$\boxed{\text{Lecture 20}}$ 8.A Generalized Eigenvectors

Last time: $T: F^2 \to F^2$ given by $T(x,y) = (y,0)$
has <u>no</u> basis such that $\mathcal{M}(T)$ is diagonal

If we use standard basis,

$$\mathcal{M}(T) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

← upper triangular but not diagonal

GOAL OF CHAPTER 8:
Consider case $F = \mathbb{C}$. Although can't always find basis s.t. $\mathcal{M}(T)$ is diagonal, can get "close" i.e. a basis so

$$\mathcal{M}(T) = \begin{bmatrix} \lambda_1 & * & & 0 \\ & \lambda_2 & * & \\ & & \ddots & * \\ 0 & & & \lambda_n \end{bmatrix}$$

← subdiagonal allowed nonzero values

$\boxed{\text{Null Spaces}}$

V is a fin. dim. v.s. Study properties of

$$\text{Null}(T^k) \text{ for } T \in \mathcal{L}(V)$$

Lemma 1 Let $T \in \mathcal{L}(V)$. Then

$$\{0\} \subseteq \mathrm{Null}(T) \subseteq \mathrm{Null}(T^2) \subseteq \mathrm{Null}(T^3) \subseteq \cdots$$

Proof Suppose $v \in \mathrm{Null}(T^k)$. So $T^k v = 0$
But then

$$T^{k+1} v = T(T^k v) = T0 = 0$$

So $v \in \mathrm{Null}(T^{k+1})$ $\square$

Lemma 2 If $\mathrm{Null}(T^k) = \mathrm{Null}(T^{k+1})$,
then
$$\mathrm{Null}(T^k) = \mathrm{Null}(T^\ell) \text{ for all } \ell \geq k$$

Proof Since $\mathrm{Null}(T^{k+n}) \subseteq \mathrm{Null}(T^{k+n+1})$
need to show
$$\mathrm{Null}(T^{k+n+1}) \subseteq \mathrm{Null}(T^{k+n})$$
for all $n \geq 1$.

Let $v \in \mathrm{Null}(T^{k+n+1})$. So

$$0 = T^{k+n+1} v = T^{k+1}(T^n v) \Rightarrow T^n v \in \mathrm{Null}(T^{k+1})$$
$$= \mathrm{Null}(T^k)$$

So $T^k(T^n v) = 0 \Rightarrow T^{k+n} v = 0$

$\Rightarrow v \in \mathrm{Null}(T^{k+n})$. $\square$

As next lemma shows, chain will stabilize by $(\dim V)$-th power

<u>Lemma</u> $\mathrm{Null}(T^{\dim V}) = \mathrm{Null}(T^\ell)$ for all $\ell \geq \dim V$

<u>Proof</u> Suppose $\mathrm{Null}(T^{\dim V}) \subsetneq \mathrm{Null}(T^{\dim V+1})$
So have

$$\{0\} \subsetneq \mathrm{Null}(T) \subsetneq \mathrm{Null}(T^2) \subsetneq \cdots \subsetneq \mathrm{Null}(T^{\dim V}) \subsetneq \mathrm{Null}(T^{\dim V+1})$$

$$0 < \dim \mathrm{Null}(T) < \dim \mathrm{Null}(T^2) < \cdots < \dim \mathrm{Null}(T^{\dim V+1})$$

This forces $\dim \mathrm{Null}(T^i) \geq i$. But this means

$\dim \mathrm{Null}(T^{\dim V+1}) \underline{\geq} n+1$. But $\mathrm{Null}(T^{\dim V+1}) \subseteq V$.

So $\dim \mathrm{Null}(T^{\dim V+1}) \underline{\leq} n$

A contradiction. So
$$\operatorname{null}(T^{\dim V}) = \operatorname{null}(T^{\dim V+1}).$$

Now apply previous Lemma 2 $\square$

Thm If $\dim V = n$, then $V = \operatorname{Null}(T^n) \oplus \operatorname{Range}(T^n)$ for all $T \in \mathcal{L}(V)$

Proof Let $v \in \operatorname{Null}(T^n) \cap \operatorname{Range}(T^n)$. So

$T^n v = 0$ and $T^n w = v$ for some w.

So $T^{2n} w = T^n(T^n w) = T^n v = 0$.

Since $\operatorname{Null}(T^n) = \operatorname{Null}(T^{2n})$, and since $w \in \operatorname{Null}(T^{2n})$

we have $w \in \operatorname{Null}(T^n)$. Hence $v = T^n w = 0$. So $v = 0$.

