

Lecture 5 2.A Span and linear independence

Goal: introduce span & linear independence in general vector space.

today's focus: span

Span

Defⁿ A linear combination of vectors v_1, \dots, v_m in V is a vector of the form
 $a_1 v_1 + a_2 v_2 + \dots + a_m v_m$ with $a_i \in F$

Ex $(1, 2, 3), (2, 4, 7) \in F^3$

linear comb.

$$2(1, 2, 3) + (-1)(2, 4, 7) = (2, 4, 6) + (-2, -4, -7) = (0, 0, -1)$$

Defⁿ Given vectors $v_1, \dots, v_m \in V$, the span of v_1, \dots, v_m , denoted $\text{span}(v_1, \dots, v_m)$, is the set of all linear combinations of v_1, \dots, v_m .

That is,

$$\text{span}(v_1, \dots, v_m) = \{a_1 v_1 + a_2 v_2 + \dots + a_m v_m \mid a_i \in F\}$$

Notation $\text{span}() = \{0\}$

Ex $(0, 0, -1) \in \text{span}((1, 2, 3), (2, 4, 7))$

Thm Let $v_1, \dots, v_m \in V$. Then

- (A) $\text{span}(v_1, \dots, v_m)$ is a subspace of V
- (B) $\text{span}(v_1, \dots, v_m)$ is smallest subspace of V to contain v_1, \dots, v_m .

Proof (A) Check three conditions:

1. $0 \in \text{span}(v_1, \dots, v_m)$ since $0 = 0 \cdot v_1 + \dots + 0 \cdot v_m$

2. Let $x, y \in \text{span}(v_1, \dots, v_m)$. So

$$x = a_1 v_1 + \dots + a_m v_m \quad \text{and} \quad y = b_1 v_1 + \dots + b_m v_m.$$

$$\begin{aligned} \text{Then } x+y &= (a_1 v_1 + \dots + a_m v_m) + (b_1 v_1 + \dots + b_m v_m) \\ &= a_1 v_1 + b_1 v_1 + \dots + a_m v_m + b_m v_m \\ &= (a_1 + b_1) v_1 + \dots + (a_m + b_m) v_m \in \text{span}(v_1, \dots, v_m) \end{aligned}$$

3. Let $x \in \text{span}(v_1, \dots, v_m)$. So $x = a_1 v_1 + \dots + a_m v_m$.

Let $c \in F$. Then

$$cx = c(a_1 v_1 + \dots + a_m v_m) = (ca_1) v_1 + \dots + (ca_m) v_m$$

So $cx \in \text{span}(v_1, \dots, v_m)$.

Thus $\text{span}(v_1, \dots, v_m)$ is a subspace

③ Suppose W is a subspace with $v_1, \dots, v_m \in W$.
 Then, for any $a_1, \dots, a_m \in F$,
 $a_1 v_1 + a_2 v_2 + \dots + a_m v_m \in W$ (since W is a subspace)

This means $\text{span}(v_1, \dots, v_m) \subseteq W$

Defⁿ If $\text{span}(v_1, \dots, v_m) = V$, say v_1, \dots, v_m span V

Ex $V = \mathbb{R}^2$ $v_1 = (1, 0)$ and $v_2 = (0, 1)$ ← standard basis elements

$$\begin{aligned} \text{Then } \text{span}(v_1, v_2) &= \{a(1, 0) + b(0, 1) \mid a, b \in \mathbb{R}\} \\ &= \{(a, b) \mid a, b \in \mathbb{R}\} = \mathbb{R}^2 \end{aligned}$$

Remark Generalizes:

$(1, 0, \dots, 0), (0, 1, 0, \dots, 0), (0, 0, 1, 0, \dots, 0), \dots, (0, \dots, 1)$
 span F^n

Defⁿ A vector space V is finite dimensional if there is v_1, \dots, v_m such that
$$V = \text{span}(v_1, \dots, v_m)$$

Ex F^n is finite dimensional

Defⁿ A v.s. is infinite dimensional if it is not finite dimension

Vector Space of Polynomials

A function $p: F \rightarrow F$ is a polynomial with coefficients in F if there exists $a_0, a_1, \dots, a_m \in F$ such that
$$p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$$

$\mathcal{P}(F) \leftarrow$ all polynomials with coefficients in F

Fact $\mathcal{P}(F)$ is a subspace of $F^F \leftarrow$ all functions $g: F \rightarrow F$

Defⁿ If $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_m z^m$ and $a_m \neq 0$, then degree of $p(z)$ is $\deg p(z) = m$

If $p(z) = 0$, then $\deg 0 = -\infty$

Defⁿ (also saw in 1B03)

$$\mathcal{P}_m(F) = \{a_0 + a_1 z + \dots + a_m z^m \mid a_i \in F\} \leftarrow \text{saw in 1B03}$$

this is a
vector space

$$= \{ \text{all polynomials of degree } \leq m \}$$

Fact $\mathcal{P}_m(F) = \text{span}\{1, z, \dots, z^m\}$ so $\mathcal{P}_m(F)$
is finite dimensional

Proof $\text{span}\{1, z, \dots, z^m\} \subseteq \mathcal{P}_m(F)$ since $1, z, \dots, z^m \in \mathcal{P}_m(F)$.

