Lecture X: Splittable Ideals (March 21, 2006)

SPEAKER AND NOTES BY: JING HE

First, we want to introduce splittable ideals. Let $\mathcal{G}(I)$ denote the minimal generators of I.

Definition 1. A monomial ideal I is *splittable* if I is the sum of two nonzero monomial ideals J and K, that is, $I = J + K$, such that

(1) $\mathcal{G}(I)$ is the disjoint union of $\mathcal{G}(J)$ and $\mathcal{G}(K)$.
(2) there is a *splitting function*

$$\mathcal{G}(J \cap K) \to \mathcal{G}(J) \times \mathcal{G}(K)$$

$$\omega \rightarrowtail (\phi(\omega), \psi(\omega))$$

satisfying

(a) for all $\omega \in \mathcal{G}(J \cap K), \omega = \operatorname{lcm}(\phi(\omega), \psi(\omega))$
(b) for every subset $S \subset \mathcal{G}(J \cap K)$, both $\operatorname{lcm}(\phi(S))$ and $\operatorname{lcm}(\psi(S))$ strictly divide $\operatorname{lcm}(S)$.

If J and K satisfy the above properties, then we shall say $I = J + K$ is a *splitting* of I.

We want to find splittings of the monomial ideal I since we can then apply the following result:

Theorem 2 (Eliahou-Kervaire, Fatabbi). *Suppose I is a splittable monomial ideal with splitting $I = J + K$. Then for all $i, j \geq 0$,*

$$\beta_{i,j}(I) = \beta_{i,j}(J) + \beta_{i,j}(K) + \beta_{i-1,j}(J \cap K).$$

We now turn our attention to edge ideals.

Definition 3. Let G be a simple graph with vertex set $V_G = \{x_1, \ldots, x_n\}$ and edge set E_G. The *edge ideal* of G, denoted $\mathcal{I}(G)$ is the ideal

$$\mathcal{I}(G) = \{x_i x_j \mid \{x_i, x_j\} \in E_G\} \subseteq k[x_1, \ldots, x_n].$$

Note an edge ideal is an ideal whose generators are square-free monomials of degree 2.

Let G be a simple graph with edge ideal $\mathcal{I}(G)$ and $e = uv \in E_G$. If we set

$$J = (uv) \text{ and } K = \mathcal{I}(G \setminus e),$$

then $\mathcal{I}(G) = J + K$. In general this may not be a splitting of $\mathcal{I}(G)$. The goal of this lecture is to determine when J and K give a splitting of $\mathcal{I}(G)$, and furthermore, how this splitting can be used to ascertain information about the numbers $\beta_{i,j}(\mathcal{I}(G))$. We begin by assigning a name to an edge for which there is a splitting.

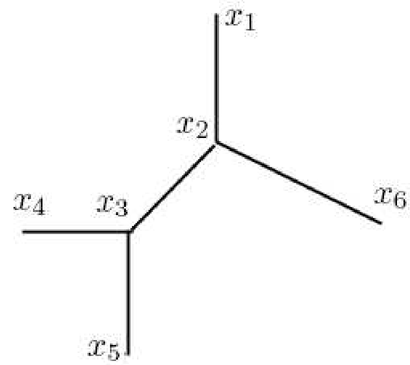
Definition 4. An edge $e = uv$ is a *splitting edge* of G if $J = (uv)$ and $K = \mathcal{I}(G \setminus e)$ is a splitting of $\mathcal{I}(G)$.

We will prove the following theorem:

Theorem 5. *An edge $e = uv$ is a splitting edge of G if and only if $N(u) \subseteq (N(v) \cup \{v\})$ or $N(v) \subseteq (N(u) \cup \{u\})$*

Note $N(u)$ is the the neighborhood of u. We illustrate the theorem using an example.

Example 6. Consider the graph G:



In G the edge x_1x_2 is a splitting edge (we remove the leaf x_1x_2 from the tree G).

To see Theorem 1,

$$\begin{aligned} I &= (x_1x_2, x_2x_3, x_3x_4, x_3x_5, x_2x_6) \\ J &= (x_1x_2) \leftarrow \text{leaf} \\ K &= (x_2x_3, x_3x_4, x_3x_5, x_2x_6) \end{aligned}$$

Then $I = J + K$. Because $N(x_1) = \{x_2\} \subseteq (N(x_2) \cup \{x_2\}) = \{x_2, x_3, x_6\}$, $\{x_1, x_2\}$ is a splitting edge.

