Lecture IV: Cohen-Macaulay Rings (Jan. 31, 2006)
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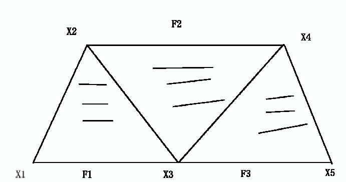
1. SHELLABLE SIMPLICIAL COMPLEXES

We begin by introducing a "nice" class of simplicial complexes which are called shellable. Recall that a simplicial complex Δ of dimension $(d-1)$ is pure if all the facets of Δ have dimension $(d-1)$ i.e., $|F| = d$ for all facets.

Definition 1.1. A pure simplicial complex Δ is *shellable* if the facets of Δ can be listed $F_1, F_2, \ldots, F_n$ such that for all $1 \leq j < i \leq n$ there exists some $v \in F_i \setminus F_j$ and some $k \in \{1, \ldots, i-1\}$ with $F_i \setminus F_k = \{v\}$.

Example 1.2. The simplicial complex $\Delta =$



is shellable since

$$x_4 \in F_2 \setminus F_1 \text{ and } \{x_4\} = F_2 \setminus F_1$$
$$x_5 \in F_3 \setminus F_1 \text{ and } \{x_5\} = F_3 \setminus F_2$$
$$x_5 \in F_3 \setminus F_2 \text{ and } \{x_5\} = F_3 \setminus F_2.$$

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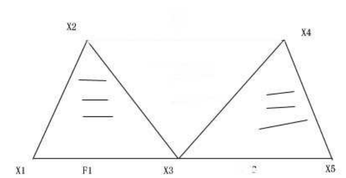


is not shellable since

$$x_1 \in F \setminus G, \text{ but } \{x_1\} \neq F \setminus G \text{ (or } G \setminus F)$$

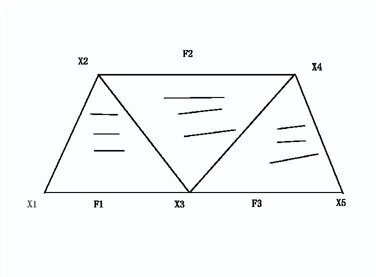
$$x_2 \in F \setminus G, \text{ but } \{x_2\} \neq F \setminus G \text{ (or } G \setminus F)$$

$$x_5 \in G \setminus F, \text{ but } \{x_5\} \neq G \setminus F \text{ (or } F \setminus G).$$

An equivalent definition for a shellable complex is given below.

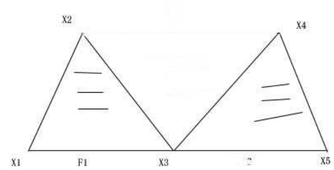
Definition 1.3. A pure simplicial complex Δ is shellable if the facets of Δ can be given a linear order $F_1, \cdots, F_n$ such that $< F_i > \cap < F_1, \ldots, F_{i-1} >$ is generated by a nonempty set of maximal proper faces F_i for $i = 1, \ldots, n$.

Example 1.4. Consider the simplicial complex



Then we have
$< F_2 > \cap < F_1 > = < \{x_2, x_3\} > \longleftarrow$ a maximal proper face of F_2
$< F_3 > \cap < F_1, F_2 > = < \{x_3\}, \{x_3, x_4\} > = < \{x_3, x_4\} > \longleftarrow$ a maximal proper face of F_3

Example 1.5. We now look at the simplicial complex:



For this example, we have

$$< F > \cap < G > = < \{x_3\} > \longleftarrow \text{not a maximal proper face of } F \text{ or } G.$$

Note that the maximal proper faces are F are $\{x_1, x_2\}, \{x_2, x_3\}, \{x_3, x_1\}$.

Recall that if Δ is a simplicial complex, then the Stanley-Reisner ideal is

$$I_\Delta = (\{x_{i_1} \cdots x_{i_r} \mid \{x_{i_1}, \ldots, x_{i_r}\} \notin \Delta\})$$

The quotient ring R/I_Δ is the Stanley-Reisner ring. The Stanley-Reisner ring of a shellable simplicial complex is a special type of ring; it is an example of a Cohen-Macaulay ring which is defined in the next section.

2. Cohen-Macaulay Rings

To define a Cohen-Macaulay (CM) ring, we need the notions of (Krull) dimension and regular sequences.

2.1. **Dimension.** Recall that a prime ideal of a ring S is an ideal $P \subsetneq S$ such that whenever $ab \in P$ then either $a \in P$ or $b \in P$. A chain of prime ideals is a strictly increasing sequence of prime ideals, i.e.

