AN APPENDIX TO A PAPER OF CATALISANO, GERAMITA, GIMIGLIANO: THE HILBERT FUNCTION OF GENERIC SETS OF 2-FAT POINTS IN $\mathbb{P}^1 \times \mathbb{P}^1$

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ABSTRACT. Implicit in the paper of Catalisano, Geramita, and Gimigliano [1] is a formula for the Hilbert function of generic sets of 2-fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. We make this result explicit in this note.

We shall use the definitions and notation as found in [1].

Let d_1, d_2 be two positive integers and consider the fat point scheme $W = d_1Q_1 + d_2Q_2 + 2R_1 + \cdots + 2R_s \subseteq \mathbb{P}^2$ where $Q_1, Q_2, R_1, \ldots, R_s$ are $s+2$ points in generic position. Proposition 2.1 of [1] describes some properties of the Hilbert function of W. Furthermore, this proposition is used (see [1, Corollary 2.2]) to compute the dimensions of the secant varieties of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded into $\mathbb{P}^{(d_1+1)(d_2+1)-1}$ by the morphism given by $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d_1, d_2)$.

By Terracini's Lemma (see §1 of [1]) there is a relationship between the dimensions of these embedded varieties and the Hilbert functions of generic sets of 2-fat points in $\mathbb{P}^1 \times \mathbb{P}^1$. Thus, implicit in [1, Corollary 2.2] is a formula for $H(Z, (d_1, d_2))$ when $Z = 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a generic set of 2-fat points and $(d_1, d_2) \geq (1, 1)$.

Because of the recent work on the Hilbert functions of points (for the reduced case, see [2, 6] and for the nonreduced case, see [3, 4]) in multiprojective spaces, it is of interest to have an explicit description of this formula. In fact, by coupling [1, Proposition 2.1] with a result from [5], we can describe $H(Z, (i, j))$ for all $(i, j) \in \mathbb{N}^2$.

Theorem 1. *Suppose $Z = 2P_1 + \cdots + 2P_s \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ is a generic set of 2-fat points. Then*

$$H(Z,(i,j)) = \begin{cases} \min\{(i+1), 2s\} & \text{if } j = 0. \\ \min\{(j+1), 2s\} & \text{if } i = 0. \\ \min\{(i+1)(j+1), 3s\} & \text{if } (i,j) \geq (1,1) \text{ and} \\ & \quad (i,j) \notin \{(2, s-1), (s-1, 2)\}. \\ 3s & \text{if } s \equiv 0 \pmod 2 \text{ and} \\ & \quad (i,j) \in \{(2, s-1), (s-1, 2)\}. \\ 3s-1 & \text{if } s \equiv 1 \pmod 2 \text{ and} \\ & \quad (i,j) \in \{(2, s-1), (s-1, 2)\}. \end{cases}$$

Proof. Suppose that $(i, j) \in \mathbb{N}^2$ with $j = 0$. Let $\pi_1 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection morphism given by $\pi_1(P \times Q) = P$. If we set $Z_1 := \pi_1(Z) \subseteq \mathbb{P}^1$, then by [5, Lemma

1991 *Mathematics Subject Classification.* 13D40,14M05.
Key words and phrases. Hilbert function, fat points, multiprojective space.
Updated: August 3, 2011.

4.1] we have

$$H(Z,(i,0)) = H(Z_1,i) \text{ for all } i \in \mathbb{N}.$$

Because the support of Z is in generic position, the first coordinates of the points $P_1,\ldots,P_s$ must all be distinct. Thus Z_1 is a set of s double points in $\mathbb{P}^1$, and hence

$$H(Z,(i,0)) = H(Z_1,i) = \min\{(i+1),2s\} \text{ for all } i \in \mathbb{N}.$$

Notice that if $(i,j) \in \mathbb{N}^2$ with $i = 0$, then the proof is the same except that the projection morphism $\pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ defined by $\pi_2(P \times Q) = Q$ is used.

So, we need to compute $H(Z,(i,j))$ when $(i,j) \geq (1,1)$. By [1, Theorem 1.1] (and the discussion at the start of §2 in [1]) we have

$$(1) \qquad \begin{array}{rcl} H(Z,(i,j)) & = & \dim_k R_{(i,j)} - \dim_k (I_Z)_{(i,j)} \\ & = & (i+1)(j+1) - \dim_k (I_W)_{i+j} \end{array}$$

where I_W is the defining ideal in $S = k[x,y,z]$ of the fat point scheme

$$W = iQ_1 + jQ_2 + 2R_1 + \cdots + 2R_s \subseteq \mathbb{P}^2$$

and $Q_1,Q_2,R_1,\ldots,R_s$ are $s+2$ points in generic position.

The value of $H(W,i+j)$ can be computed using [1, Proposition 2.1]. In particular,

$$(2) \qquad H(W,i+j) = \min\{\dim_k S_{i+j}, \deg W\}$$

except when

$$s = 2a+1 \text{ and } (i,j) = (2,2a) \text{ or } (2a,2).$$

(Note that although [1, Proposition 2.1] assumes that $i \geq j$, if $(i,j) \in \mathbb{N}^2$ with $i < j$, then we can swap the roles of i and j to compute $H(W,i+j)$.) In the two exceptional cases, [1, Proposition 2.1] demonstrated that

$$(3) \qquad h^0(\mathcal{I}_W(i+j)) = \dim_k (I_W)_{i+j} = 1.$$

Using the fact that $\deg W = 3s + \binom{i+1}{2} + \binom{j+1}{2}$, we can use (2) and (3) to derive the following formula for $\dim_k (I_W)_{i+j}$:

$$\dim_k (I_W)_{i+j} = \begin{cases} \max\{0,(i+1)(j+1)-3s\} & \text{if } (i,j) \geq (1,1) \text{ and} \\ & \quad (i,j) \notin \{(2,s-1),(s-1,2)\}. \\ 0 & \text{if } s \equiv 0 \pmod 2 \text{ and} \\ & \quad (i,j) \in \{(2,s-1),(s-1,2)\}. \\ 1 & \text{if } s \equiv 1 \pmod 2 \text{ and} \\ & \quad (i,j) \in \{(2,s-1),(s-1,2)\}. \end{cases}$$

We now substitute the above values into (1). If $(i,j) \geq (1,1)$ and $(i,j) \notin \{(2,s-1),(s-1,2)\}$, then

$$\begin{array}{rcl} H(Z,(i,j)) & = & (i+1)(j+1) - \max\{0,(i+1)(j+1)-3s\} \\ & = & \min\{(i+1)(j+1),3s\}. \end{array}$$

When $(i,j) \in \{(2,s-1),(s-1,2)\}$, then $(i+1)(j+1) = 3s$. So, if $s \equiv 0 \pmod 2$, then

$$H(Z,(i,j)) = 3s - \dim_k (I_W)_{i+j} = 3s.$$

However, if $s \equiv 1 \pmod 2$, then

$$H(Z,(i,j)) = 3s - \dim_k (I_W)_{i+j} = 3s - 1.$$

This completes the proof. □

Acknowledgments. I would like to thank A.V. Geramita for his suggestion to write this appendix. As well, I would like to acknowledge the financial support provided by NSERC.

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