

Lecture 10 3.B Null spaces and Range

Last time: $\mathcal{L}(V, W)$ \leftarrow all linear maps between v.s. V and W
 $\mathcal{L}(V, W)$ is also a vector space

Today : Given any $T \in \mathcal{L}(V, W)$, show T defines
Subspaces of V and W .

Null Space

Defⁿ For $T \in \mathcal{L}(V, W)$, the null space of T (or
kernel of T), denote $\text{null}(T)$, is

$$\text{null}(T) = \{v \in V \mid Tv = 0\} \subseteq V$$

null space \leftarrow all things in V that are sent to 0 in W by T

Ex 1 If $T=0$ is the zero map, i.e. $Tv=0$ for all $v \in V$
Then $\text{null}(T) = V$

Ex 2 Let $T \in \mathcal{L}(F^\infty, F^\infty)$ defined by
 $T(a_1, a_2, \dots) = (a_2, a_3, a_4, \dots)$

Then

$$\text{null}(T) = \{(a, 0, 0, 0, \dots) \mid a \in F\} \subsetneq F^\infty$$

Ex 3 Let $T \in \mathcal{L}(F^3, F^2)$ given by

$$T(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, 2x_1 + 4x_2 + 6x_3)$$

So

$$\text{null}(T) = \left\{ (x_1, x_2, x_3) \in F^3 \mid \begin{array}{l} x_1 + 2x_2 + 3x_3 = 0 \\ 2x_1 + 4x_2 + 6x_3 = 0 \end{array} \right\}$$

System of lin equations

Use RREF results to find spanning set.

$$\text{SLE} \Leftrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 6 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \leftarrow \begin{array}{l} x_2, x_3 \text{ free} \\ x_1 = -2x_2 - 3x_3 \end{array}$$

$$\text{So } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{So } \text{null}(T) = \text{Span}((-2, 1, 0), (-3, 0, 1))$$

RREF connection: If A is $m \times n$, then $\text{null}(A)$ corresponds to the null space of the linear map $Tx = Ax$

Thm $\text{null}(T)$ is a subspace of V

Proof Check 3 subspace conditions

1. $0 \in \text{null}(T)$ Since all linear maps satisfy $TO=0$

2. Let $u, v \in \text{null}(T)$. Then \Downarrow
(see 3.11)

$$\begin{aligned} T(u+v) &= Tu + Tv \quad (\text{by additivity}) \\ &= 0 + 0 \quad (\text{since } u, v \in \text{null}(T)) \\ &= 0 \end{aligned}$$

So $u+v \in \text{null}(T)$

3. Let $u \in \text{null}(T)$ and $\lambda \in F$. Then $T(\lambda u) = \lambda T(u)$
(by homogeneity). Since $u \in \text{null}(T)$, $T(u) = 0$. So

$$T(\lambda u) = \lambda T(u) = \lambda \cdot 0 = 0. \quad \text{So } \lambda u \in \text{null}(T)$$

□

Defⁿ A function $T: V \rightarrow W$ is injective (or one-to-one) if $Tu = Tv$ implies $u = v$.

$\text{null}(T)$ "measures" injectivity

Thm $\text{null}(T) = \{0\}$ if and only if T injective

Proof: (\Rightarrow) Suppose $Tu = Tv$. Then $Tu - Tv = 0$
But $Tu - Tv = T(u - v)$. So $T(u - v) = 0$ implies
 $u - v \in \text{null}(T) = \{0\}$. So $u - v = 0 \Leftrightarrow u = v$

(\Leftarrow) Let T be injective, and suppose $u \in \text{null}(T)$, i.e. $Tu = 0$. We also know that $T0 = 0$ for any linear map. Since T is injective, and because $Tu = T0 = 0$, this means $u = 0$. see 3.11

So $\text{null}(T) = \{u \mid Tu = 0\} = \{0\}$. □

Range

Defⁿ For $T \in \mathcal{L}(V, W)$, range of T (or image of T) is
$$\text{range}(T) = \{Tu \mid u \in V\} \subseteq W$$

Ex 1 If $T=0$ is the zero map, then
$$\text{range}(T) = \{Tu = 0 \mid u \in V\} = \{0\} \subseteq W \quad \leftarrow \text{not onto}$$

Ex 2 Let $T \in \mathcal{L}(F^\infty, F^\infty)$ defined by
$$T(a_1, a_2, a_3, \dots) = (a_2, a_3, a_4, \dots)$$

