Lecture 26   6.A Inner Product Spaces

From 1B03/2LA3:

dot product: if $\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ in $\mathbb{R}^n$

then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$

norm of $\vec{v} \in \mathbb{R}^n$: $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

In Chapter 6, we introduce INNER PRODUCT SPACES
i.e., vector spaces w/ operations like a dot product
so we can define norm

Inner Products

Def^n An inner product on V is a function that takes an ordered pair (u, v) with $u, v \in V$ and maps this pair to a number $\langle u, v \rangle \in \mathbb{F}$ such that

1. (positivity) $\langle v, v \rangle \geq 0$ ← i.e $\langle v, v \rangle$ is a real number and positive
2. (definiteness) $\langle v, v \rangle = 0 \iff v = 0$
3. (addition in first slot)
$$\langle v + w, u \rangle = \langle v, u \rangle + \langle w, u \rangle$$
4. (homogenity in first slot)
$$\langle \lambda v, u \rangle = \lambda \langle v, u \rangle \text{ for } \lambda \in \mathbb{F}$$
5. (conjugate similarity)
$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$ ← complex conjugate

NOTES 1 If $F = \mathbb{R}$, last condition is

$$\langle u, v \rangle = \langle v, u \rangle$$

2 Why we need #5, i.e., why $\langle u, v \rangle \neq \langle v, u \rangle$ if $F = \mathbb{C}$

If $\langle u, v \rangle = \langle v, u \rangle$, we would have

$$0 < \langle iu, iu \rangle \text{ by \#2 and \#1 for } u \neq 0$$

$$= i\langle u, iu \rangle \text{ by \#4}$$

$$= i\langle iu, u \rangle \text{ (if we allowed } \langle u, v \rangle = \langle v, u \rangle)$$

$$= i^2\langle u, u \rangle = -1\langle u, u \rangle < 0$$ A contradiction

So $\langle u, v \rangle \neq \langle v, u \rangle$ if $F = \mathbb{C}$.

Examples of inner products

Ex 1 From 1B03, the dot product in $\mathbb{R}^n$ is an inner product

$$\langle \vec{u}, \vec{v} \rangle = u_1v_1 + u_2v_2 + \dots + u_nv_n = \vec{u} \cdot \vec{v}$$

Ex 2 EUCLIDEAN INNER PRODUCT on F^n is

$$\langle (u_1, \dots, u_n), (v_1, \dots, v_n) \rangle = u_1\overline{v_1} + u_2\overline{v_2} + \dots + u_n\overline{v_n}$$

Note If $F = \mathbb{R}$, then this is the same def^n as Ex 1

Ex 3 If $p, q \in \mathcal{P}(\mathbb{R})$, then

$$\langle p, q \rangle = \int_{-1}^{1} (p(x)q(x))\,dx$$

Proof Check the conditions:

1. $\langle p, p \rangle = \int_{-1}^{1} (p(x))^2\,dx \geq 0$ since $p(x)^2 \geq 0$ for all x

(i.e. area under $p(x)^2$ is positive on $[-1, 1]$)

2. $\langle p, p \rangle = 0 \iff \int_{-1}^{1} (p(x))^2\,dx = 0 \iff p(x) = 0$

some real analysis

3. $\langle p+q, r\rangle = \int_{-1}^{1} (p(x)+q(x))r(x)\,dx$

a prop from calculus

$= \int_{-1}^{1} (p(x)r(x) + q(x)r(x))\,dx = \int_{-1}^{1} p(x)\,r(x)\,dx + \int_{-1}^{1} q(x)r(x)\,dx$

$= \langle p, r\rangle + \langle q, r\rangle$

4. $\langle \lambda p, q\rangle = \int_{-1}^{1} (\lambda p(x) q(x))\,dx$

$= \lambda \int_{-1}^{1} (p(x) q(x))\,dx = \lambda \langle p, q\rangle$

5. $\langle p, q\rangle = \int_{-1}^{1} (p(x) q(x))\,dx = \int_{-1}^{1} q(x)\,p(x)\,dx$

$= \langle q, p\rangle$

$\boxed{\text{Inner Product Spaces}}$

Def$^{\underline{n}}$ A vector space V with an inner product is an inner product space

Ex F^n is an inner product space

(Basic Prop) Let V be an inner product space:

1. If we fix $u \in V$, then the map $T: V \to F$ given by
$$T(v) = \langle v, u \rangle$$
is a linear map

2. $\langle 0, u \rangle = 0$ for any $u \in V$

3. $\langle u, 0 \rangle = 0$ for any $u \in V$

4. $\langle u, v+w \rangle = \langle u, v \rangle + \langle u, w \rangle$

5. $\langle u, \lambda v \rangle = \overline{\lambda} \langle u, v \rangle$

complex conjugate

Proof (of 4)

$$\langle u, v+w \rangle = \overline{\langle v+w, u \rangle}$$

$$= \overline{\langle v, u \rangle + \langle w, u \rangle}$$

$$= \overline{\langle v, u \rangle} + \overline{\langle w, u \rangle}$$

$$= \langle u, v \rangle + \langle w, u \rangle \qquad \square$$

gives a "flavor" of the type of proofs. "Flip" around so we can look at the first coordinate