We will use the Definition to check that this is a splitting directly.

Note: If M, N are monomial ideals, $M \cap N = \{\text{lcm}(m_1, m_2) | m_1 \in M, m_2 \in N\}$.

Since $J \cap K = (x_1x_2x_3, x_1x_2x_3x_4, x_1x_2x_3x_5, x_1x_2x_6)$, we have $J \cap K = (x_1x_2x_3, x_1x_2x_6)$. Hence

$$\begin{aligned} \mathcal{G}(J \cap K) &= \{x_1x_2x_3, x_1x_2x_6\}, \\ \mathcal{G}(J) &= \{x_1x_2\}, \\ \mathcal{G}(K) &= \{x_2x_3, x_3x_4, x_3x_5, x_2x_6\}, \\ \mathcal{G}(J) \times \mathcal{G}(K) &= \{(x_1x_2, x_2x_3), (x_1x_2, x_3x_4), (x_1x_2, x_3x_5), (x_1x_2, x_2x_6)\} \end{aligned}$$

We define our splitting function $\mathcal{G}(J \cap K) \longrightarrow \mathcal{G}(J) \times \mathcal{G}(K)$ as follows:

$$x_1x_2x_3 \mapsto (x_1x_2, x_1x_3), \phi(x_1x_2x_3) = x_1x_2, \psi(x_1x_2x_3) = x_2x_3$$

$$x_1x_2x_6 \mapsto (x_1x_2, x_2x_6), \phi(x_1x_2x_6) = x_1x_2, \psi(x_1x_2x_6) = x_2x_6$$

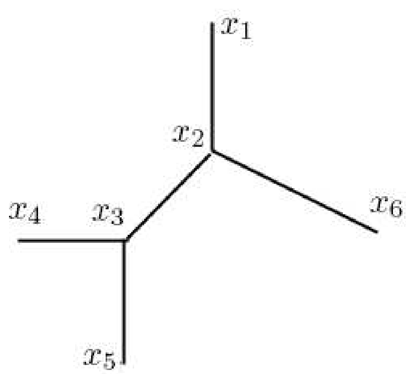
For example, just let $S = \{x_1x_2x_3, x_1x_2x_6\}$, $\mathrm{lcm}(S) = \{x_1x_2x_3x_6\}$, $\phi(S) = \{\phi(x_1x_2x_3), \phi(x_1x_2x_6)\} = \{x_1x_2, x_1x_2\}$, i.e. $\phi(S) = \{x_1x_2\}$. Obviously $\mathrm{lcm}(\phi(S))$ divides $\mathrm{lcm}(S)$. Readers can show $\mathrm{lcm}(\psi(S))$ divides $\mathrm{lcm}(S)$ by yourself.

Lemma 7. *Let $e = uv \in E_G$. Set $\tilde{N}(u) := N(u)\setminus\{v\} = \{u_1, \cdots, u_n\}$ and $\tilde{N}(v) := N(v)\setminus\{u\} = \{v_1, \cdots, v_m\}$. Then*

$$J \cap K = uv((u_1, \cdots, u_n, v_1, \cdots, v_m) + \mathcal{I}(H))$$

where $\mathcal{I}(H)$ is the edge ideal of $H = G \setminus \{u, v, u_1, \cdots, u_n, v_1, \cdots, v_m\}$.

Example 8. See the same graph in the previous example:



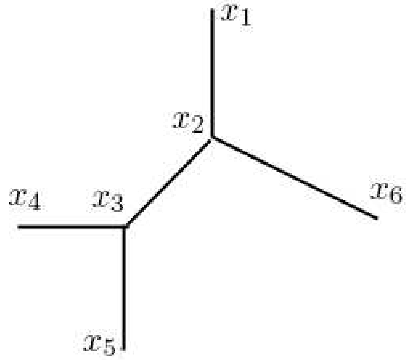
If $e = x_1x_2 \in E_G, \tilde{N}(x_1) := N(x_1) \setminus \{x_2\} = \emptyset, \tilde{N}(x_2) = N(x_2) \setminus \{x_1\} = \{x_3, x_6\}, \mathcal{I}(H) = \{0\}$. Use the above lemma, we have the same result we deduced before, i.e.

