Lecture VIII: Introduction to reduced simplicial homology (March 7, 2006)
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From last week: if $I \subseteq R = k[x_1, \ldots, x_n]$ is a homogeneous ideal, then associated to I is a minimal free graded resolution of the form:

$$0 \longrightarrow \bigoplus_j R(-j)^{\beta_{l,j}(I)} \longrightarrow \bigoplus_j R(-j)^{\beta_{l-1,j}(I)} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}(I)} \longrightarrow I \longrightarrow 0$$

where $\beta_{i,j}(I)$ is the i,jth graded Betti number of I.

Question Suppose $I = I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex Δ. How is $\beta_{i,j}(I_\Delta)$ related to Δ?

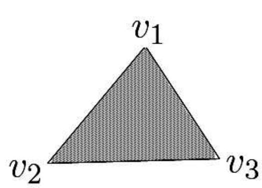
Mel Hochster found a connection using simplicial homology. We save the connection until the next lecture. Today, we introduce the language of reduced simplicial homology.

1. Orientation

A face of dimension q is sometimes called a q-simplex. We put an orientation on each simplex. An oriented 0-simplex is just a vertex $[v]$. An oriented 1-simplex is a directed edge $[v_1, v_2] \Leftrightarrow$

want to distinguish $[v_1, v_2]$ from $[v_2, v_1] \Leftrightarrow$

We make the convention that $[v_1, v_2] = -[v_2, v_1]$. An oriented 3-simplex is a triangle with vertices in some order.



Let $[v_1, v_2, v_3]$ denote the ordered vertices. Note that $[v_1, v_2, v_3] = [v_2, v_3, v_1] = [v_3, v_1, v_2]$ since they all go in same direction around the triangle. Because the order of the vertices $[v_1, v_3, v_2] = [v_3, v_2, v_1] = [v_2, v_1, v_3]$ go in the reverse direction, we set

$$[v_1, v_2, v_3] = [v_2, v_3, v_1] = [v_3, v_1, v_2] = -[v_1, v_3, v_2] = -[v_3, v_2, v_1] = -[v_2, v_1, v_3].$$

Observe

$$[v_i, v_j, v_k] = \begin{cases} [v_1, v_2, v_3] & \text{if } \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \text{ is an even permutation,} \\ -[v_1, v_2, v_3] & \text{if } \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix} \text{ is an odd permutation.} \end{cases}$$

In general, if F is a q-simplex whose vertices have been ordered

$$[v_1, v_2, \ldots, v_{q+1}]$$

then

$$[v_{i_1}, v_{i_2}, \ldots, v_{i_{q+1}}] = \begin{cases} [v_1, v_2, \ldots, v_{q+1}] & \text{if } \begin{pmatrix} 1 & 2 & \ldots & q+1 \\ i_1 & i_2 & \ldots & i_{q+1} \end{pmatrix} \text{ is an even permutation,} \\ -[v_1, v_2, \ldots, v_{q+1}] & \text{if } \begin{pmatrix} 1 & 2 & \ldots & q+1 \\ i_1 & i_2 & \ldots & i_{q+1} \end{pmatrix} \text{ is an odd permutation.} \end{cases}$$

2. Boundaries

The boundary of the 0-simplex $[v]$ is empty, i.e.

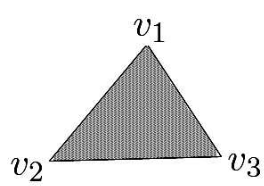
$$\partial_0([v]) = 0.$$

The boundary of the 1-simplex $[v_1, v_2]$ is

$$\partial_1([v_1, v_2]) = [v_2] - [v_1].$$

This is simply the formal difference of end point and initial point. The boundary of the 2-simplex $[v_1, v_2, v_3]$ is

$$\partial_2([v_1, v_2, v_3]) = [v_2, v_3] - [v_1, v_3] + [v_1, v_2].$$



Since $-[v_1, v_3] = [v_3, v_1]$, notice that the "sum" corresponds to the boundary of the triangle by traveling around the edges.

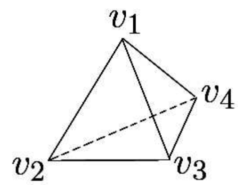
The boundary of the q-simplex $[v_1, v_2, \ldots, v_{q+1}]$ is

$$\partial_q([v_1, v_2, \ldots, v_{q+1}]) = \sum_{i=1}^{q+1} (-1)^{i+1} [v_1, \ldots, \hat{v}_i, \ldots, v_{q+1}]$$

where by $\hat{v}_i$ we mean v_i is removed.

