Lecture VII: Minimal Free Resolutions (Feb. 28, 2006)

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Throughout this lecture I is a homogeneous ideal of $R = k[x_1, \ldots, x_n]$ where k is a field. Our goal is to to describe the resolution of an ideal. The resolution encodes many invariants of I and R/I.

1. Some Linear Algebra

Recall that a function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if $T(\overrightarrow{x} + \overrightarrow{y}) = T(\overrightarrow{x}) + (\overrightarrow{y})$ for all $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{R}^n$, and $T(c\overrightarrow{x}) = cT(\overrightarrow{x})$ for all $c \in \mathbb{R}$ and $\overrightarrow{x} \in \mathbb{R}^n$.

Theorem 1.1. *If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation, then there exists a $m \times n$ matrix A such that $T(c\overrightarrow{x}) = A\overrightarrow{x}$. Precisely*

$$A = \begin{bmatrix} T(e_1) & T(e_2) & \cdots & T(e_n) \end{bmatrix}$$

where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$.

Example 1.2. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + x_2 \\ 4x_1 + 5x_2 \end{bmatrix}.$$

Then $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$. So

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

2. Linear Algebra over Polynomial Rings

If we replace $\mathbb{R}$ by $R = k[x_1, \ldots, x_n]$, we can derive similar results. Let

$$R^n = \left\{ \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \;\middle|\; f_i \in R = k[x_1, \ldots, x_n] \right\}.$$

Note that R^n is a free R-module under the operation

$$g\begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} = \begin{bmatrix} gf_1 \\ \vdots \\ gf_n \end{bmatrix} \text{ with } g \in R \text{ and } \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} \in R^n.$$

Definition 2.1. A function $T : R^n \to R^m$ is an R-module homomorphism if $T(\underline{x} + \underline{y}) = T(\underline{x}) + (\underline{y})$ for all $\underline{x}, \underline{y} \in R^n$. $T(c\underline{x}) = cT(\underline{x})$ for all $c \in R$ and $\underline{x} \in R^n$

Theorem 2.2. *If $T : R^n \to R^m$ is an R-module homomorphism, then there exists an $m \times n$ matrix with entries in R such that $T(\underline{x}) = A\underline{x}$. In particular,*

$$A = \begin{bmatrix} T(\underline{e}_1) & T(\underline{e}_2) & \cdots & T(\underline{e}_n) \end{bmatrix}$$

where $\underline{e}_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} \in R^n$.

Example 2.3. If $R = k[x, y, z]$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is the R-module homomorphism given by

$$T\left(\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}\right) = \begin{bmatrix} f_1x + f_2z \\ -f_1x + f_3z \\ -f_2x - f_3y \end{bmatrix}$$

then

$$T\left(\begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}\right) = \begin{bmatrix} x & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}.$$

3. Building a minimal free resolution

Suppose $I = (F_{0,1}, \ldots, F_{0,t_0})$ is an ideal of R. We can construct an R-module homomorphism

$$\varphi_0 : R^{t_0} \to I \subseteq R^1$$

by

$$\varphi_0\left(\begin{bmatrix} G_1 \\ \vdots \\ G_{t_0} \end{bmatrix}\right) = G_1F_{0,1} + \cdots + G_{t_0}F_{0,t_0} = \begin{bmatrix} F_{0,1} & F_{0,2} & \cdots & F_{0,t_0} \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \\ \vdots \\ G_{t_0} \end{bmatrix}.$$

Definition 3.1. Let M be a R-module and suppose $\{F_1, \ldots, F_t\} \subseteq M$. A syzygy of $F_1, \ldots, F_t$ is a t-tuple. $(G_1, \ldots, G_t) \in R^t$ such that

$$G_1F_1 + \cdots + G_tF_t = 0.$$

We make some observations about the map φ_0:

(1)

$$\ker\varphi_0 = \left\{ \begin{bmatrix} G_1 \\ \vdots \\ G_{t_0} \end{bmatrix} \middle| \varphi_0\left(\begin{bmatrix} G_1 \\ \vdots \\ G_{t_0} \end{bmatrix}\right) = G_1F_{0,1} + \cdots + G_{t_0}F_{0,t_0} = 0 \right\}$$
$$= \{\text{all syzygies of } F_{0,1}, \dots, F_{0,t_0}\}$$
$$= \text{first syzygy module of } I.$$

(2) $\ker\varphi_0$ is a finitely generated submodule of R^{t_0}, i.e., there exists $\underline{F}_{1,1}, \dots, \underline{F}_{1,t_1}$, such that

$$\ker\varphi_0 =< \underline{F}_{1,1}, \dots, \underline{F}_{1,t_1} >= \{G_1\underline{F}_{1,1} + \cdots + G_{t_1}\underline{F}_{1,t_1} \mid G_i \in R\}.$$

(3) $\ker\varphi_0$ is like the null space of matrix A, i.e.

