1.

What are the solutions to

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 5 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} ?$$

A) $\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$

B) $\begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \\ -2 \end{bmatrix}$

C) $\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$

D) $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} -2 \\ 4 \\ -2 \\ 0 \end{bmatrix}$

E) $\begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

Continued on page 3

2. For what value of k is the matrix A not invertible, where

$$A = \begin{bmatrix} k & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 0 & 2 & 4 & 5 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

A) 2 B) -1 C) -3 D) 4 E) -5

3. A and B are invertible 3x3 matrices. det(B)=6 and

$$\det(2AB^{-1}) = \det I$$

What is $\det(A^2)$?

A) 1 B) 3 C) 9/16 D) 36 E) -9

Continued on page 4

4. Find the eigenvalue of A where

$$A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$$

The eigenvalue is:

A) 4 B) 2 C) 3 D) 1 E) 5

5. Which of the following is <u>not</u> equivalent to the others for an nxn matrix A?

A) $Ax = 0$ has infinitely many solutions
B) 0 is an eigenvalue of A
C) $\dim(\text{null } A) > 0$
D) There is at least one 0 on the main diagonal of A
E) $\det(A) = 0$

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6. The 2x2 matrix A has eigenvalues 1 and -1, with corresponding eigenvectors $(4,1)$ and $(2,1)$. Which of the following could be the matrix A^3?

A) $\begin{bmatrix} 4 & -2 \\ 1 & -1 \end{bmatrix}$ B) $\begin{bmatrix} 2 & -6 \\ 3 & -3 \end{bmatrix}$ C) $\begin{bmatrix} 3 & -8 \\ 1 & -3 \end{bmatrix}$

D) $\begin{bmatrix} 8 & 1 \\ 1 & -6 \end{bmatrix}$ E) $\begin{bmatrix} 64 & 8 \\ 1 & 1 \end{bmatrix}$

7. Find a basis for the eigenspace associated with the eigenvalue $\lambda = 2$ for the matrix

$$A = \begin{bmatrix} 3 & -2 & -3 \\ -3 & 8 & 9 \\ 2 & -4 & -4 \end{bmatrix}$$

A) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ B) $\left\{ \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\}$ C) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$

D) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right\}$ E) $\left\{ \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix} \right\}$

Continued on page 6

8. Which of the following statements are true?
i) If a matrix is invertible, it must be diagonalizable
ii) If a matrix is diagonalizable, it must be invertible
iii) Every nxn matrix with n different eigenvalues is diagonalizable

A) i) only B) iii) only C) ii) and iii)
D) i) and iii) E) All of these are true

9. A is a 3x3 matrix of rank 2. The system of equations

$$Ax = [3\ \ 5\ \ 7]^T$$

has infinitely many solutions, including $x = [1\ \ 2\ \ 3]^T$ and $x = [4\ \ 4\ \ 4]^T$.
A basis for the null space of A is:

A) $\{(1,1,1),(2,3,5)\}$ B) $\{(1,2,3),(3,5,7)\}$ C) $\{(3,2,1)\}$
D) $\{(3,5,7)\}$ E) $\{(2,3,4)\}$

Continued on page 7

10. For what value of k are the following polynomials linearly dependent?

$$1+3x^3 \qquad x-2x^3 \qquad 5+2x+kx^3$$

A) 7 B) 1 C) -12 D) 0 E) 11

11. Consider the 3 vectors in M_{22},

$$M_1 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad M_3 = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Which of the following statements is correct?

A) The matrices are not linearly independent, and form a basis for M_{22}
B) The matrices are linearly independent, and form a basis for M_{22}
C) The matrices are not linearly independent, and do not form a basis for M_{22}
D) The matrices are linearly independent, and do not form a basis for M_{22}

Continued on page 8

12. In $\mathbb{R}^3$, consider the following vectors

$$\mathbf{v}_1 = (h, 1, 0), \quad \mathbf{v}_2 = (4, 1, h), \quad \mathbf{v}_3 = (1, -1, -3),$$

where $h \in \mathbb{R}$. For which values of h does the equation $\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v}_3$ hold?

A) $h = 0$ B) $h = 2$ C) $h = 1$
D) $h = -2$ E) all h

13. Let $\mathbf{v} = (1, 1, 0)$ and $\mathbf{u} = (0, 3, 1)$ be two vectors in $\mathbb{R}^3$. Find two vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ such that $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$.

