Lecture VI: Facet Ideals (Feb. 14, 2006)

SPEAKER AND NOTES BY: JING HE

Goal:

(1) Introduce facet ideals and Stanley-Reisner ideals;

(2) Analyze the relationship between facet ideals and Stanley-Reisner ideals.

Recall: Definition (Simiplicial complex) Let $V = \{x_1, \dots, x_n\}$ be a finite set, then an (abstract) *simplicial complex* Δ is a subset of $P(V)$ (If V is a set, then the power set of V, denoted $P(V)$ is the set of all subsets of V) such that

- $\{x_i\} \in \Delta$ for each $i = 1, 2, \dots, n$;
- if $F \in \Delta$ and $G \subseteq F$, then $G \in \Delta$.

An element of Δ is called a face of Δ. The maximal faces of Δ under inclusion are called facets of Δ.

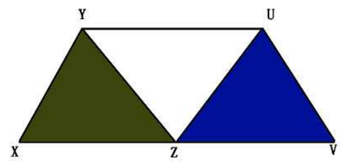
Denote: $\Delta =< F_1, \dots, F_q >$, where F_i is facet of Δ.

We introduce facet ideals; facet ideals were first defined by Sara Faridi.

Definition 1. Let k be a field and $R = k[x_1, \dots, x_n]$ be a polynomial ring.

- We define the facet ideal of Δ, denoted by $\mathcal{F}(\Delta)$ to be the ideal of R generated by square-free monomials $x_{i_1} \cdots x_{i_s}$, where $\{x_{i_1}, \dots, x_{i_s}\}$ is a facet of Δ, i.e.
$\mathcal{F}(\Delta) = (\{x_{i_1} \cdots x_{i_s} \mid \{x_{i_1}, \dots, x_{i_s}\} \in \Delta \text{ is a facet}\})$.
- Let $I = (M_1, \dots, M_q)$ be an ideal of R, where $M_1, \dots, M_q$ are square-free monomials in $x_1, \dots, x_n$ that form a minimal set of generators for I. We define the facet complex of I, denoted by $\delta_{\mathcal{F}}(I)$ to be the simplicial complex over a set of vertices $x_1, \dots, x_n$ with facets $F_1, \dots, F_q$, where for each i, $F_i = \{x_j \mid x_j \mid M_i, 1 \leq j \leq n\}$.

Example 2. Let $\Delta =< xyz, uvz, yu >$



then $\mathcal{F}(\Delta) = (xyz, uvz, yu)$ is the facet ideal of Δ in $R = k[x, y, z, u, v]$.

$$\{\text{simplical complexes}\} \longleftrightarrow \{\text{monomial ideals}\}$$

$$\Delta \longmapsto \mathcal{F}(\Delta)$$

$$\delta_{\mathcal{F}}(I) \longleftarrow I$$

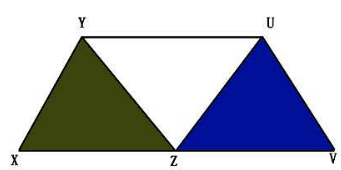
Facet ideals give a one-to-one correspondence between simplicial complexes and square-free monomial ideals.

Definition 3. A vertex cover for Δ is a subset P of V that intersects every facet of Δ, i.e. $P \cap F_i \neq \emptyset$, where F_i is any facet of Δ. If P is a minimal element of the set of vertex covers of Δ, then P is called a *minimal vertex cover.*

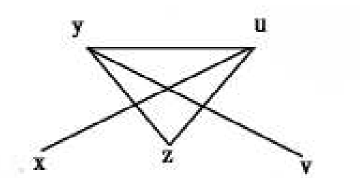
Example 4. Let Δ be the simplicial complex as in the above example. Then the vertex covers of Δ are $\{y, u\}, \{y, z\}, \{u, z\}, \{x, u\}, \{y, v\}, \{x, y, u\}, \{v, y, u\}, \ldots$.
So $\{y, u\}, \{y, z\}, \{u, z\}\{x, u\}, \{y, v\}$ are the minimal vertex covers of Δ.

Definition 5. Let Δ_M be the simplicial complex whose facets are the minimal vertex covers of Δ. We call Δ_M the *cover complex.*

Example 6. If $\Delta =< xyz, uvz, yu >$ is the simplicial complex of the above example,



then the cover complex of Δ is $\Delta_M = < yu, yz, uz, xu, yv >$. This simplicial complex looks like:



Proposition 7. *If Δ is a simplicial complex, then Δ_M is a dual of Δ; i.e. $(\Delta_M)_M = \Delta$.*

See the proof of Proposition 10 in [1].

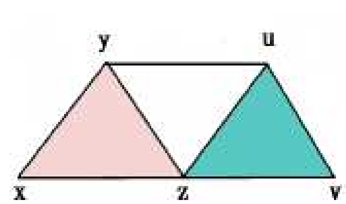
Now we introduce the nonface ideal:

Definition 8. Let k be a field, and let $R = k[x_1, \ldots, x_n]$ be a polynomial ring.