Hence $\operatorname{Null}(T^n) \oplus \operatorname{Range}(T^n)$ is a direct sum. Since

$$\begin{aligned}\dim V &= \dim(\operatorname{Null}(T^n)) + \dim \operatorname{Range}(T^n) \quad \text{(FT of linear maps)}\\ &= \dim(\operatorname{Null}(T^n) \oplus \operatorname{Range}(T^n)) \quad \text{(since we have a direct sum)}\end{aligned}$$

So, we have $V = \operatorname{Null}(T^n) \oplus \operatorname{Range}(T^n)$ $\square$

Generalized Eigenvectors

Given $T \in \mathcal{L}(V)$, want to rewrite V as

$$V = U_1 \oplus U_2 \oplus \cdots \oplus U_t$$

So T is invariant on each U_i. We can do this if we can find a basis of eigenvectors of V (point of chapter 5)

__Ex__ Consider $T \in \mathcal{L}(\mathbb{C}^2)$ given by

$$T(x,y) = (y,0)$$

Can't find a basis of eigenvectors $\Rightarrow$ not "enough" eigenvectors

Need "more" eigenvectors

__Def__$^{\underline{n}}$ Suppose $T \in \mathcal{L}(V)$ and λ an eigenvalue of T. Then v is a generalized eigenvector of T corresponding to λ if

$$(T - \lambda I)^j v = 0 \quad \text{for some } j \geq 1$$

__Note__ $\{0\} \subseteq \underset{\overset{\|}{E(\lambda, I)}}{\mathrm{Null}(T - \lambda I)} \subseteq \mathrm{Null}((T-\lambda I)^2) \subseteq \cdots$

"more" vectors

Ex Let $T(x,y)=(y,0)$. Then $\lambda=0$ is the eigenvalue (the only eigenvalue)

Saw $(x,0)$ is an eigenvalue of $\lambda=0$ since

$$T(x,0) = (0,0) = 0\cdot(x,0)$$

Any $(x,y)\in F^2$ is a generalized eigenvalue of $\lambda=0$ since

$$(T-0I)^2(x,y) = T^2(x,y) = T(T(x,y))$$
$$= T(y,0) = (0,0) = 0(x,y)$$

So $\mathrm{Null}((T-\lambda I)^2) = F^2$

$$\{0\} \subsetneq \underset{\substack{\parallel \\ \mathrm{span}((1,0))}}{\mathrm{Null}(T-\lambda I)} \subsetneq \underset{\substack{\parallel \\ \mathbb{C}^2}}{\mathrm{Null}((T-\lambda I)^2}$$

Def$^{\underline{n}}$ Let $T \in \mathcal{L}(V)$ and λ an eigenvalue.
Then generalized eigenspace of T is

$$G(\lambda,T) = \{ v \mid (T-\lambda I)^j v = 0 \text{ for some } j \geq 1 \}$$

Thm Suppose $T \in \mathcal{L}(V)$ and λ an eigenvalue.
Then

$$G(\lambda,T) = \mathrm{Null}\left((T-\lambda I)^{\dim V}\right)$$

Proof By def$^{\underline{n}}$, $\mathrm{Null}(T-\lambda I)^{\dim V} \subseteq G(\lambda, I)$.

Suppose $v \in G(\lambda,T)$. Then

$v \in \mathrm{Null}((T-\lambda I)^j)$ for some $j \geq 1$

If $1 \leq j \leq \dim V$, then since $\mathrm{Null}(T-\lambda I)^j \subseteq \mathrm{Null}(T-\lambda I)^{\dim V}$ (by Lemma 1), we have $v \in \mathrm{Null}(T-\lambda I)^{\dim V}$

If $j > \dim V$. But then

$v \in \mathrm{Null}(T-\lambda I)^j = \mathrm{Null}(T-\lambda I)^{\dim V}$ by Lemma 3

So $v \in \mathrm{Null}(T-\lambda I)^{\dim V}$ $\square$

Ex For $T \in \mathcal{L}(\mathbb{C}^2)$ with $T(x,y) = (y,0)$ and $\lambda = 0$

$$G(0,T) = \text{Null}\left((T - 0I)^2\right) = \text{Null}(T^2) = \mathbb{C}^2$$

Key ideas: • properties of $\text{Null}(T^k)$
• Generalized eigenvectors
• Generalized eigenspace $G(\lambda, T)$