$$\mathrm{Nul}(A) = \{\overrightarrow{x} \in \mathbb{R}^n \mid A\overrightarrow{x} = \overrightarrow{0}\}.$$

We can now define a map $\varphi_1 : R^{t_1} \to \ker\varphi_0 \subseteq R^{t_0}$ by

$$\varphi_1\left(\begin{bmatrix} G_1 \\ \vdots \\ G_{t_1} \end{bmatrix}\right) = G_1\underline{F}_{1,t_1} + \cdots + G_{t_1}\underline{F}_{1,t_1} = \begin{bmatrix} \underline{F}_{1,1} & \underline{F}_{1,2} & \cdots & \underline{F}_{1,t_1} \end{bmatrix} \begin{bmatrix} G_1 \\ \vdots \\ G_{t_1} \end{bmatrix}$$

We make some further observations:

(1) $\ker\varphi_1$ is called the second syzygy module.
(2) $\ker\varphi_1$ measures the relations among the generators of $\ker\varphi_0$.
(3) $\ker\varphi_1$ is finitely generated, i.e., there exist $F_{2,1}, \dots, F_{2,t_2} \in \ker\varphi_1$ such that

$$\ker\varphi_1 =< \underline{F}_{2,1}, \dots, \underline{F}_{2,t_2} >$$

We can repeat the above step to now create a map $\varphi_2 : R^{t_2} \to \ker\varphi_1 \subseteq R^{t_1}$. In fact, we continue to reiterate this process. Eventually, this process will stop because of the following theorem:

Theorem 3.2 (Hilbert Syzygy Theorem). *If $R = k[x_1, \dots, x_n]$, then there exists an $l \le n$ such that $\ker\varphi_l \cong 0$, i.e., the l^{th} syzygy module is 0.*

4. The Resolution of an ideal

We now tie the above ideas together to describe the resolution of an ideal. Associated to any ideal $I \subseteq R = k[x_1, \dots, x_n]$ is a minimal free resolution of the form

$$0 \longrightarrow R^{t_l} \xrightarrow{\varphi_l} R^{t_{l-1}} \xrightarrow{\varphi_{l-1}} \cdots \xrightarrow{\varphi_2} R^{t_1} \xrightarrow{\varphi_1} R^{t_0} \xrightarrow{\varphi_0} I \longrightarrow 0$$

where

- $l \le n$,
- $\mathrm{Im}\,\varphi_{i+1} = \ker\varphi_i$, and

- each φ_i is represented by $t_{i-1} \times t_i$ matrix with entries in R.

Definition 4.1. The ith Betti number of I, denoted $\beta_i(I)$, equals t_i, the rank of R appearing in the ith step of the resolution. The number $\beta_i(I)$ is the number of minimal generators of $\ker \varphi_{i-1}$.

Example 4.2. Let $R = k[x, y, z]$ and $I = (x^2, y^2, z)$. Then the minimal resolution is

$$0 \longrightarrow R \xrightarrow{\begin{bmatrix} z \\ -y^2 \\ x^2 \end{bmatrix}} R^3 \xrightarrow{\begin{bmatrix} y^2 & z & 0 \\ -x^2 & 0 & z \\ 0 & -x^2 & -y^2 \end{bmatrix}} R^3 \xrightarrow{[x^2 \ y^2 \ z]} I \longrightarrow 0$$

Example 4.3. Suppose $F_1, \ldots, F_n$ is a regular sequence on R. If $I = (F_1, \ldots, F_n)$, then the minimal free resolution has form

$$0 \longrightarrow R^{\binom{n}{n}} \longrightarrow R^{\binom{n}{n-1}} \longrightarrow \cdots \longrightarrow R^{\binom{n}{3}} \longrightarrow R^{\binom{n}{2}} \longrightarrow R^{\binom{n}{1}} \longrightarrow I \longrightarrow 0.$$

Note that we have omitted the maps (although they are easy to write down in this case). In many situations we are primarily interested in the Betti numbers.

5. The Graded Resolution

Return to the example $I = (x^2, y^2, z)$ in $R = k[x, y, z]$. The elements in each matrix defining a map are in fact homogeneous elements. The degrees of these elements are also of interest. We can modify the construction so that we can extract this information.

Definition 5.1. Let M and N be graded R-modules, i.e.

$$M = \bigoplus_{i \in \mathbb{Z}} M_i \text{ and } N = \bigoplus_{i \in \mathbb{Z}} N_i$$

An R-module homomorphism $\varphi : M \longrightarrow N$ is graded of degree 0 if $\varphi(M_a) \subseteq N_a$ for all a. i.e., degree a elements of M are mapped to elements of degree a of N.

Definition 5.2. If $R = k[x_1, \ldots, x_n]$, then the graded R-module shifted by $a \in \mathbb{N}$ is $R(-a)$ where

$$R(-a)_i = R_{i-a},$$

that is, the degree i part of $R(-a)$ equals the degree $i - a$ part of R.

