Lecture III: Stanley-Reisner Correspondence (Jan. 24, 2006)

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1. INTRODUCTION

So far we have only studied the combinatorial properties of simplicial complexes. We will now introduce some algebra.

2. STANLEY-REISNER RING

We begin by recalling some results from abstract algebra.

- $k[x_1, \ldots, x_n]$ is the set of all polynomials in the variables, $x_1, \ldots, x_n$ with coefficients in the field k (usually we will take k to be $\mathbb{C}$ or $\mathbb{R}$).
- $k[x_1, \ldots, x_n]$ is a ring (it is equipped with an addition and multiplication).
- A subset $I \subseteq R = k[x_1, \ldots, x_n]$ is an ideal if
  - (i) $F - G \in I$ for all $F, G \in I$.
  - (ii) $FG \in I$ for all $F \in I$ and all $G \in R$.
- I is generated by $F_1, \ldots, F_r$, usually written $I = (F_1, \ldots, F_r)$, if

$$I = \{G_1F_1 + \cdots + G_rF_r \mid G_i \in k[x1, \ldots, x_n]\}.$$

Definition 2.1. Suppose Δ is a simplicial complex on the vertex set $V = \{x_1, \ldots, x_n\}$ The Stanley-Reisner ring is the quotient ring

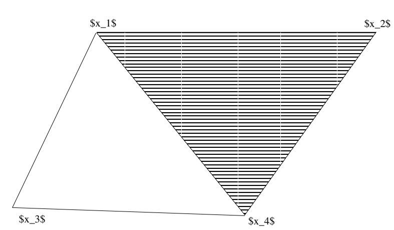
$$k[x_1, \ldots, x_n]/I_\Delta$$

where

$$I_\Delta = (\{x_{i_1} \cdots x_{i_r} \mid \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta\}).$$

That is, the generators of I_Δ correspond to the nonfaces of Δ.

Example 2.2. If Δ is the simplicial complex



then the nonfaces are

$$\{x_2, x_3\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_2, x_3, x_4\}, \{x_1, x_2, x_3, x_4\}.$$

Thus, the Stanley-Reisner ideal I_Δ is given by

$$I_\Delta = (x_2x_3, x_1x_3x_4, x_1x_2x_3, x_2x_3x_4, x_1x_2x_3x_4).$$

Note that our set of generators is not a minimal set of generators. For example, since $x_2x_3 \in I_\Delta$, it follows from the definition of an ideal that $x_1x_2x_3x_4 = (x_1x_4)(x_2x_3) \in I_\Delta$. Thus, if remove the generator $x_1x_2x_3x_4$ from the above list, the remaining generators still generate I_Δ.

More precisely, I_Δ is generated by the minimal nonfaces of Δ. So, in our example,

$$I_\Delta = (x_2x_3, x_1x_3x_4).$$

Definition 2.3. A monomial is a term of the form $m = x_1^{a_1} \cdots x_n^{a_n}$ with $a_i \in \mathbb{N}$. A monomial is square-free if $0 \leq a_i \leq 1$ for all i. An ideal I is a (square-free) monomial ideal if I is generated by (square-free) monomial ideals.

Remark 2.4. I_Δ is always a square-free monomial ideal. In fact, we have a bijection:

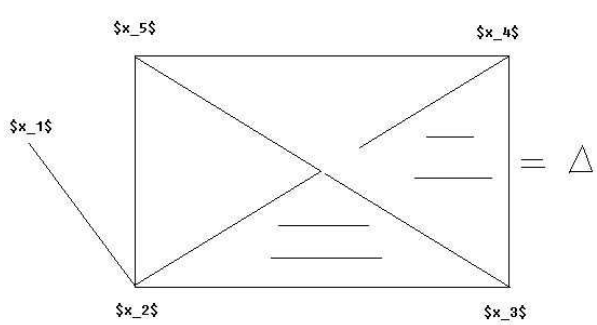
$$\{\text{simplicial complexes}\} \longleftrightarrow \{\text{square-free monomial ideals}\}$$

given by $\Delta \longmapsto I_\Delta$.

Example 2.5. We show how one can associate to a square-free monomial ideal a simplicial complex. Let

$$I = (x_1x_3, x_1x_4, x_1x_5, x_2x_3x_5, x_2x_4x_5, x_3x_4x_5).$$

Then the generators correspond to the minimal nonfaces. So, the square-free monomials of $R = k[x_1, x_2, x_3, x_4, x_5]$ that are not in I correspond to the faces of the simplicial complex Δ. For example, x_1x_2 is not in I, so there is an edge between x_1 and x_2. For this ideal we get the simplicial complex:



The ring R/I_Δ encodes information about the simplicial complex Δ. In the next sections we show how some of ths information is encoded.

