MIDTERM 2 SOLUTIONS

A1. a. A g-cage is a 3-regular graph where the girth is g

b. A magic labelling is a labelling of the q edges of a graph with the numbers 1,…,q so that each vertex, the sum of the labels of the edges adjacent to the vertex are all the same.

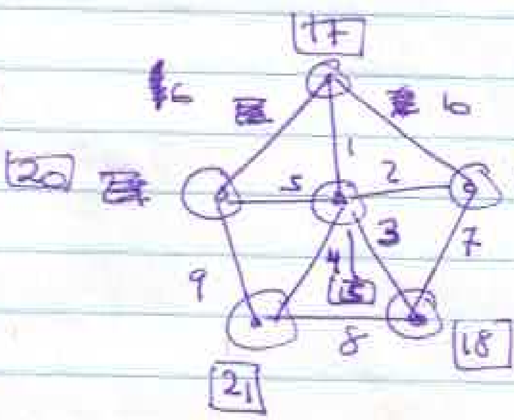
c. An antimagic labelling is a labelling of the q-edges with 1,…,q so that the sum of labels at each vertex is different.

d. A directed graph is a graph where each edge is given a direction. Equivalently, each edge is an ordered pair

e. Kirchhoff's current law says that in a conservative labelling of a directed graph, the sum of the labels of the edges entering a vertex equals the sum of the labels of the edges leaving the vertex.

A2. This is $D_6 = 6!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!}\right] = \boxed{265}$

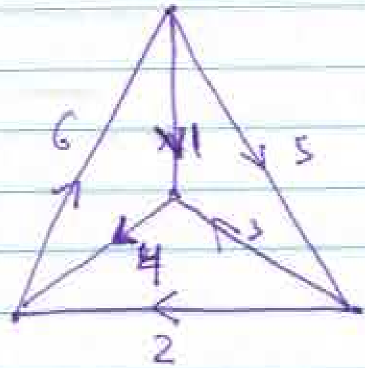
A3.



Not unique

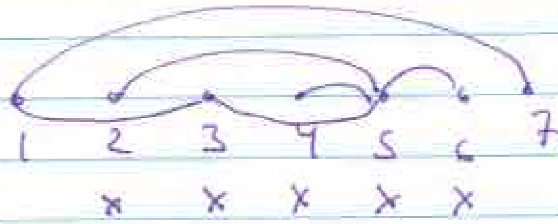
← Sum of labels all different

A4.

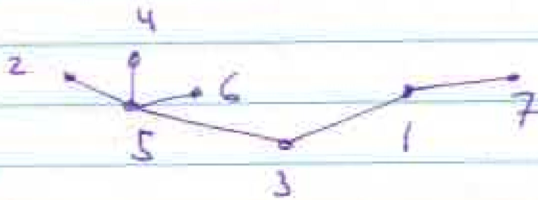


$\boxed{\text{not unique.}}$

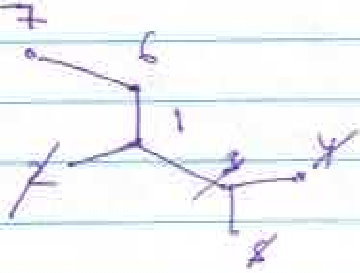
A5. $(\cancel{5},\cancel{5},\cancel{5},\cancel{3},\cancel{1})$



or redrawn as



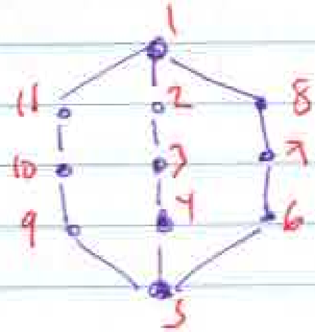
A6.



$\boxed{(1, 3, 3, 1, 6)}$

Prufer Seq.