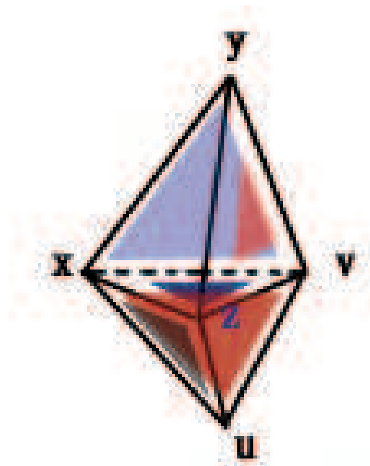
- Let $V = \{x_1, \ldots, x_n\}$ and Δ be a simplicial complex over V. We define the *nonface ideal*, or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, to be the ideal of R generated by square-free monomials $x_{i_1} \cdots x_{i_s}$, where $\{x_{i_1}, \ldots, x_{i_s}\}$ is not a face of Δ.
- Let $I = (M_1, \ldots, M_q)$ be an ideal in R, where $M_1, \ldots, M_q$ are square-free monomials in $x_1, \ldots, x_n$ that form a minimal set of generators for I. We define the *nonface complex*, or the *Stanley-Reisner complex* of I, denoted by $\delta_N(I)$, to be the simplicial complex over $\{x_1, \ldots, x_n\}$, where $\{x_{i_1}, \ldots, x_{i_s}\}$ is a face of $\delta_N(I)$ if and only if $x_{i_1} \cdots x_{i_s} \notin I$.

Note: To simplify notation: $\Delta_N = \delta_N(\mathcal{F}(\Delta))$.

Example 9. Let $I = (xyz, uvz, yu)$. Then this is the facet ideal of



but it's also the Stanley-Reisner ideal of



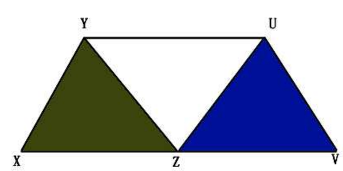
The faces xyz and zuv are missing.

Definition 10. Let I be a square-free monomial ideal in the polynomial ring $k[x_1, \dots, x_n]$. Then the *Alexander dual* of Δ_N is the simplicial complex

$\Delta_N^\vee = \{F \subset V \mid F^c \notin \Delta_N\}$, where $F^c = V - F$.

We also have $\Delta_N^{\vee\vee} = \Delta_N$.

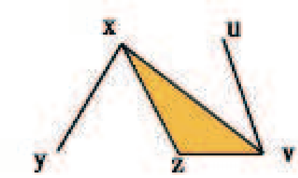
Example 11. Let $I = (xyz, uvz, yu)$.



Then the facet ideal of Δ_M (Δ_M is the cover complex of Δ) is

$$J = \mathcal{F}(\Delta_M) = (yu, yz, uz, xu, yv)$$

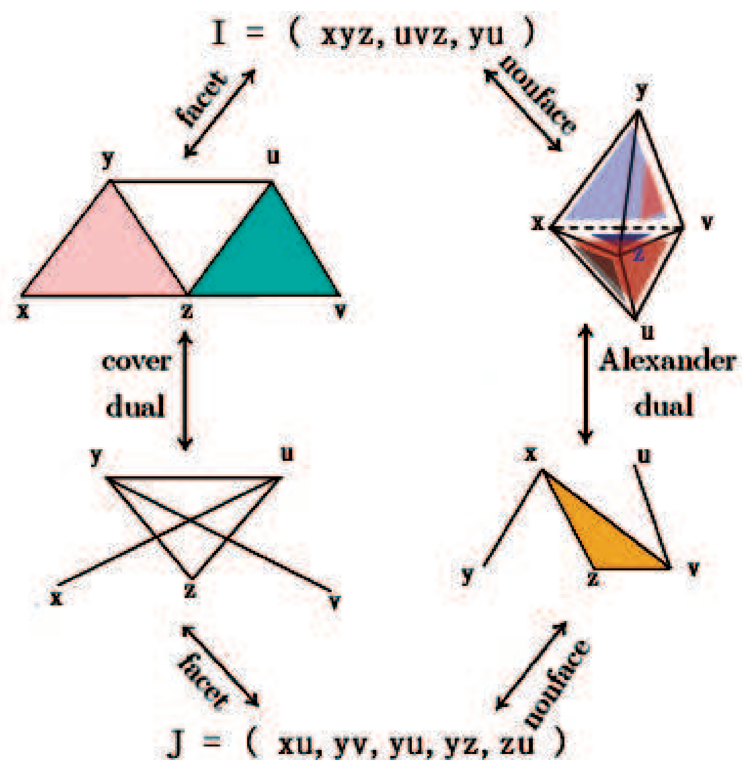
Let $\Delta_N = \delta_N(I)$. This is the simplicial complex in the precious example. Then the Alexander dual of Δ_N is $\Delta_N^\vee =< xy, xvz, uv >$. It looks like:



The ideal J is also the nonface ideal of $\Delta_N^\vee$. Since the nonface ideal of $\Delta_N^\vee$ is (yu, yz, uz, xu, yv). So the facet ideal of Δ_M space, i.e. $\mathcal{F}(\Delta_M)$, is equal to the nonface ideal of $\Delta_N^\vee$.

The following picture summarizes the relationships between facet ideals and nonface ideals.

Example 12. Let $I = (xyz, uvz, yu)$.



References

[1] Sara Faridi, *Simplicial trees are sequentially Cohen-Macaulay*, J. Pure and Applied Algebra, **190** (2004), no. 1-3, 121-136.