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \subseteq S$$

We say n is the length of the chain.

Definition 2.1. The (Krull) dimension of R, denoted $\dim R$, is

$$\dim R = \sup\{n \mid P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \text{ is a chain of prime ideals in } R\}$$

Example 2.2. If $R = k[x_1, \ldots, x_n]$, then $\dim R = n$

Example 2.3. Let $I = (x_1x_3, x_1x_4, x_2x_3, x_2x_4) = (x_1, x_2) \cap (x_3, x_4)$ in $R = k[x_1, x_2, x_3, x_4]$. We will compute the dimension of R/I.

First, recall that $\mathcal{P}$ is a prime ideal in R/I if and only if there exists a prime ideal $I \subseteq P \subsetneq R$ such that $\mathcal{P} = P/I$. Also, note that if P is any prime ideal with $I \subseteq P$, then either

(1) $x_1, x_2 \in P$ or
(2) $x_3, x_4 \in P$.

Set $\mathcal{P}_0 = (x_1, x_2)/I$, $\mathcal{P}_1 = (x_1, x_2, x_3)/I$, and $\mathcal{P}_2 = (x_1, x_2, x_3, x_4)/I$. Then $\mathcal{P}_0 \subsetneq \mathcal{P}_1 \subsetneq \mathcal{P}_2$ is a chain of prime ideals in R/I, so it follows that $\dim R/I \geq 2$.

Suppose there is a chain $\mathcal{Q}_0 \subsetneq \mathcal{Q}_1 \subsetneq \cdots \subsetneq \mathcal{Q}_n \subsetneq R/I$ with $n \geq 3$. So $\mathcal{Q}_i = Q_i/I$ for some prime ideal $I \subsetneq Q_i \subsetneq R$. Thus, we have a chain

$$Q_0 \subsetneq Q_1 \subsetneq \cdots \subsetneq Q_n \subsetneq R.$$

Suppose we are in case (1), i.e., $x_1, x_2 \in Q_0$. Then

$$(0) \subsetneq (x_1) \subsetneq Q_0 \subsetneq \cdots \subsetneq Q_n.$$

is a chain of length $n + 2 \geq 3 + 2 = 5$ in R. This contradicts the fact that $\dim R = 4$. A similar argument for case (2) will give us a similar conclusion. Thus $\dim R/I \not> 2$. Hence, $\dim R/I = 2$.

2.2. **Regular sequence.** A zero divisor of a ring R is an element $a \in R$ such that $a \neq 0$ and there exists $0 \neq b \in R$ such that $ab = 0$.

Definition 2.4. Let $I \subset R = k[x_1, \cdots, x_n]$. An element $F \in R$ is a regular element on R/I if $\bar{F} = (F + I)$ is not a zero divisor of R/I. Equivalently, F is regular on R/I if whenever $FG \in I$, then $G \in I$.

Example 2.5. Consider any $x_i \in R = k[x_1, \ldots, x_n]$. Then x_i is regular on $R = R/(0)$ since R is a domain.

Example 2.6. Suppose $I = (xyz) \subseteq k[x, y, z]$. Then xy is not regular on R/I since $\overline{xy} \neq \overline{0} \in R/I$ and $\overline{z} \neq \overline{0} \in R/I$ but $\overline{xy}(\overline{z}) = \overline{xyz} = \overline{0}$ in R/I.

Example 2.7. Let $I = (x_1, x_2) \cap (x_3, x_4) \subset k[x_1, x_2, x_3, x_4] = R$. We show that $(x_1 + x_3)$ is regular on R/I.

Suppose $(x_1 + x_3)G \in J = (x_1, x_2) \cap (x_3, x_4)$. So $(x_1 + x_3)G \in (x_1, x_2)$ and $(x_1 + x_3)G \in (x_3, x_4)$. Both (x_1, x_2) and (x_3, x_4) are prime ideals. Also $(x_1 + x_3) \notin (x_1, x_2)$ and (x_3, x_4). So $G \in (x_1, x_2) \cap (x_3, x_4) = I$.

Definition 2.8. A sequence $F_1, \ldots, F_m$ of R is called a regular sequence on R/I if

(1) $\overline{F_1}$ is regular on R/I, and
(2) $\overline{F_i}$ is regular on $R/(I, F_1, \cdots, F_{i-1})$.

Example 2.9. If $R = k[x_1, \ldots, x_n]$ and $I = (0)$, then $x_1, \ldots, x_n$ is a regular sequence on R/I since

(1) $\overline{x_1}$ is regular on $R/(0)$.
(2) $\overline{x_i}$ is regular on $R/(x_1, \ldots, x_{i-1}) \cong K[x_i, \cdots, x_n]$.