Example 5.3. $1 \in R(-5)$ has $\deg 1 = 5$, since $1 \in R(-5)_5 = R_{5-5} = R_0$. Similarly, $x^2 + y^2 \in R(-5)$ has $\deg x^2 + y^2 = 7$, since $x^2 + y^2 \in R(-5)_7 = R_2$.

Definition 5.4. Let $d_1, \ldots, d_t \in \mathbb{N}$. Then

$$R(-d_1) \oplus \ldots \oplus R(-d_t) = \{(G_1, \ldots, G_t) \mid G_i \in R(-d_i)\}.$$

Definition 5.5. $(G_1, \ldots, G_t)$ is homogeneous of degree d in $R(-d_1)\oplus \ldots \oplus(-d_t)$ if $G_i \in R(-d_i)_d$ for each i

Example 5.6. $(x^2+y^2, zxy) \in R(-5) \oplus R(-4)$ is homogeneous of degree 7.

6. Graded Resolution Construction

Let $I = (F_{0,1}, \ldots, F_{0,t_0})$ be a homogeneous ideal with degree $\deg F_{0,i} = d_{0,i}$. Define a map

$$\varphi_0 : R(-d_{0,1}) \oplus R(-d_{0,2}) \oplus \cdots \oplus R(-d_{0,t_0}) \longrightarrow (F_{0,1}, \ldots, F_{0,t_0}) \subseteq R$$

by

$$\varphi_0((G_1, \ldots, G_{t_0})) = G_1 F_{0,1} + \cdots + G_{t_0} F_{0,t_0}.$$

Then the map φ_0 has degree 0. To see this, note that if $(G_1, \ldots, G_{t_0})$ is homogeneous of degree d in $R(-d_{0,1}) \oplus \ldots \oplus R(-d_{0,t_0})$ then $\deg G_i = d - d_{0,i}$ in R. So

$$\varphi((G_1, \ldots, G_{t_0})) = G_1 F_{0,1} + \cdots + G_{t_0} F_{0,t_0} \text{ is homogeneous of } \deg d.$$

One can show that $\ker \varphi_0 =< \underline{F}_{1,1}, \ldots, \underline{F}_{1,t_1} >$ is generated by homogeneous elements in $R(-d_{0,1}) \oplus \cdots \oplus R(-d_{0,t_0})$ of degree $d_{1,1}, \ldots, d_{1,t_1}$, respectively. Repeat the above ideas to get a map

$$\varphi_1 : R(-d_{1,1}) \oplus \cdots \oplus R(-d_{1,t_1}) \to \ker \varphi_1 \subseteq R(-d_{0,1}) \oplus \cdots \oplus R(-d_{0,t_0})$$

defined by

$$(G_1, \ldots, G_t) \mapsto \begin{bmatrix} \underline{F}_{1,1} & \ldots & \underline{F}_{1,t_1} \end{bmatrix} \begin{bmatrix} G_1 \\ \vdots \\ G_t \end{bmatrix}.$$

Again, $\ker \varphi_1$ is generated by homogeneous elements. We continue to reiterate this process until $\ker \varphi_l = 0$ for some l (which is guaranteed by the Hilbert Syzygy Theorem).

So, associated to any homogeneous ideal $I \subseteq R = k[x_1, \ldots, x_n]$ is a minimal graded free resolution of the form

$$0 \longrightarrow \mathcal{F}_l \xrightarrow{\varphi_l} \mathcal{F}_{l-1} \xrightarrow{\varphi_{l-1}} \cdots \longrightarrow \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_0 \xrightarrow{\varphi_0} I \longrightarrow 0$$

where

- $l \leq n$
- φ_i is a matrix with entries in R
- $\mathcal{F}_i = R(-d_{i,1}) \oplus \cdots \oplus R(-d_{i,t_i})$

Definition 6.1. The i,jth graded Betti number, denoted $\beta_{i,j}(I)$, equals the number of times $R(-j)$ appears in $\mathcal{F}_i$. Equivalently, $\beta_{i,j}(I)$ is the number of minimal generators of degree j of $\ker \varphi_{i-1}$.

Problems from Lecture 7

1. Let F be a homogeneous polynomial of degree d in the polynomial ring $R = k[x_1, \ldots, x_n]$. If $I = (F)$, then find the minimal free graded resolution of I.

2. Let $R = k[x_1, x_2]$.
   (*i*) Describe the minimal free graded resolution of $I = (x_1, x_2)$.
   (*ii*) Describe the minimal free graded resolution of $I = (x_1^a, x_2^b)$ where a and b are any positive integers.
   (*iii*) Let F and G be a regular sequence on R and suppose $\deg F = a$ and $\deg G = b$. Describe the minimal free graded resolution of $I = (F, G)$. (Hint: Compare to part (*ii*))

3. Let $I = (x_1x_2, x_2x_3)$ in $R = k[x_1, x_2, x_3]$. Describe the minimal free graded resolution of I.