A) $\mathbf{w}_1 = \frac{1}{2}(3, 3, 2)$ and $\mathbf{w}_2 = \frac{1}{2}(-3, 3, 0)$ B) $\mathbf{w}_1 = \frac{1}{2}(3, 3, 0)$ and $\mathbf{w}_2 = \frac{1}{2}(-3, 3, 2)$
C) $\mathbf{w}_1 = \frac{1}{2}(3, 3, 0)$ and $\mathbf{w}_2 = (-3, 3, 2)$ D) $\mathbf{w}_1 = (3, 3, 0)$ and $\mathbf{w}_2 = \frac{1}{2}(-3, 3, 2)$
E) $\mathbf{w}_1 = (3, 3, 0)$ and $\mathbf{w}_2 = (-3, 3, 2)$

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14. In $\mathbb{R}^3$, what is the area of the triangle with vertices $(1,1,1)$, $(0,0,0)$ and $(0,0,1)$?

A) 3 B) $\dfrac{1}{\sqrt{2}}$ C) $\sqrt{3}$ D) 1 E) $\dfrac{\sqrt{3}}{2}$

15. Let $\mathbf{u} = (1,1,1)$ and $\mathbf{v} = (0,1,0)$ be two vectors in $\mathbb{R}^3$. What are all the vectors $\mathbf{w} \in \mathbb{R}^3$ that lie in the same plane as $\mathbf{u}$ and $\mathbf{v}$, are orthogonal to $\mathbf{v}$ and have unit norm?

A) $\frac{1}{\sqrt{2}}(1,0,1)$ and $\frac{1}{\sqrt{2}}(-1,0,-1)$ only B) $(0,0,0)$ only

C) There are no such vectors D) $\frac{1}{2}(1,0,1)$ and $\frac{1}{2}(-1,0,-1)$ only

E) $\frac{1}{\sqrt{6}}(1,2,1)$ only

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16. Let A be a 3×3 matrix with only one eigenvalue λ with algebraic multiplicity 3. Which of the following statements is true?

i) A is always diagonalizable.
ii) A is diagonalizable if and only if the eigenspace corresponding to λ has dimension 3.
iii) A can never be diagonalizable.

A) i) and iii) only B) ii) and iii) only C) i) only
D) i), ii) and iii) E) ii) only

17. Let A be an $m \times n$ matrix. Which of the following statements is true?

i) The column space of A is a subspace of $\mathbb{R}^n$.
ii) If $m = n$, then the row space and the column space of A are both $\mathbb{R}^n$ if and only if A is invertible.
iii) The dimensions of the row space and of the column space are always the same.

A) i), ii) and iii) B) ii) only C) i) and iii) only
D) i) only E) ii) and iii) only

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18. Which of the following statements are always true:

i) $\frac{1}{i} = -i$

ii) $\arg(\bar{z}) = -\arg(z)$

iii) $\cos(\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2}$

A) i) is true, ii) and iii) are not

B) ii) and iii) are true, i) is not

C) i) and ii) are true, iii) is not

D) None of the statements are true

E) All three are true

19. Let

$$A = \begin{bmatrix} 1 & 5 & 2 & 6 & 7 \\ 1 & 0 & 2 & 1 & 7 \\ 1 & 3 & 2 & 4 & 7 \end{bmatrix}$$

What is the dimension of the row space of A?

A) 2 B) 4 C) 1 D) 5 E) 3

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20. Let W be the set of all vectors $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that $x_1 + x_4 = 0$. If we are considering the usual addition and scalar multiplication of $\mathbb{R}^4$, which of the following statements is true?

A) W is closed under addition and scalar multiplication, therefore it is a subspace.
B) W does not contain the vector $(1, 0, 0, 1)$, therefore it cannot be a subspace.
C) W is not closed under addition, therefore it is not a subspace.
D) W is not closed under scalar multiplication, therefore it is not a subspace.
E) W contains the zero vector, therefore it is not a subspace.

21. A student is given a complex number, and asked to find the fourth roots. If one of the roots is given by:

$$1 - \sqrt{3}i$$

which of the following must also be a root?

A) $-1 - \sqrt{3}i$ B) $-\sqrt{3} - i$ C) -1 D) $-2 + 3i$ E) $1 + \sqrt{3}i$

Continued on page 13

22. Suppose A and B are skew-symmetric invertible square matrices. Which of the following statements must hold?
(Recall that a matrix M is skew-symmetric if and only if $M^T = -M$).