Theorem 2.10. *All maximal regular sequence have same length, and any regular sequence can be extended to a maximal regular sequence.*

Definition 2.11. The depth of R/I, denoted depth (R/I), is the length of the longest maximal sequence on R contained in $\mathfrak{m} = (x_1, x_2, \ldots, x_n)$.

Theorem 2.12. *For any ideal* $I \subseteq k[x_1, \ldots, x_n] = R$*,* $\operatorname{depth}(R/I) \leq \dim(R/I)$.

Definition 2.13. A ring R/I is *Cohen-Macaulay* if $\operatorname{depth}(R/I) = \dim(R/I)$.

Example 2.14. $R = k[x_1, \ldots, x_n]$ is Cohen-Macaulay since $\operatorname{depth}(R/I) = \dim(R/I) = n$.

Example 2.15. If $I = (x_1, x_2) \cap (x_3, x_4) \subseteq R = k[x_1, \ldots, x_4]$, then R/I is not Cohen-Macaulay. We saw that $\dim R/I = 2$ and $x_1 + x_3$ is regular on R/I. So $1 \leq \operatorname{depth}(R/I)$. We want to show that $\operatorname{depth}(R/I) = 1$.

Take any $G \in \mathfrak{m} = (x_1, x_2, x_3, x_4)$. Need to show G cannot be regular on $R/(I, x_1 + x_3)$. We can write G as

$$G = G_1(x_1, x_2, x_3, x_4)x_1 + G_2(x_2, x_3, x_4)x_2 + G_3(x_3, x_4)x_3 + G_4(x_4)x_4.$$

Suppose $\overline{G} \neq \overline{(0)}$ in $R/(I, x_1 + x_3)$. This implies that $G \notin (I, x_1 + x_3)$. Note that $\overline{(x_1)} \neq \overline{(0)} \in R/(I, x_1 + x_3)$. But $Gx_1 = G_1 x_1^2 + G_2 x_1 x_2 + G_3 x_3 x_1 + G_4 x_4 x_1$.

Now

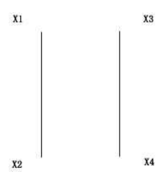
$$\begin{aligned} x_1^2 = x_1(x_1 + x_3) - x_1x_3 &\in (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_1 + x_3) \\ x_1x_2 = x_2(x_1 + x_3) - x_2x_3 &\in (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_1 + x_3) \\ x_3x_1 &\in (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_2 + x_3) \\ x_4x_1 &\in (x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_1 + x_3). \end{aligned}$$

So $Gx_1 \in (I, x_1 + x_3)$ but $G \notin (I, x_1 + x_3)$. Thus G is not regular. Therefore we cannot extend the length of the regular sequence. So $\text{depth}(R/I) = 1$.

We now relate Cohen-Macaulay with the notion of shellable introduced at the beginning of this talk.

Theorem 2.16. *Suppose that Δ is a shellable simplicial complex. If R/I_Δ is the associated Stanley-Reisner ring, then R/I_Δ is Cohen-Macaulay.*

Example 2.17. Let Δ be the simplicial complex



Then $I_\Delta = (x_1x_3, x_1x_4, x_2x_3, x_3x_4)$. This simplicial complex Δ is not shellable since R/I_Δ is not Cohen-Macaulay as shown in Example 2.15.

Problems from Lecture 4

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1. Let Δ be a pure simplicial complex. Prove that Δ is shellable if and only if the facets of Δ can be ordered $F_1, \ldots, F_s$ such that for all $1 \leq j < i \leq s$, there is a $v \in F_i \backslash F_j$ and $k < i$ with $F_i \cap F_j \subset F_i \cap F_k = F_i \backslash \{v\}$.

2. Suppose that $F_1, \dots, F_m$ are elements of R that form a regular sequence on R/I where I is ideal R. Show that $F_1^{t_1}, \dots, F_m^{t_m}$ is also a regular sequence on R/I for all positive integers $t_1, \dots, t_m$.

3. This example shows that the order of the sequence $\{F_1, \dots, F_m\}$ is important when defining a regular sequence. Let $R = k[x_1, x_2, x_3]$ with k a field. Set $F_1 = x_1$, $F_2 = x_2(1 - x_1)$ and $F_3 = x_3(1 - x_1)$. Show
   (a) F_1, F_2, F_3 is a regular sequence on R.
   (b) F_2, F_3, F_1 is not a regular sequence on R.