Example 2.1. Consider the 3-simplex $[v_1, v_2, v_3, v_4]$. Then

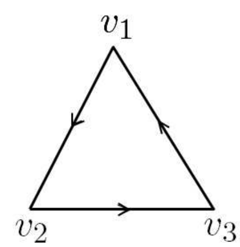
$$\partial_3([v_1, v_2, v_3, v_4]) = [v_2, v_3, v_4] - [v_1, v_3, v_4] + [v_1, v_2, v_4] - [v_1, v_2, v_3].$$



3. Chains and Cycles

Let k be a field and let $F_q(\Delta)$ denote all q-simplexes of Δ. Let $k^{F_q(\Delta)}$ be the vector space over k whose basis elements are the oriented q-simplexes of $F_q(\Delta)$. Elements of $k^{F_q(\Delta)}$ are called q-chains.

Example 3.1. Using the simplicial complex



we have

$$\begin{aligned} F_{-1}(\Delta) &= \{\emptyset\} = \{[0]\} \\ F_0(\Delta) &= \{[v_1], [v_2], [v_3]\} \\ F_1(\Delta) &= \{[v_1, v_2], [v_2, v_3], [v_3, v_1]\}. \end{aligned}$$

Thus, we get

$$\begin{aligned} k^{F_{-1}(\Delta)} &= \{c[0] \mid c \in k\} \cong \mathbb{Q} \\ k^{F_0(\Delta)} &= \{c_1[v_1] + c_2[v_2] + c_3[v_3] \mid c_i \in \mathbb{Q}\} \cong \mathbb{Q}^3 \\ k^{F_1(\Delta)} &= \{c_1[v_1, v_2] + c_2[v_2, v_3] + c_3[v_3, v_1] \mid c_i \in \mathbb{Q}\} \cong \mathbb{Q}^3 \end{aligned}$$

We make the convention that $k^{F_q(\Delta)} = 0$ for $q > \dim \Delta$ and $q < -1$.

Fact. $\dim_k k^{F_q(\Delta)} = |F_q(\Delta)| = \#$ q-simplexes.

The boundary gives a map $\partial_q : k^{F_q(\Delta)} \longrightarrow k^{F_{q-1}(\Delta)}$ as follows:

$$\partial_q(\sum_i m_i[v_{1,i}, \ldots, v_{(q+1),i}]) = \sum_i m_i \partial_q([v_{1,i}, \ldots, v_{(q+1),i}])$$

Example 3.2. Let Δ be as above. Then

$$7[v_1, v_2] + 2[v_2, v_3] + 3[v_3, v_1] \in k^{F_1(\Delta)} \text{ is a 1-chain}$$

Then

$$\begin{aligned}
\partial_1(7[v_1, v_2] + 2[v_2, v_3] + 3[v_3, v_1]) &= 7\partial_1([v_1, v_2]) + 2\partial_1([v_2, v_3]) + 3\partial_1([v_3, v_1]) \\
&= 7([v_2] - [v_1]) + 2([v_3] - [v_2]) + 3([v_1] - [v_3]) \\
&= 7[v_2] - 7[v_1] + 2[v_3] - 2[v_2] + 3[v_1] - 3[v_3] \\
&= -4[v_1] + 5[v_2] - [v_3] \in k^{F_0(\Delta)}.
\end{aligned}$$

The elements of $\ker \partial_q$ are called q-cycles. To see why this is an appropriate name, return to the above example.

Example 3.3. Note that $v = [v_1, v_2] + [v_2, v_3] + [v_3, v_1] \in k^{F_1(\Delta)}$ forms a cycle in Δ. Then

$$\partial_1(v) = \partial_1([v_1, v_2]) + \partial_2([v_2, v_3]) + \partial_3([v_3, v_1]) = [v_2] - [v_1] + [v_3] - [v_2] + [v_1] - [v_3] = 0.$$

So v is a in the kernel of ∂_1. In other words, a cycle is sent to 0.

Definition 3.4.

$$\begin{aligned}
\ker \partial_q &= \text{group of } q\text{-cycles.} \\
\operatorname{Im} \partial_q &= \text{group of } (q-1)\text{-boundaries.}
\end{aligned}$$

Theorem 3.5. *For all q,* $\operatorname{Im} \partial_{q+1} \subseteq \ker \partial_q$.