$$J \cap K = x_1x_2(x_3, x_6) = (x_1x_2x_3, x_1x_2x_6).$$

Corollary 9. *Let $e = uv \in E_G$, $J = (uv)$ and $K = \mathcal{I}(G \setminus e)$. Then*

$$\begin{aligned} \mathcal{G}(J \cap K) &= \{uvu_i | u_i \in \tilde{N}(u) \setminus (\tilde{N}(u) \cap \tilde{N}(v))\} \cup \{uvv_i | v_i \in \tilde{N}(v) \setminus (\tilde{N}(u) \cap \tilde{N}(v))\} \\ &\quad \cup \{uvz_i | z_i \in (\tilde{N}(u) \cap \tilde{N}(v))\} \cup \{uvm | m \in \mathcal{I}(H)\}. \end{aligned}$$

Example 10. Using the same graph as before,



if $e = x_1x_2$, then $J = (x_1x_2)$, and $K = (x_2x_3, x_3x_4, x_3x_5, x_2x_6)$. We have $\mathcal{I}(H) = (0)$, $\tilde{N}(x_1) = \emptyset$, $\tilde{N}(x_2) = \{x_3, x_6\}$, then $\tilde{N}(x_1) \setminus (\tilde{N}(x_1) \cap \tilde{N}(x_2)) = \emptyset$, $\tilde{N}(x_2) \setminus (\tilde{N}(x_1) \cap \tilde{N}(x_2)) = \tilde{N}(x_2)$, $\mathcal{I}(H) = (0)$. So $\mathcal{G}(J \cap K) = x_1x_2(x_3, x_6)$

Now we give the proof of Theorem 5 using the previous lemmas.

Proof. (of Theorem 5) Without loss of generality, we shall assume that $N(u) \subseteq (N(v) \cup \{v\})$. This condition and Corollary 3.3 then imply that

$$\mathcal{G}(J \cap K) = \{uvv_i \mid v_i \in \tilde{N}(v)\} \cup \{uvm \mid m \in \mathcal{I}(H)\}.$$

To show that $e = uv$ is splitting edge, it suffices to verify that the function

$$\mathcal{G}(J \cap K) \to \mathcal{G}(J) \times \mathcal{G}(K)$$

defined by

$$\omega \mapsto (\phi(\omega), \psi(\omega)) = \begin{cases} (uv, vv_i), & \text{if } \omega = uvv_i; \\ (uv, m), & \text{if } \omega = uvm \end{cases} \tag{0.1}$$

is a splitting function satisfying conditions (a) and (b) of Definition 2.1. Indeed, condition (a) is immediate.

So, suppose $S \subseteq \mathcal{G}(J \cap K)$. From our description of $\mathcal{G}(J \cap K)$ it follows that all the elements of S are divisible by uv. Thus $\operatorname{lcm}(S)$ will also have this property. Furthermore, since $\operatorname{lcm}(\phi(s)) = uv$, it must be the case that uv strictly divides $\operatorname{lcm}(S)$ since $\operatorname{lcm}(S)$ must have degree at least three.

Again it follows from our description of the generators that we can write S as

$$S = \{uvv_{i_1}, \cdots, uvv_{i_j}\} \cup \{uvm_1, \cdots, uvm_t\}$$

where $\{uvv_{i_1}, \cdots, uvv_{i_j}\} \subseteq \tilde{N}(v)$ and $m_i \in \mathcal{I}(H)$. Thus, $\operatorname{lcm}(S) = uvv_{i_1}v_{i_2} \cdots v_{i_j}M$. It is thus clear that $\operatorname{lcm}(\psi(S))$ strictly divides $\operatorname{lcm}(S)$. Condition (b) now follows.

