$\boxed{\text{Lecture 13}}$ 3.D Invertibility and Isomorphisms II

Last time: • Two finite dimensional vector spaces V and W isomorphic $\iff \dim V = \dim W$

• Given $T \in \mathcal{L}(V,W)$ with a basis $v_1, \dots, v_n$ and $w_1, \dots, w_m$,

$$\mathcal{M}(T) = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & & \vdots \\ A_{m1} & & A_{mn} \end{bmatrix} \quad \text{with } Tv_j = A_{1j}w_1 + \cdots + A_{mj}w_m$$

This gives a map:

$$\mathcal{M} : \mathcal{L}(V,W) \longrightarrow F^{m,n} \quad \leftarrow \text{all } m \times n \text{ matrices}$$

$$T \longmapsto \mathcal{M}(T)$$

Thm $\mathcal{M}$ is an isomorphism, i.e $\mathcal{L}(V,W)$ is isomorphic to $F^{m,n}$.

Proof: $\mathcal{M}$ is a linear map:

$$\mathcal{M}(T+S) = \mathcal{M}(T) + \mathcal{M}(S) \quad \text{and} \quad \mathcal{M}(\lambda S) = \lambda \mathcal{M}(S)$$

Need to check that $\mathcal{M}$ is injective and surjective (left for you!)

Cor $\dim \mathcal{L}(V,W) = (\dim V)(\dim W)$

Proof $\dim \mathcal{L}(V,W) = \dim F^{m,n} = mn = (\dim V)(\dim W)$

$\square$

"BIG IDEA" $\mathcal{L}(V,W)$ is the "same" as the set of $m \times n$ matrices.

$\boxed{\text{Linear maps as matrix multiplication}}$

Def$^{\underline{n}}$ Let $v \in V$ with $v_1, \ldots, v_n$ a basis for V. The matrix of v is

$$\mathcal{M}(v) = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \quad \text{where } v = c_1 v_1 + \cdots + c_n v_n$$

Ex 1 Let $v = 4 + 3x + 17x^2 \in P_2(\mathbb{R})$ with basis $\{1, x, x^2\}$. Then

$$\mathcal{M}(v) = \begin{bmatrix} 4 \\ 3 \\ 17 \end{bmatrix} \quad \text{Since } v = 4 \cdot \underline{1} + 3 \cdot \underline{x} + 17 \cdot \underline{x^2}$$

Ex2 Consider basis $\{(1,1),(1,2)\}$ of F^2 and let $(3,4) \in F^2$.

$$(3,4) = 2(1,1) + 1(1,2).$$

Then $M((3,4)) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Ex3 Consider standard basis $\{e_1, \ldots, e_n\}$ of F^n.
For any $v = (a_1, \ldots, a_n) \in F^n$

$$v = a_1 e_1 + a_2 e_2 + \cdots + a_n e_n$$

So $M(v) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

Remark · we view elements of F^n as "rows", but $M(v)$ allows us to view elements as "columns", or as $n \times n$ matrices

· Have an isomorphism

$$M : V \longrightarrow F^{n,1} \quad \leftarrow n \times 1 \text{ matrices}$$

$$v \longmapsto M(v) \qquad \text{if } \dim V = n$$

Thm Let $T \in \mathcal{L}(V,W)$ and $v \in V$. Fix a basis $v_1, \ldots, v_n$ for V and $w_1, \ldots, w_m$ for W. Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$$

matrix of the vector Tv in W $(m \times 1)$

matrix of linear map $(m \times n)$

matrix of $v \in V$ $(n \times 1)$

Ex Let $V = \mathcal{P}_3(\mathbb{R})$ with basis $1, x, x^2, x^3$

$W = \mathcal{P}_2(\mathbb{R})$ with basis $1, x, x^2$

and $D \in \mathcal{L}(V,W)$ given by

$$D(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = a_1 + 2a_2 x + 3a_3 x^2$$

Previous lecture

$$\mathcal{M}(D) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Let $v = 2+3x+8x^2+11x^3 \in \mathcal{P}_3(\mathbb{R})$

Then $\mathcal{M}(v) = \begin{bmatrix} 2 \\ 3 \\ 8 \\ 11 \end{bmatrix}$

$$\begin{tikzcd}
v = 2+3x+8x^2+11x^3 \arrow[rr, mapsto, "D"] \arrow[d, phantom, sloped, "\in"] \arrow[ddd, mapsto] & & 3+16x+33x^2 \arrow[ddd, mapsto] \\
\mathcal{P}_3(\mathbb{R}) \arrow[rr, "D"] \arrow[d] & & \mathcal{P}_2(\mathbb{R}) \arrow[d] \arrow[u, phantom, sloped, "\in"] \\
\mathbb{R}^4 \arrow[rr, "\mathcal{M}(D)"] & & \mathbb{R}^3 \arrow[d, phantom, sloped, "\ni"] \\
\mathcal{M}(v) = \begin{bmatrix} 2 \\ 3 \\ 8 \\ 11 \end{bmatrix} \arrow[u, phantom, sloped, "\in"] \arrow[rr, mapsto] & & \mathcal{M}(D)\begin{bmatrix} 2 \\ 3 \\ 8 \\ 11 \end{bmatrix} = \begin{bmatrix} 3 \\ 16 \\ 33 \end{bmatrix}
\end{tikzcd}$$

$$\underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 8 \\ 11 \end{bmatrix}}_{\mathcal{M}(D)\mathcal{M}(v)} = \underbrace{\begin{bmatrix} 3 \\ 16 \\ 33 \end{bmatrix}}_{=\mathcal{M}(Dv)}$$

<u>Observation</u> $\mathcal{M}(T)$ depends upon choice of basis of V and W. Want to simplify $\mathcal{M}(T)$ by picking "good" bases $\Big\}$ long term goal!

$\boxed{\text{Operators}}$

Defⁿ A linear map from V to itself is an operator, i.e. an element $T \in \mathcal{L}(V,V)$

Write $\mathcal{L}(V) = \mathcal{L}(V,V)$

Thm Suppose V is finite dimensional and $T \in \mathcal{L}(V)$. The following are equivalent

(a) T invertible
(b) T injective
(c) T surjective

Proof (a)$\Rightarrow$(b) T invertible implies T is injective (and surjective!)

(b)$\Rightarrow$(c) Since T injective, $\dim \operatorname{Null}(T) = 0$. So

$$\dim V = 0 + \dim \operatorname{range} T.$$

Since $\operatorname{range}(T) \subseteq V$ and $\dim V = \dim \operatorname{range}(T)$, $V = \operatorname{range}(T)$.

So T is surjective

(c) $\Rightarrow$ (a) Since T surjective, range $T = V$. So

$$\dim \operatorname{Null}(T) = \dim V - \dim \operatorname{range}(T)$$
$$= \dim V - \dim V = 0$$

So $\operatorname{Null}(T) = \{0\}$, so T is injective. So T injective and surjective implies T invertible $\square$

Remark: require hypothesis "fin. dim."

Ex

- $x^3 \in \mathcal{L}(\mathcal{P}(\mathbb{R}))$ given by

$$p \longmapsto x^3 p$$

is injective but not surjective (since nothing maps to 1)

- $S \in \mathcal{L}(F^\infty)$ given by

$$(a_1, a_2, a_3, \dots) \longmapsto (a_2, a_3, a_4, \dots)$$

this is surjective, but not injective since

$(0,0,0,\dots)$ and $(1,0,0,\dots)$ both in $\operatorname{Null}(S)$.

Key ideas

* linear maps as matrix multiplication
* operators
* invertibility of operators