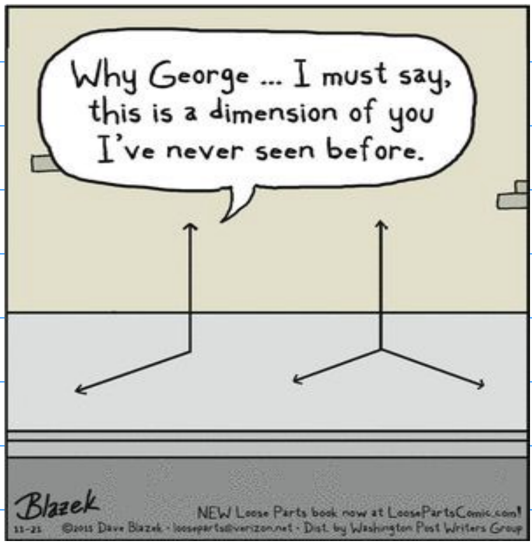
Lecture 8 2.C Dimension



Goal: Introduce dimension
"size" of a vector space

Thm Let V be a fin. dim. v.s. If $v_1, \dots, v_p$ and $w_1, \dots, w_r$ are basis of V, then $p = r$

Proof: Since $v_1, \dots, v_p$ is linearly indep and because $V = \operatorname{span}(w_1, \dots, w_r)$, then $p \le r$. By the same arg, $r \le p$ ∎

Consequence: All bases of V have the same size!

Def^n The dimension of a fin. dim. v.s. V is the length of any basis of V, denoted $\dim V$

<u>Ex</u>
- $\dim F^n = n$
- $\dim \{0\} = 0$ since $\operatorname{span}() = \{0\}$
- $\dim \mathcal{P}_m(F) = m+1$ since $1, z, \ldots, z^m$ is a basis

<u>Thm</u> If $U \subseteq V$ is a subspace of V, then
$$\dim U \leq \dim V$$

<u>Proof</u> Let $u_1, \ldots, u_t$ be a basis of U. Then $u_1, \ldots, u_t$ is linearly indep in V. Let $w_1, \ldots, w_p$ be a basis for V, (so $p = \dim V$). Then $\dim U = t \leq p = \dim V$. $\boxtimes$

Connecting dim-span-linear indep.

<u>Thm</u> Suppose $\dim V = n$. If $v_1, \ldots, v_n$ is a linearly independent list in V, then $v_1, \ldots, v_n$ is also a basis of V.

Proof Need to show $V = \text{span}(v_1, .., v_n)$. Can extend $v_1, .., v_n, u_1, .., u_t$ to a basis of V. Since all bases of V have the same length, and because $\dim V = n$, we have $n = n+t \iff t = 0$. So $v_1.., v_n$ is a spanning, i.e it is a basis $\boxtimes$

Thm Suppose $\dim V = n$. If $v_1, .., v_n$ is a spanning set of V, then $v_1, .., v_n$ is a basis of V.

Proof Since $v_1, .., v_n$ spans V, some subset of $v_1, .., v_n$ is a basis of V. Since $\dim V = n$, a basis must have n elements. So a subset of size n of $v_1, .., v_n$ is a basis of V. But the only subset of size n is $v_1, .., v_n$ $\boxtimes$

Ex $\{(1,2), (2,3)\}$ in F^2. We know $\dim F^2 = 2$ and $(1,2)$ and $(2,3)$ are linearly indep.

Thus $\{(1,2), (2,3)\}$ is a basis for F^2

CAUTION dimension will depend upon F

Consider

$$\mathbb{C}' = \{a+bi \mid a,b \in \mathbb{R}\}.$$

This is a vector space over $\mathbb{C}$ and $\mathbb{R}$.