$\boxed{\text{Norm and Orthogonality}}$

Def$^{\underline{n}}$ If $v \in V$, norm of v is

$$\|v\| = \sqrt{\langle v, v \rangle}$$

Ex Consider $\mathcal{P}(\mathbb{R})$ with inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x) \, dx$$

If $p \in \mathcal{P}(\mathbb{R})$

$$\|p(x)\| = \sqrt{\int_{-1}^{1} (p(x))^2 \, dx} = \sqrt{\langle p, p \rangle}$$

Def$^{\underline{n}}$ $u, v \in V$ are orthogonal if $\langle u, v \rangle = 0$

(Properties of norm and orthogonality)

① $\|v\| = 0 \iff v = 0$

② $\|\lambda v\| = |\lambda| \, \|v\|$ where $|\lambda| = \sqrt{a^2+b^2}$ if $\lambda = a+bi$

③ u is orthogonal to itself if and only if $u = 0$
i.e. $\langle u, u \rangle = 0 \iff u = 0$

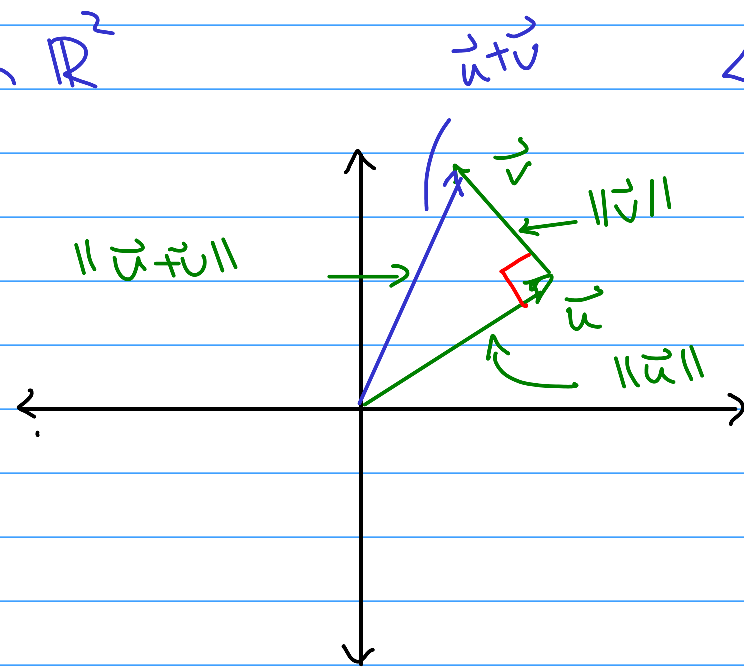
Proof TEXT

(Pythagorean Thm) Suppose $u, v \in V$ an inner prod space. with u, v orthogonal. Then

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

Picture in $\mathbb{R}^2$

$\langle u, v \rangle = \vec{u} \cdot \vec{v}$.
$= u_1 v_1 + u_2 v_2$



Proof $\|u+v\|^2 = \langle u+v, u+v\rangle$ (def^n of norm)

$$= \langle u+v, u\rangle + \langle u+v, v\rangle$$

$$= \langle u,u\rangle + \langle v,u\rangle + \langle u,v\rangle + \langle v,v\rangle$$

$$= \|u\|^2 + \langle v,u\rangle + \langle u,v\rangle + \|v\|^2$$

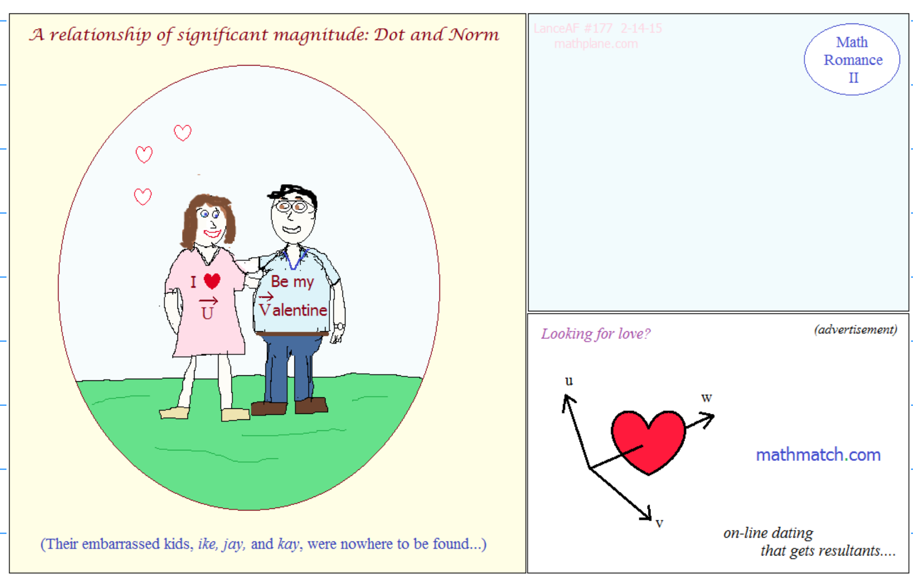
But $\langle u,v\rangle = 0$ so $0 = \langle u,v\rangle = \overline{\langle v,u\rangle} = 0$

So $\langle v,u\rangle = 0$. So

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2 \qquad \square$$

Key ideas

* inner products
* inner product spaces
* norm
* orthogonal



(Their embarrassed kids, *ike*, *jay*, and *kay*, were nowhere to be found...)