Let $p(z) \in \mathcal{P}_m(F)$. Then $p(z) = a_0 + a_1 z + \dots + a_m z^m$
 $= a_0 \cdot 1 + a_1 \cdot z + \dots + a_m \cdot z^m$
 $\in \text{span}(1, z, \dots, z^m)$.

So $\mathcal{P}_m(F) = \text{span}(1, z, \dots, z^m)$. □

Fact $\mathcal{P}(F)$ is infinite dimensional

Proof Suppose $\mathcal{P}(F) = \text{span}\{p_1(z), \dots, p_t(z)\}$

Let $m = \max\{\deg p_1(z), \dots, \deg p_t(z)\}$

Now $z^{m+1} \in \mathcal{P}(F)$. But $z^{m+1} \notin \text{span}\{p_1(z), \dots, p_t(z)\}$

Since all $p_i(z)$ have $\deg p_i(z) \leq m$, and so

$$\deg(a_1 p_1(z) + \dots + a_m p_m(z)) \leq m.$$

So $\mathcal{P}(F) \neq \text{span}(p_1(z), \dots, p_t(z))$ for any finite $p_1(z), \dots, p_t(z)$ □

Example: F^∞

$$F^\infty = \{ (x_1, x_2, x_3, \dots) \mid x_i \in F \}$$

Claim F^∞ is infinite dimensional

Proof Suppose $F^\infty = \text{span}(v_1, \dots, v_t)$.

Write v_i 's as

$$v_1 = (a_{11}, a_{12}, a_{13}, \dots)$$

$$v_2 = (a_{21}, a_{22}, a_{23}, \dots)$$

\vdots

$$v_t = (a_{t1}, a_{t2}, a_{t3}, \dots)$$

Consider any $(\underbrace{b_1, b_2, \dots, b_t}_t, 0, 0, \dots) \in F^\infty$

Since we are assuming $F^\infty = \text{span}(v_1, \dots, v_t)$, there exists c_1, \dots, c_t such that

$$c_1 v_1 + c_2 v_2 + \dots + c_t v_t = (b_1, b_2, \dots, b_t, 0, 0, \dots, 0)$$

Note that if we restrict to first t entries, get a system of linear equations:

$$c_1(a_{11}, \dots, a_{1t}) + c_2(a_{21}, \dots, a_{2t}) + \dots + c_t(a_{t1}, \dots, a_{tt}) = (b_1, \dots, b_t)$$

$$\Leftrightarrow \begin{bmatrix} a_{11} & a_{21} & \dots & a_{t1} \\ \vdots & \vdots & & \vdots \\ a_{1t} & a_{2t} & \dots & a_{tt} \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_t \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_t \end{bmatrix}$$

So, this system has a solⁿ for all $(b_1, \dots, b_t) \in F^t$

From 1B03, this means the $t \times t$ matrix is invertible matrix

But now consider

$$\underbrace{(0, 0, \dots, 0)}_t, 1, 0, \dots \in F^\infty$$

← entry $t+1$

There must be d_1, \dots, d_t such that

$$d_1 v_1 + d_2 v_2 + \dots + d_t v_t = (0, \dots, 0, 1, 0, \dots, 0, \dots)$$

But this means that d_1, \dots, d_t satisfies

$$\begin{bmatrix} a_{11} & \dots & a_{t1} \\ \vdots & & \vdots \\ a_{1t} & & a_{tt} \end{bmatrix} \begin{bmatrix} d_1 \\ \vdots \\ d_t \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

The matrix is invertible, so only solⁿ is
 $d_1 = d_2 = \dots = d_t = 0$

Contradiction

But then

$$(0, 0, \dots) = d_1 v_1 + \dots + d_t v_t = (0, 0, \dots, 1, 0, 0, \dots)$$

Thus, F^∞ is infinite dimensional.

□

- Key results
- * defⁿ of linear combination
 - * defⁿ of span
 - * finite dimensional vs infinite dim.
 - * $\mathcal{P}(F)$ and F^∞