Then

$$\text{range}(T) = \{(a_2, a_3, \dots) \mid a_i \in F\} = F^\infty \quad \leftarrow \text{onto}$$

Ex 3 Let $T \in \mathcal{L}(F^3, F^2)$ given by
$$T(x_1, x_2, x_3) = (x_1 + 2x_2 + 3x_3, 2x_1 + 4x_2 + 6x_3)$$

$$\begin{aligned} \text{range}(T) &= \{(x_1 + 2x_2 + 3x_3, 2x_1 + 4x_2 + 6x_3) \mid (x_1, x_2, x_3) \in F^3\} \\ &= \{x_1(1, 2) + x_2(2, 4) + x_3(3, 6) \mid (x_1, x_2, x_3) \in F^3\} \\ &= \text{Span}((1, 2), (2, 4), (3, 6)) = \text{Span}((1, 2)) \neq F^2 \\ &= \text{column space of } A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} \quad \leftarrow \text{from 1303} \end{aligned}$$

If A defines linear map $T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ by $Tx = Ax$, then $\text{range}(T) = \text{col}(A)$

Thm $\text{range}(T)$ is a subspace of W

Proof (see text) Check 3 subspace criteria

Defⁿ A function $T: V \rightarrow W$ is surjective (or onto) if $\text{range}(T) = W$

Ex Above, Ex 2 is surjective, but Ex 1 and 3 are not.

Linking Ideas

(Fundamental Theorem of Linear maps)

Suppose V is a finite dimensional V.S. and $T \in \mathcal{L}(V, W)$.

Then $\text{range } T$ is fin. dim. and

$$\dim V = \dim \text{null}(T) + \dim \text{range}(T)$$

Proof Let u_1, \dots, u_n be a basis of $\text{null}(T) \subseteq V$
(so $\dim \text{null}(T) = n$).

Extend to a basis of V , i.e.
 $u_1, \dots, u_n, v_1, \dots, v_m$

So $\dim V = n + m$

Need to show $\text{range}(T)$ is fin. dim and $\dim \text{range}(T) = m$

For any $v \in V$,

$$v = a_1 u_1 + \dots + a_n u_n + b_1 v_1 + \dots + b_m v_m$$

So

$$\begin{aligned} Tv &= a_1 \overset{\rightarrow=0}{Tu_1} + \dots + a_n \overset{=0}{Tu_n} + b_1 Tv_1 + \dots + b_m Tv_m \\ &= b_1 Tv_1 + \dots + b_m Tv_m \end{aligned}$$

Thus $\text{range } T = \text{span}(Tv_1, \dots, Tv_m)$ & this implies $\text{range}(T)$ is fin. dim.

We show Tv_1, \dots, Tv_m linearly indep. (and hence $\dim \text{range}(T) = m$)

Suppose

$$c_1 Tv_1 + \dots + c_m Tv_m = 0$$

$$\Leftrightarrow T(c_1 v_1 + \dots + c_m v_m) = 0$$

This implies $c_1 v_1 + \dots + c_m v_m \in \text{null}(T) = \text{span}(u_1, \dots, u_n)$

Hence $C_1 v_1 + \dots + C_m v_m = d_1 u_1 + \dots + d_n u_n$ for some d_i

\Leftrightarrow

$$C_1 v_1 + \dots + C_m v_m + (-d_1) u_1 + \dots + (-d_n) u_n = 0$$

But $v_1, \dots, v_m, u_1, \dots, u_n$ is linearly indep, so

$$C_1 = \dots = C_m = d_1 = \dots = d_n = 0.$$

In particular, $C_1 = \dots = C_m = 0$, so Tv_1, \dots, Tv_m linearly indep.

□

In Math 1B03, learned about rank-nullity theorem.

If A is $m \times n$ matrix, then

$$\text{rank}(A) + \dim \text{Nul}(A) = n$$

\uparrow $\dim \text{Col}(A)$

Same result !! A defines a linear map

$T: \mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $Tx = Ax$ (i.e. $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$)

$$\text{Then } n = \underline{\dim \mathbb{R}^n} = \underline{\dim \text{null}(T)} + \underline{\dim \text{range}(T)} \leftarrow \text{Fund Thm}$$

$$= \dim \text{Nul}(A) + \dim \text{Col}(A)$$

$$= \dim \text{Nul}(A) + \text{rank}(A)$$

Key ideas: * $\text{null}(T)$

* $\text{range}(T)$

* Fun & Thm of Linear maps.