As a vector space over $\mathbb{C}$, scalar mult:

$$(a+bi)(c+di) = (ac-bd) + (bc+ad)i$$

As a vector space over $\mathbb{R}$, scalar mult:

$$t(c+di) = (ct) + (dt)i$$

subscript to track of scalars

Over $\mathbb{C}$ $\quad \mathbb{C}' = \operatorname{span}_{\mathbb{C}}(1) = \{c \cdot 1 \mid c \in \mathbb{C}\}$

Over $\mathbb{R}$ $\quad \mathbb{C}' = \operatorname{span}_{\mathbb{R}}(1,i) = \{c \cdot 1 + d \cdot i \mid c,d \in \mathbb{R}\}$

So $\dim_{\mathbb{R}} \mathbb{C}' = 2$ but $\dim_{\mathbb{C}} \mathbb{C}' = 1$

to keep track of coefficients

Note $\dim_F F = 1$ since $F = \operatorname{span}_F\{1\}$

Thm If $U_1, U_2 \subseteq V$ of fin. dim v.s V, then

$$\dim(U_1+U_2) = \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)$$

Proof (main ideas).

Let $m = \dim(U_1 \cap U_2)$. So
$U_1 \cap U_2 = \operatorname{span}(u_1, .., u_m)$

Extend $u_1 .., u_m$ to a basis of U_1 and U_2, i.e

$U_1 = \operatorname{Span}(u_1, .., u_m, v_1, .., v_p)$

$U_2 = \operatorname{Span}(u_1, .., u_m, w_1, .., w_r)$

Claim $u_1, .., u_m, v_1 .., v_p, w_1 .., w_r$ is a basis $U_1 + U_2$

The result follows from the claim:

$$\begin{aligned}\dim(U_1+U_2) &= m + p + r \\ &= 2m + p + r - m \\ &= (m+p) + (m+r) - m \\ &= \dim U_1 + \dim U_2 - \dim(U_1 \cap U_2)\end{aligned}$$

See the book for details of the claim $\boxtimes$

<u>Cor</u> Suppose $V = U \oplus W$. Then
$$\dim V = \dim U + \dim W.$$

<u>Proof</u>: $V = U \oplus W \iff V = U + W$ <u>and</u> $U \cap W = \{0\}$. So

So $\dim V = \dim(U+W)$
$= \dim U + \dim W - \dim U \cap W$
$= \dim U + \dim W$ $\boxtimes$

Ex Let $U = \{p \in \mathcal{P}_4(\mathbb{R}) \mid \int_{-1}^{1} p(x)\,dx = 0\}$

Find a basis for U.

Note $\dim U < \dim \mathcal{P}_4(\mathbb{R}) = 5$ since $1 \notin U$.

$$\int_{-1}^{1} 1\,dx = x \Big|_{-1}^{1} = 1-(-1) = 2 \neq 0$$

(If $\dim U = 5$, then $U = \mathcal{P}_4(\mathbb{R})$ which is false)

Note that:

$$\int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1} = \frac{(1)^2}{2} - \frac{(-1)^2}{2} = 0$$

$$\int_{-1}^{1} \left(\frac{1}{3} - x^2\right) dx = \frac{1}{3}x - \frac{x^3}{3}\Big|_{-1}^{1} = 0$$

$$\int_{-1}^{1} x^3\,dx = \frac{x^4}{4}\Big|_{-1}^{1} = 0$$

$$\int_{-1}^{1} \left(\frac{1}{5} - x^4\right) dx = \frac{1}{5}x - \frac{x^5}{5}\Big|_{-1}^{1} = \left(\frac{1}{5} - \frac{1}{5}\right) - \left(\frac{-1}{5} - \frac{(-1)^5}{5}\right) = 0$$

So $x,\ \frac{1}{3} - x^2,\ x^3,\ \frac{1}{5} - x^4 \in U$

These elements are linearly independent in U (all have different degrees) [CHECK!]

So $4 = \dim \operatorname{span}(x, \tfrac{1}{3} - x^2, x^3, \tfrac{1}{5} - x^4) \leq \dim U \leq 4$

Hence $\dim \operatorname{span}(x, \dots, \frac{1}{5} - x^4) = \dim U = 4$

$$\Rightarrow \underbrace{\operatorname{span}\left(x, \frac{1}{3} - x^2, x^3, \frac{1}{5} - x^4\right)}_{\text{this is a basis for } U} = U$$

Key Ideas
* dimension
* relationship between dim, span and linear independence.