3. Dimension

Definition 3.1. An ideal P of a ring R (here R is any ring) is a prime ideal if whenever $ab \in P$, then either $a \in P$ or $b \in P$.

Definition 3.2. The (Krull) dimension of a ring, denoted $\dim R$, is the length of the longest chain of prime ideals in R, i.e.

$$\dim R = \sup\{d \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_d \subsetneq R, \text{ with } P_i \text{ prime}\}.$$

Example 3.3. In $R = k[x_1, \ldots, x_n]$ any ideal generated by a subset of $\{x_1, \ldots, x_n\}$, e.g. $I = (x_{i_1}, \ldots, x_{i_r})$ is a prime ideal. So $\dim K[x_1, \ldots, x_n] \geq n$. Since

$$(0) \subsetneq (x_1) \subsetneq (x_1, x_2) \subsetneq \cdots \subsetneq (x_1, x_2, \ldots, x_n)$$

is a chain of length n of prime ideals.

Remark 3.4. $\dim k[x_1, \ldots, x_n] = n$, but this is nontrivial to show.

If R/I is a quotient ring, the prime ideals of R/I have the form P/I where P is a prime ideal of R and $I \subseteq P$. Also $P_1/I \subseteq P_2/I_1 \iff I \subseteq P_1 \subseteq P_2$. Thus

$$\dim R/I = \sup\{d \mid I \subsetneq P_0 \subsetneq P_1 \subsetneq, \cdots, \subsetneq P_d \subsetneq R\}.$$

Theorem 3.5. *If Δ is a simplicial complex, then*

$$\dim R/I_\Delta = \dim \Delta + 1$$

where by $\dim R/I_\Delta$ *we mean the dimension of the ring, and* $\dim \Delta$ *we mean the dimension of the simplicial complex.*

4. Hilbert Series.

Definition 4.1. A polynomial $F \in R = k[x_1, \ldots, x_n]$ is homogeneous if all its terms have the same degree.

Example 4.2. The following are examples in $R = k[x_1, x_2, x_3]$:

- $3x_1x_2x_3 + 4x_1^2x_2 + 7x_3^3$ ←homogeneous.
- $3x_1x_2x_3 + 4x_3^7$ ← not homogeneous.
- a monomial is always homogenous.

Set R_i = all homogenous polynomial at degree i. Then R is a graded ring, i.e.

$$R = \bigoplus_{i \in \mathbb{N}} R_i \text{ and } R_iR_j \subseteq R_{i+j}.$$

The set R_i is a vector space over k. A basis for R_i is the set of all monomials of degree i.

Example 4.3. If $R = k[x_1, x_2], R_3$ is the k-vector space with basis $B = \{x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$.

Lemma 4.4. *If* $R = k[x_1, \dots x_n]$ *then* $\dim_k R_i = \binom{n-1+i}{i}$.

An ideal I is homogenous if I is generated by homogenous elements. Let $I_i = I \cap R_i$. This is the set of all homogenous elements of degree i in I. Furthermore, I_i is a subspace of R_i. If I is a homogenous ideal of $R = k[x_1, \dots, x_n]$, then R/I is also a graded ring. That is

$$R/I = \bigoplus_{i \in \mathbb{N}} (R/I)_i.$$

Here $(R/I)_i$ is the k-vector space R_i/I_i.

Note: $\dim_k(R/I)_i = \dim_k R_i - \dim_k I_i$.

We can encode the information about dimensions into a generating function.

Definition 4.5. Let I be a homogenous ideal of $R = K[x_1, \dots, x_n]$. The Hilbert series of R/I is the formal power series.

$$HS(R/I, t) = \sum_{i \in \mathbb{N}} (\dim_k(R/I)_i)\, t^i.$$

Example 4.6. Suppose $R = k[x_1, x_2]$ and $I = (0)$. So $R/I = R$ and $\dim_k(R/I)_i = \dim_k R_i$. Since a basis for R_i is given by $\{x_1^i, x_1^{i-1}x_2, \dots, x_1x_2^{i-1}, x_2^i\}$, we have $\dim_k R_i = (i+1)$. So

$$HS(R, t) = \sum_{i \in \mathbb{N}} (i+1)t^i = 1 + 2t + 3t^2 + 4t^3 + \cdots$$

But $1/(1-t)^2 = 1 + 2t + 3t^2 + 4t^3 + \cdots$, so

$$HS(R, t) = 1/(1-t)^2.$$

When I is a homogenous ideal of $R = k[x_1, \dots, x_n]$, the $HS(R/I, t)$ is always a rational function.