(i) A^{-1} is skew-symmetric
(ii) A^{-1} is symmetric
(iii) If $AB = BA$, then AB is skew-symmetric
(iv) If $AB = BA$, then AB is symmetric

A) (i) only B) (ii) and (iv) only C) (ii) and (iii) only
D) (ii) only E) (i) and (iv) only

23. Divide:

$$\frac{1-3i}{2+i}$$

A) $-\frac{1}{2} - 3i$ B) $-2 - 3i$ C) $5 - 3i$ D) $-\frac{1}{5} - \frac{7}{5}i$ E) $-1 - 7i$

Continued on page 14

24. Gordon eats his lunch at any one of 3 different restaurants: Al's Chinese, BurgerBoom, or Chicken Mountain. On day t, the probability that he eats at each is given by a_t, b_t, and c_t respectively. He never eats at the same place two times in a row. After he eats at Al's, he is equally likely to eat at BurgerBoom or Chicken Mountain the next time. Likewise after he eats at BurgerBoom, he is equally likely to eat at Al's or Chicken Mountain the next time. If a dynamical system for (a_t, b_t, c_t) has a steady state vector of $(6/15, 4/15, 5/15)$, what is the probability that after he eats at Chicken Mountain, he eats at Al's the next time?

A) 2/3 B) 1/3 C) 1/2 D) 4/5 E) 1/10

25. What point on the line

$$\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

is closest to the point (2, 2, 2)?

A) $(-1/3, 1, 1)$ B) $(-1, 1, -1)$ C) $(1/5, 1, 13/5)$
D) $(1/2, 1, 7/2)$ E) $(1, 1, 5)$

Continued on page 15

26. Suppose V is a vector space of dimension 3, and $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are elements of V. Suppose $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = V$, and $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 = 0$. Which of the following are bases of V?

- $S_0 := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$
- $S_1 := \{\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$
- $S_2 := \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_4\}$
- $S_3 := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4\}$
- $S_4 := \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$

A) S_1, S_2, S_3, S_4 only B) S_3 only C) S_2, S_3, S_4 only D) S_1, S_2, S_3 only E) S_0 only

27. What is the dimension of the following subspace of the vector space M_{22} of 2x2 real matrices:

$$\text{span}\left(\left\{\begin{bmatrix}1 & 1\\1 & 1\end{bmatrix}, \begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}, \begin{bmatrix}2 & 3\\4 & 5\end{bmatrix}\right\}\right)?$$

A) 3 B) 2 C) 1 D) It has infinite dimension. E) 4

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28. Let $B = (x^2 + x + 1, x^2 + x - 1, x^2 - x + 1)$, which is an ordered basis of the vector space P_2 of polynomials of degree at most 2. What is the co-ordinate vector with respect to B of the quadratic polynomial $(x + 1)^2$?

A) $\begin{bmatrix} 3/2 \\ 1/2 \\ 0 \end{bmatrix}$ B) $\begin{bmatrix} 0 \\ 1/2 \\ -3/2 \end{bmatrix}$ C) $\begin{bmatrix} 0 \\ 1/2 \\ -3/2 \end{bmatrix}$

D) $\begin{bmatrix} -1/2 \\ 0 \\ 3/2 \end{bmatrix}$ E) $\begin{bmatrix} 3/2 \\ 0 \\ -1/2 \end{bmatrix}$

29. Let M_{22} be the set of 2x2 invertible matrices, with the usual scalar multiplication and a new vector addition operation given by the following:

If $\mathbf{u} = A$, $\mathbf{v} = B$ in M_{22}, then $\mathbf{u} + \mathbf{v} = AB$.

The fourth axiom of real vector spaces states that there must exist a unique element, $\mathbf{0}$, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ for all $\mathbf{u}$ in our set. What is this $\mathbf{0}$ for our given system?

A) No such matrix B) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ C) I

D) A^{-1} E) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

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30. Let A be the matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Find A^{-1}.

A) $\begin{bmatrix} 0 & 1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$ B) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$ C) $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

D) $\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ E) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}$

31. What value appears in row 1, column 2 of the matrix obtained by the product AB where

$$A = \begin{bmatrix} 2+3i & 2 \\ 0 & 1-7i \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 5-7i & 1-i \\ 0 & 3i \end{bmatrix}$$

A) $3-3i$ B) $2-2i$ C) $5+7i$ D) $4+10i$ E) $3-i$

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32. Given the vectors:

$$\mathbf{u}_1 = (-1, 2, 0)$$
$$\mathbf{u}_2 = (7, -9, -1)$$
$$\mathbf{u}_3 = (10, 0, 2)$$

The Gram-Schmidt process is used on this basis of $\mathbb{R}^3$ to produce an orthogonal basis. The first two vectors are:

$$\mathbf{v}_1 = (-1, 2, 0)$$
$$\mathbf{v}_2 = (2, 1, -1)$$

What is the third vector in the new basis?