Here is a the main idea behind the proof. Suppose $v = [v_1, v_2, \dots, v_{q+1}, v_{q+2}] \in k^{F_{q+2}(\Delta)}$ is a $(q+1)$-simplex. Then $\partial_{q+1}(v)$ is the boundary of v. This boundary forms a "cycle" in Δ. Since all cycles are sent to 0, we get

$$\partial_q(\partial_{q+1}(v)) = 0 \Leftrightarrow \operatorname{Im}\partial_{q+1} \subseteq \ker \partial_q.$$

Definition 3.6. The q^{th} reduced homology of Δ over k is

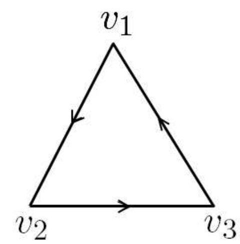
$$\tilde{H}_q(\Delta, k) = \frac{\ker \partial_q}{\operatorname{Im}\partial_{q+1}}.$$

The homology of Δ measures the "holes" in the simplicial complex. To see this, suppose

$$\tilde{H}_q(\Delta, k) \neq 0 \Leftrightarrow \operatorname{Im}\partial_{q+1} \subsetneq \ker \partial_q.$$

So, there is a q-chain that forms a "cycle" in Δ, but this q-chain is not the boundary of a $(q+1)^{th}$-simplex, i.e. the boundary is there, but not the face itself.

Example 3.7. Consider $\Delta =$



Since we have no faces of dimension 2 or bigger, $k^{F_q(\Delta)} = 0$ for $q \geq 2$. We have a series of maps

$$0 \xrightarrow{\partial_2} k^{F_1(\Delta)} \xrightarrow{\partial_1} k^{F_0(\Delta)} \xrightarrow{\partial_0} k^{F_{-1}(\Delta)} \longrightarrow 0$$

Note that $\operatorname{Im} \partial_2 = (0)$. So

$$\tilde{H}_1(\Delta, k) = \frac{\ker \partial_1}{\mathrm{Im} \partial_2} = \ker \partial_1.$$

Now $[v_1, v_2] + [v_2, v_3] + [v_3, v_1] \in k^{F_1(\Delta)}$ and in $\ker \partial_1$. So

$$T = \{c([v_1, v_2] + [v_2, v_3] + [v_3, v_1]) \mid c \in k\} \subseteq \ker \partial_1.$$

We claim that in fact $\ker \partial_1 = T$. So, suppose $v = m_1[v_1, v_2] + m_2[v_2, v_3] + m_3[v_3, v_1] \in \ker \partial_1$. This implies that $\partial(v) = m_1[v_2] - m_1[v_1] + m_2[v_3] - m_2[v_2] + m_3[v_1] - m_3[v_3] = 0$ which means that $m_1 = m_2 = m_3$. So, we get that $v \in T$.

Hence $\tilde{H}_1(\Delta, k) \cong \ker \partial_1 \cong k$. Notice that Δ has a "hole". It has the boundary for $[v_1, v_2, v_3]$, but no $[v_1, v_2, v_3]$.

Theorem 3.8.

$$\tilde{H}_0(\Delta, k) + 1 = \textit{number of connected components of } \Delta.$$

Example 3.9. In our previous examples, $\tilde{H}_0(\Delta, k) = 0$, since Δ is connected.

Problems from Lecture 8

1. Let $[v_1, v_2, v_3, v_4, v_5]$ be an oriented 4-simplex. Show that

$$\partial_3(\partial_4([v_1, v_2, v_3, v_4, v_5])) = 0.$$

2. Let Δ be a simplicial complex with facets $\{\{v_1, v_2\}, \{v_2, v_3\}, \{v_3, v_4\}, \{v_4, v_1\}\}$ and let $k = \mathbb{Q}$. Suppose we have put an orientation on the faces so that oriented 1-simplexes are: $[v_1, v_2], [v_2, v_3], [v_3, v_4], [v_4, v_1]$. Prove that

$$\widetilde{H}_i(\Delta, k) = \begin{cases} \mathbb{Q} & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

3. Let rank $\widetilde{H}_i(\Delta, k) = \dim_k \ker \partial_i - \dim_k \operatorname{Im} \partial_{i+1}$. Let Δ be a simplicial complex of dimension d and f_i the number of i-faces of Δ. If k is a field, then prove that

$$\sum_{i=-1}^{d} (-1)^i \operatorname{rank} \widetilde{H}_i(\Delta, k) = -1 + \sum_{i=0}^{d} (-1)^i f_i$$

(Hint: $\dim_k k^{F_i(\Delta)} - \dim_k \ker \partial_i = \dim_k \operatorname{Im} \partial_i$ where $F_i(\Delta)$ is the set of all faces of dimension i of Δ.)