Theorem 4.7. *Suppose* R/I *has dimension* d*. Then there exists a unique polynomial* $h(t) = h_0 + h_1t + \cdots + h_lt^l \in \mathbb{Z}[t]$ *such that* $h(1) \neq 0$ *and*

$$HS(R/I, t) = \frac{h_0 + h_1t + \cdots + h_lt^l}{(1-t)^d}$$

Definition 4.8. Suppose the Hilbert series of R/I is $HS(R/I, t) = \frac{h_0 + h_1t + \cdots + h_lt^l}{(1-t)^d}$. Then the h-vector of R/I is the tuple $h(R/I) = (h_0, h_1, \dots, h_l)$.

The f-vector can be used to compute the Hilbert series of R/I_Δ.

Theorem 4.9. *Let Δ be a simplicial complex with f-vector $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$. Then*

$$HS(R/I_\Delta, t) = \sum_{i=-1}^{d-1} \frac{f_i t^{i+1}}{(1-t)^{i+1}}.$$

Example 4.10. Consider the simplicial complex Δ:

The f-vector of Δ is $f(\Delta) = (4, 5, 1)$. So, the Hilbert series of R/I_Δ is given by

$$\begin{aligned} HS(R/I_\Delta, t) &= \frac{f_{-1}t^0}{(1-t)^0} + \frac{f_0 t^1}{(1-t)} + \frac{f_1 t^2}{(1-t)^2} + \frac{f_2 t^3}{(1-t)^3} \\ &= 1 + \frac{4t}{(1-t)} + \frac{5t^2}{(1-t)^2} + \frac{1t^3}{(1-t)^3} \\ &= \frac{(1-t)^3 + 4t(1-t)^2 + 5t^2(1-t) + t^3}{(1-t)^3} \\ &= \frac{1+t+t^3}{(1-t)^3}. \end{aligned}$$

The h-vector of Δ is then given by $h(\Delta) = (1, 1, 0, 1)$.

Finally there is a relationship between the f-vectors and h-vectors:

Lemma 4.11.

$$h_j = \sum_{i=0}^{j} (-1)^{j-1} \binom{d-i}{j-i} f_{i-1} \text{ and } f_{j-1} = \sum_{i=0}^{j} \binom{d-i}{j-i} h_i.$$

Problems from Lecture 3

1. Let Δ be a simplex, i.e., $\Delta = \langle F \rangle$ has a single facet. If $\dim \Delta = d-1$, determine the h-vector of Δ. (Hint: You had to calculate the f-vector for last week's problems.)

2. What is the Stanley-Reisner ideal I_Δ associated to the simplicial complex Δ on the vertex set $V_\Delta = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ where Δ has the maximal facets:

$$\{x_1, x_2, x_3, x_4\}, \{x_3, x_5\}, \{x_4, x_5\}, \{x_5, x_6\}.$$

3. Let Δ be a simplicial complex, and let $\sigma \in \Delta$ be a face. The *star* of σ is
$$\text{star}(\sigma) = \{G \mid \sigma \cup G \in \Delta\}.$$
Show that $\text{star}(\sigma)$ is a simplicial complex.
4. Let Δ be a simplicial complex, and let $\sigma \in \Delta$ be a face. After relabeling the vertices, we can assume $\sigma = \{x_1, \ldots, x_r\}$. Let $f = \Pi_{i=1}^r x_i = x_1 x_2 \cdots x_r$ be a polynomial in $k[x_1, \ldots, x_n]$. Show that the Stanley-Reisner ideal of the simplicial complex $\text{star}(\sigma)$ satisfies
$$I_{\text{star}(\sigma)} = (I_\Delta : f).$$
Recall that if I and J are ideals of ring R, then
$$I : J = (F \in R \mid FG \in I \text{ for all } G \in J)$$
Hint: Since I_Δ is a monomial ideal and f is a monomial, then $(I_\Delta : f)$ is generated by square-free monomials. Since $I_{\text{star}(\sigma)}$ is also generated by square-free monomials, it suffices to show that every square-free monomial of $I_{\text{star}(\sigma)}$ is also in $(I_\Delta : f)$ and conversely.