A7. we need 8 vertices to make a cycle of size 8



← has 11 vertices, girth 8.
If we add any edge, make a smaller cycle
12 edges

~~A~~ 8. Fix some $n \geq 2$. By def^{n}, $r = r(2,n)$ is the smallest r such that K_r ~~be~~ either has a red K_2 or blue K_n when we colour edges of K_r red or blue.

Give K_r any colouring. If it has a red edge, this is a K_2. So, suppose it has no red edge. It must have ~~to~~ be all coloured blue. So K_r will have a K_n if and only if $n \leq r$. The smallest r with this property is ~~b~~ $r=n$. So $r(2,n)=n$.

B.1 Do induction on $m+n$. The smallest value allowed is $(m,n)=(2,2)$, i.e. $m+n=4$. Then

$$r(2,2) = 2 \leq \binom{2+2-2}{2-1} = \binom{2}{1} = 2$$

Now suppose true for all (i,j) with $i+j < m+n$.

Then for (m,n)
$$\begin{aligned} r(m,n) &\leq r(m-1,n) + r(m,n-1) \\ &\leq \binom{m-1+n-2}{m-2} + \binom{m+n-1-2}{m-1} \text{ by induction} \\ &= \binom{m+n-3}{m-2} + \binom{m+n-3}{m-1} \\ &= \binom{m+n-2}{m-1} \text{ by Pascal's Identity} \end{aligned}$$

So the result holds by induction

$\boxed{B2}$ Consider path P_n. ~~with~~

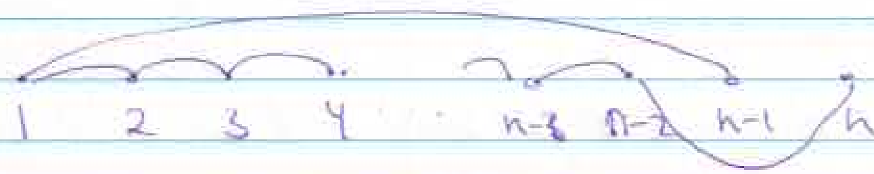
Case 1 $n=2m$ is even. Label n edges in order

At vertex i, sum is $(i-1)+i = 2i-1$ except at $i=1$, when sum is 1 and at n when it is n.
Since $1, 3, 5, 7, \ldots, 2n-1$ and n are all distinct (n is even, so does not appear in $1, \ldots, 2n-1$) we have an antimagic labelling

Case 2 $n=2m+1$ is odd. Label n edges as

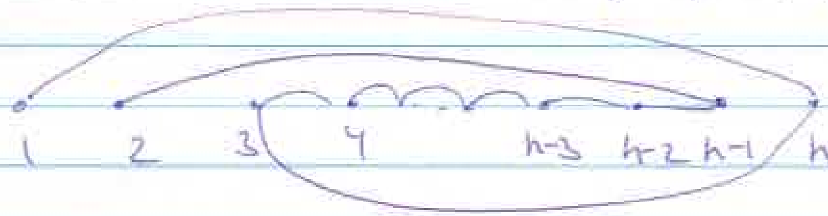
At 1, sum is 1
At i, sum is $(2i-1)$ for $i=2, \ldots, n-3$
At $n-2$, sum is $n-3+n = 2n-3$ which does not appear in
$1, 3, \ldots, 2(n-3)-1 = 2n-7$
At ~~n-2~~, $n-1$, sum is $2n-1$, again, not seen before
At n, sum is $n-1$, and since this is even, haven't seen before.
So P_n is magic

$\boxed{B3}$ Draw the tree associated to $(1,2,3,\ldots,n-3,n-2)$



So we get a path P_{n-1}

If we draw the tree associated to $(n, n-1, n-2, \ldots, 4, 3)$, get



Again, we get path P_{n-1}. So clear we get isomorphic tree

We can find another sequence simply by taking any other labelling of P_{n-1}

$\boxed{B4}$ See proof on pg 117 of text