($\Rightarrow$)We prove the contrapositive. Suppose that $e = uv$ is an edge such that $N(u) \nsubseteq (N(v) \cup \{v\})$ and $N(v) \nsubseteq (N(u) \cup \{u\})$. Hence, there exists vertices x and y such that $ux, vy \in E_G$, but $uy, vx \notin E_G$. We now show that no splitting function can exist. Suppose that there was a splitting function

$$\mathcal{G}(J \cap K) \to \mathcal{G}(J) \times \mathcal{G}(K).$$

Since $\mathcal{G}(J) = \{uv\}$ it follows that our splitting function has the form

$$\omega \mapsto (\phi(\omega), \psi(\omega)) = (uv, \psi(\omega)).$$

By Corollary it follows that $uvx, uvy \in \mathcal{G}(J \cap K)$. By condition (a) of Definition 1, we must have $uvx = \operatorname{lcm}(\phi(uvx), \psi(uvx)) = \operatorname{lcm}(uv, \psi(uvx))$. Thus $\psi(uvx) = x, vx, ux$ or uvx. But since $\psi(uvx) \in \mathcal{G}(K)$ and $vx \notin E_G$, this forces $\psi(uvx) = ux$. By a similar argument, $\psi(uvy) = vy$.

Now consider the subset $S = \{uvx, uvy\} \subseteq \mathcal{G}(J \cap K)$. This set fails to satisfy condition (b) of Definition 1 since $\operatorname{lcm}(S) = \operatorname{lcm}(\psi(S)) = uvxy$, contradicting the fact that this function is a splitting function. Thus $e = uv$ is not a splitting edge. □

When $e = uv$ is a splitting edge, we can derive the following identity for the graded Betti numbers of $J \cap K$. The proof for the following two results can be found in the paper of Hà and Van Tuyl [1]

Lemma 11. *Let G be a simple graph with edge ideal $\mathcal{I}(G) \subseteq R$. Suppose that $e = uv$ is a splitting edge with $N(u) \subseteq (N(v) \cup \{v\})$. If $\tilde{N}(v) = \{v_1, \cdots, v_n\}$, $J = (uv)$, and $K = \mathcal{I}(G \setminus e)$, then for $i \geq 1$ and all $j \geq 0$*

$$\beta_{i-1,j}(J \cap K) = \beta_{i,j-2}(R/((v_1, \cdots, v_n) + \mathcal{I}(H)))$$

where $\mathcal{I}(H)$ is the edge ideal of $H = G \setminus \{u, v, v_1, \cdots, v_n\}$.

Corollary 12. *Under the same hypotheses as in the previous Lemma,*

$$\beta_{i-1,j}(J \cap K) = \sum_{l=0}^{i} \binom{n}{l} \beta_{i-l-1,j-2-l}(\mathcal{I}(H));$$

here $\beta_{-1,0}(\mathcal{I}(H)) = 1$ and $\beta_{-1,j}(\mathcal{I}(H)) = 0$ if $j > 0$.

By now applying the result of Eliahou-Kervaire and Fatabbi, we get

Theorem 13. *Let $e = uv$ be a splitting edge of G, and set $H = G \setminus (N(u) \cap N(v))$. If $n = |N(u) \cup N(v)| - 2$, then for all $i \geq 1$ and all $j \geq 0$*

$$\beta_{i,j}(\mathcal{I}(H)) = \beta_{i,j}(\mathcal{I}(G \setminus e)) + \sum_{l=0}^{i} \binom{n}{l} \beta_{i-l-1,j-2-l}(\mathcal{I}(H))$$

where $\beta_{-1,0}(\mathcal{I}(G)) = 1$ and $\beta_{-1,j}(\mathcal{I}(H)) = 0$ if $j > 0$.

The above formula is recursive in the case that G is a forest since the subgraphs $G \backslash e$ and H are forests, and a leaf is always a splitting edge.

Corollary 14. *Let $e = uv$ be any leaf of a forest G. If $\deg v = n$, and $N(v) = \{u, v_1, \cdots, v_{n-1}\}$, then for $i \geq 1$ and $j \geq 0$*

$$\beta_{i,j}(\mathcal{I}(G)) = \beta_{i,j}(\mathcal{I}(T)) + \sum_{l=0}^{i} \binom{n-1}{l} \beta_{i-l-1,j-2-l}(\mathcal{I}(H))$$

where $T = G \setminus e = G \setminus \{u\}$ and $H = G \setminus \{u, v, v_1, \cdots, v_{n-1}\}$. Here $\beta_{-1,0}(\mathcal{I}(H)) = 1$ and $\beta_{-1,j}(\mathcal{I}(H)) = 0$ if $j > 0$.

References

[1] H. T. Hà, A. Van Tuyl, Splittable ideals and the resolutions of monomial ideals. (2005) Preprint. `math.AC/0503203`