..,w_m$ is a basis for W. The matrix of T with respect to these bases is the $m\times n$ matrix $M(T)$ where entry A_{ij} satisfies

$$Tv_j = A_{1j}w_1 + A_{2j}w_2 + \cdots + A_{ij}w_i + \cdots + A_{mj}w_m$$

elements in F

basis for W

Set up $\quad T \in \mathcal{L}(V,W)$

$V = \text{span}(v_1, \ldots, v_n)$ and $W = \text{span}(w_1, \ldots, w_m)$

"Picture"

$$Tv_1 = \underline{A}_{11} w_1 + \underline{A}_{21} w_2 + \cdots + \underline{A}_{m1} w_m$$

$$Tv_2 = A_{12} w_1 + A_{22} w_2 + \cdots + A_{m2} w_m$$

$$\vdots$$

$$Tv_n = A_{1n} w_1 + \cdots + A_{mn} w_n$$

column k contains the coeffecients of Tv_k

So

$$\mathcal{M}(T) = \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & & v_n \end{array} \\ \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & & \vdots \\ \vdots & & & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \begin{array}{c} w_1 \\ w_2 \\ \vdots \\ w_m \end{array} \end{array}$$

$m \times n$

"1B03 Flashback" If $V = \mathbb{R}^n$, $W = \mathbb{R}^m$ with standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, and $T: \mathbb{R}^n \to \mathbb{R}^m$, then

$$\mathcal{M}(T) = [T(e_1)\ T(e_2)\ \cdots\ T(e_n)] \leftarrow \text{standard matrix}$$

<u>Ex 1</u> Suppose $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ is given by

$$T(x,y,z) = (x+2y+3z,\ 4x+5z)$$

Let e_1, e_2, e_3 be standard basis of $\mathbb{R}^3$
e_1', e_2' be " " " $\mathbb{R}^2$

$$Te_1 = T(1,0,0) = (1,4) = 1\cdot(1,0)+4(0,1) = 1\cdot e_1' + 4e_2'$$
$$Te_2 = T(0,1,0) = (2,0) = \qquad = 2\cdot e_1' + 0\cdot e_2'$$
$$Te_3 = T(0,0,1) = (3,5) = 3(1,0)+5(0,1) = 3\cdot e_1' + 5\cdot e_2'$$

$$M(T) = \begin{array}{cc} & \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix} \begin{array}{c} e_1' \\ e_2' \end{array} \end{array}$$

<u>Ex</u> $D \in \mathcal{L}(\mathcal{P}_3(\mathbb{R}), \mathcal{P}_2(\mathbb{R}))$ be the differentiation

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$$

Use basis $\{1, x, x^2, x^3\}$ and $\{1, x, x^2\}$ for $\mathcal{P}_3(\mathbb{R})$ and $\mathcal{P}_2(\mathbb{R})$

$$D(1) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$D(x) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2$$
$$D(x^2) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2$$
$$D(x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2$$

$$\mathcal{M}(D) = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccc} 1 & x & x^2 & x^3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ x \\ x^2 \end{array}$$

$\boxed{\text{Matrix Operations \& Linear Maps}}$

<u>Thm</u> Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$

(A) $\mathcal{M}(S+T) = \mathcal{M}(S) + \mathcal{M}(T)$ (B) $\mathcal{M}(\lambda S) = \lambda \mathcal{M}(S)$

this is the matrix of $S+T \in \mathcal{L}(V, W)$

sum of the matrices of S and T

matrix of the linear map λS

scalar multiple of matrix $\mathcal{M}(S)$

Def^n $F^{m,n}$ ← all $m \times n$ matrices w/ coeff in F

Thm $F^{m,n}$ is a vector space with $\dim F^{m,n} = mn$

Proof Saw this in 1B03. Recall that a basis is the set of matrices

$$B_{ij} = \begin{array}{c c} & j \\ i & \begin{bmatrix} 0 & 0 & 0 \\ \cdots 0 \cdots & 1 & \cdots 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \quad \leftarrow 1 \text{ in spot } (i,j)$$

There are mn such matrices $\square$

Matrix Multiplication ← I'm assuming you remember how to multiply matrices

Thm If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, then

$$\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$$

matrix of $ST \in \mathcal{L}(U,W)$ (the composition)

product of the two corresponding matrices.

Remarks:

① The defⁿ of matrix multiplication comes from this result. We define matrix mult so that this result is true.

② Note $\mathcal{M}(S)\mathcal{M}(T)$ defined. If $\dim U = n$, $\dim V = p$, $\dim W = q$, then

$\mathcal{M}(S)$ is $q \times p$ and $\mathcal{M}(T) = p \times n$

So $\mathcal{M}(S)\mathcal{M}(T)$ is a $q \times n$ matrix.

Key ideas:
* matrix associated to $T \in \mathcal{L}(U, W)$
* matrix operations ↔ operations on linear maps