A) $(-2, 1, 3)$ B) $(10/15, 0, 2/15)$ C) $(-3, 1, 1)$
D) $(2, 1, 5)$ E) $(6, 3, 0)$

33. Let W be the subspace of $\mathbb{R}^4$ spanned by the orthogonal vectors: $\{(0, 1, 1, 1), (1, 1, 0, -1)\}$ Compute the orthogonal projection of the vector $\mathbf{u} = (2, 1, 2, 0)$ onto this subspace.

A) $(3, 6, 3, 0)$ B) $(1, 2, 1, 0)$ C) $(2, 0, 1, -1)$
D) $(-1, -5, -1, 0)$ E) $(3, 5, -2, 1)$

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34. Which of the following **IS** an orthogonal set of vectors, but is **NOT** orthonormal?

A) $\left\{\frac{1}{\sqrt{2}}(0,1,0,-1), \frac{1}{\sqrt{2}}(0,1,0,1), \frac{1}{\sqrt{2}}(2,0,2,0)\right\}$

B) $\left\{\frac{1}{2}(0,\sqrt{3},0,1), \frac{1}{2}(0,1,0,\sqrt{3}), \frac{1}{2\sqrt{2}}(2,0,2,0)\right\}$

C) $\left\{\frac{1}{5}(0,3,4,0), \frac{1}{5}(4,0,0,3), \frac{1}{5}(-3,0,0,4)\right\}$

D) $\left\{(1,0,0,0), \frac{1}{\sqrt{2}}(0,1,1,0), \frac{1}{\sqrt{2}}(0,0,1,1)\right\}$

E) $\left\{\frac{1}{4}(0,3,4,0), \frac{1}{4}(4,0,0,3), \frac{1}{4}(3,0,0,4)\right\}$

35. Given the complex number

$$z = -2 + 2i$$

write its complex conjugate in polar form.

A) $2\sqrt{2}e^{-i\pi/4}$ B) $2\sqrt{2}e^{5i\pi/4}$ C) $2\sqrt{2}e^{-5i\pi/4}$
D) $-2e^{-i\pi/4}$ E) $-2e^{i\pi/4}$

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36. Which of the following commands used directly in the MatLab workspace produces a function that corresponds to:

$$f(x) = \begin{cases} x^2 & x > 2 \\ 1 - x & x \leq 2 \end{cases}$$

A) `f= @(x) (x>2)*x^2+(x<=2)*(1-x)`
B) `f = {x^2 1-x x=2 }`
C) `f:= (x)-> case[(>2)(x^2) (<=2)(1-x)`
D) `f= @(x) (x>2)=x^2 (x<=2)=(1-x)`
E) `f:= (x)->if(x>2) x^2 else(1-x)`

37. Let V be a vector space and let W_1 and W_2 be two subspaces. Which of the following statements is always true?
i) The subset $W_1 \cap W_2$ of all vectors $\mathbf{v} \in V$ such that $\mathbf{v} \in W_1$ and $\mathbf{v} \in W_2$ is a subspace.
ii) The subset $W_1 \cup W_2$ of all vectors $\mathbf{v} \in V$ such that $\mathbf{v} \in W_1$ or $\mathbf{v} \in W_2$ (or both) is a subspace.
iii) $W_1 \cap W_2$ defined as in i) is never empty.

A) ii) only B) i) and iii) only C) i), ii) and iii)
D) ii) and iii) only E) i) only

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38. Suppose A is a 3x3 matrix and

$$A\begin{bmatrix}1\\1\\-1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Which of the following statements must hold?

(i) For some $\mathbf{b}$, $A\mathbf{x}=\mathbf{b}$ has no solution for $\mathbf{x}$.

(ii) For all $\mathbf{b}$, $A\mathbf{x}=\mathbf{b}$ has no solution for $\mathbf{x}$.

(iii) For all $\mathbf{b}$, either $A\mathbf{x}=\mathbf{b}$ has no solution for $\mathbf{x}$ or it has infinitely many solutions for $\mathbf{x}$.

A) (i) and (ii) only B) (i), (ii), and (iii) C) (iii) only D) (i) only E) (i) and (iii) only

END OF TEST QUESTIONS

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